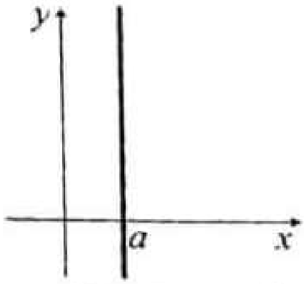
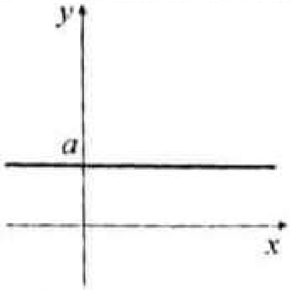
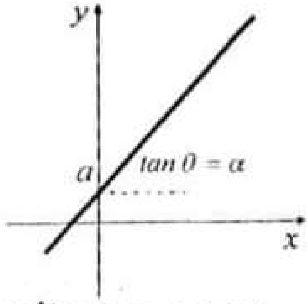
Quelques Courbes du Plan et leurs Équations



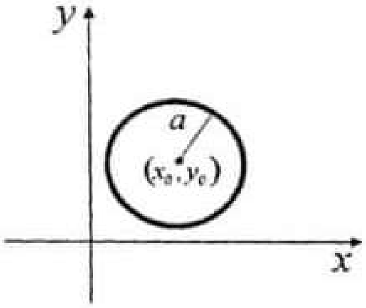
Droite: $x = a$.



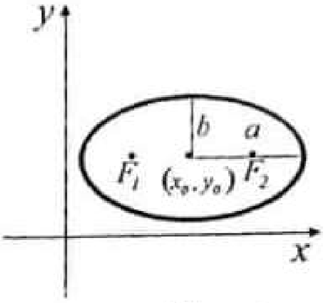
Droite: $y = a$.



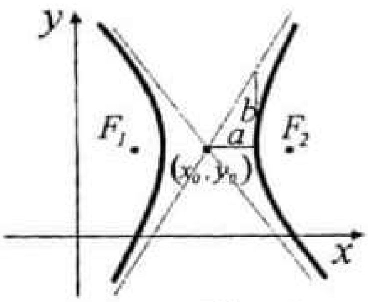
Droite: $y = \alpha x + a$.



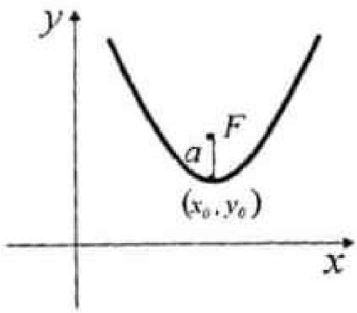
Cercle: $(x - x_0)^2 + (y - y_0)^2 = a^2$.



Ellipse: $\dfrac{(x - x_0)^2}{a^2} + \dfrac{(y - y_0)^2}{b^2} = 1$.

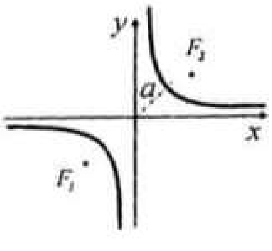


Hyperbole: $\dfrac{(x - x_0)^2}{a^2} - \dfrac{(y - y_0)^2}{b^2} = 1$.

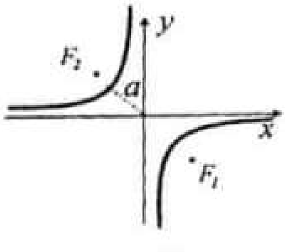


Parabole: $y - y_0 = \dfrac{(x - x_0)^2}{4a}$.

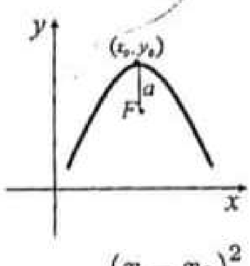
Autres positions de l'hyperbole et de la parabole



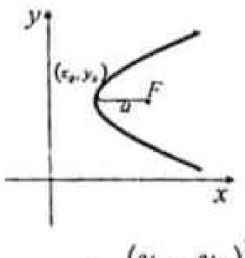
$xy = \dfrac{a}{\sqrt{2}}$.



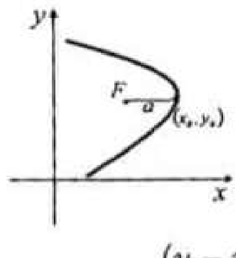
$xy = -\dfrac{a}{\sqrt{2}}$.



$y - y_0 = -\dfrac{(x - x_0)^2}{4a}$.



$x - x_0 = \dfrac{(y - y_0)^2}{4a}$.



$x - x_0 = -\dfrac{(y - y_0)^2}{4a}$.

Transformations Conformes du Plan complexe

***Si une transformation de la variable z du plan complexe garde les angles inchangés elle est appelée* transformation conforme.**

(Une transformation $z = f(z)$ est conforme si f est une fonction holomorphe (Voir Chap.III))

UNIVERSITE MED KHIDER
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FONCTIONS COMPLEXES ÉLÉMENTAIRES

Fonction exponentielle:

$$e : \mathbb{C} \to \mathbb{C}$$
$$z = x + iy \to e^z = e^x (\cos y + \sin x)$$

Propriétés de l'exponentielle :

$$1) \forall z_1, z_2 \in \mathbb{C},\ e^{z_1+z_2} = e^{z_1} e^{z_2} \quad 2) \forall z_1, z_2 \in \mathbb{C},\ \frac{e^{z_1}}{e^{z_2}} = e^{z_1 - z_2}$$
$$3) \forall z \in \mathbb{C},\ \frac{1}{e^z} = e^{-z} \quad 4) |e^z| = e^x,\ arg(e^z) = y$$

e^z est une fonction périodique, de période $2\pi i$, e^z est une fonction holomorphe.
Fonction logarithme : On appelle logarithme de z et on note logz , tout nombre complexe qui s'écrit :

$$\log z = \log |z| + i \arg z$$

Propriétés du logarithme :

$$1) \log (z_1 z_2) = \log (z_1) + \log (z_2) \quad 2) \log \left(\frac{1}{z}\right) = -\log (z)$$

- logz est une fonction holomorphe et $f'(z) = \frac{1}{z}$, $z \neq 0$
- Détermination principale du logarithme d'un nombre complexe z, $z \neq 0$, est le nombre :

$$\log z = \log |z| + i \arg z,\ 0 \leq \arg z \leq 2\pi$$

Fonctions trigonométriques :

$$1) \sin z = \frac{e^{iz} - e^{-iz}}{2i} \quad 2) \cos z = \frac{e^{iz} + e^{-iz}}{2}$$
$$3) tgz = \frac{\sin z}{\cos z} \quad 4) \cot gz = \frac{\cos z}{\sin z}$$

Fonctions hyperboliques :

$$1) shz = \frac{e^z - e^{-z}}{2} \quad 2) chz = \frac{e^z + e^{-z}}{2}$$
$$3) thz = \frac{shz}{chz} \quad 4) \coth z = \frac{chz}{shz}$$

Fonctions trigonométriques inverses :

$$1) \arcsin z = \frac{1}{i} \log \left(z + \sqrt{1 - z^2}\right) \quad 3) arctgz = \frac{1}{2i} \log \left(\frac{1 + iz}{1 - iz}\right)$$
$$2) \arccos z = \frac{1}{i} \log \left(z + \sqrt{z^2 - 1}\right) \quad 4) \operatorname{arccot} gz = \frac{1}{2i} \log \left(\frac{z + i}{z - i}\right)$$

Faculté des Sciences
Exactes et Sciences
de la Vie

2ème Année Physique (LMD)
Module: fonctions de la variable complexe
Ann Univ: 2019/2020

Solutions de la serie 4.

Ex 1 : $\int_C (3x+y)\,dx + (2y-x)\,dy$

① $C = \{(x,y) \in \mathbb{R}^2,\ y = x^2+1,\ 1 \leq y \leq 5,\ 0 \leq x \leq 2\}$

$\gamma(t) = \begin{cases} x = t \\ y = t^2+1 \end{cases} \quad 0 \leq t \leq 2 \qquad dx = dt,\quad dy = 2t\,dt$

$$\int_C (3x+y)\,dx + (2y-x)\,dy = \int_0^2 (3t+t^2+1)\,dt + (2t^2+2-t)\,2t\,dt$$

$$= \int_0^2 (3t+t^2+1+4t^3+4t-2t^2)\,dt$$

$$= \int_0^2 (4t^3 - t^2 + 7t + 1)\,dt$$

$$= \left(4 \cdot \frac{t^4}{4} - \frac{t^3}{3} + \frac{7}{2}t^2 + t\right|_0^2$$

$$= 16 - \frac{8}{3} + \frac{7 \cdot 4}{2} + 2 = 18 + 14 - \frac{8}{3} = \frac{88}{3}$$

② $C = \{(x,y) \in \mathbb{R}^2,\ y = 2x+1,\ 0 \leq x \leq 2,\ 1 \leq y \leq 5\}$

$\gamma(t) = \begin{cases} x = t \\ y = 2t+1 \end{cases} \quad dx = dt,\quad dy = 2\,dt, \quad 0 \leq t \leq 2$

$$\int_C (3x+y)\,dx + (2y-x)\,dy = \int_0^2 (3t+2t+1)\,dt + (4t+2-t)\,2\,dt$$

$$= \int_0^2 (5t+8t-2t+1+4)\,dt$$

$$= \int_0^2 (11t+5)\,dt = \frac{11}{2}t^2 + 5t\Big|_0^2 = \frac{11}{2}\,4 + 10 = 32$$

① $C = |z| = 1$: $z(t) = 1e^{it}$. $dz = ie^{it}dt$. $0 \leqslant t \leqslant 2$

$$\oint_C \bar{z}^2 dz = \oint_0^{2\pi} (e^{-it})^2 \cdot (ie^{it}) dt = \int_0^{2\pi} ie^{-2it+it} dt$$

$$= \int_0^{2\pi} i e^{-it} dt = -e^{-it}\Big|_0^{2\pi} = 0$$

② $C = |z-1| = 1 \Rightarrow z(t) = 1 + e^{it}$. $\dot{z}(t) = ie^{it}dt$. $t \in [0$

$$\oint_C \bar{z}^2 dz = \int^{2\pi} \overline{(1+e^{it})^2}\,(ie^{it})\, dt$$

$$= i\int_0^{2\pi} (1+e^{-it})^2 (e^{it}) dt = i\oint_0^{2\pi} \left(1+2e^{-it}+e^{-2it}\right)(e^{it})\, dt$$

$$= i\int_0^{2\pi} (e^{it}+2+e^{-it})\, dt = i\left(\frac{1}{i}e^{it}+2t-\frac{1}{i}e^{-it}\right|_0^{2\pi}$$

$= 4\pi i$

③ $\oint |z|^2 dz$, carré de sommets : $(0,0)$. $(1,0)$. $(1,1)$. $(0,1)$

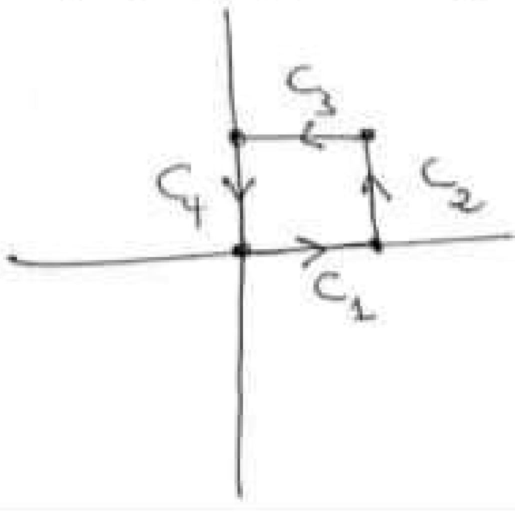
$C = C_1 + C_2 + C_3 + C_4$

$C_1 = \{(x,y) \in \mathbb{R}^2,\ y = 0,\ 0 \leqslant x \leqslant 1\}$

$C_2 = \{(x,y) \in \mathbb{R}^2,\ x = 1\ \ 0 \leqslant y \leqslant 1\}$

$C_3 = \{(x,y) \in \mathbb{R}^2,\ y = 1\ \ 0 \leqslant x \leqslant 1$

$C_4 = \{(x,y) \in \mathbb{R}^2,\ x = 0,\ 0 \leqslant y \leqslant 1.\}$



① $C = |z| = 1$:

$$\oint_C \bar{z}^2\,dz = \int_0^{2\pi} (e^{-it})^2 \cdot (ie^{it})\,dt = \int_0^{2\pi} ie^{-2it+it}\,dt \qquad z(t) = 1e^{it}, \quad dz = ie^{it}\,dt, \quad 0 \leqslant t \leqslant 2\pi$$

$$= \int_0^{2\pi} i\,e^{-it}\,dt = -e^{-it}\Big|_0^{2\pi} = 0$$

② $C = |z-1| = 1 \Rightarrow z(t) = 1 + e^{it}$. $\dot{z}(t) = ie^{it}dt$. $t \in [0, 2\pi]$

$$\oint_C \bar{z}^2\,dz = \int^{2\pi} \overline{(1+e^{it})^2}\,(ie^{it})\,dt$$

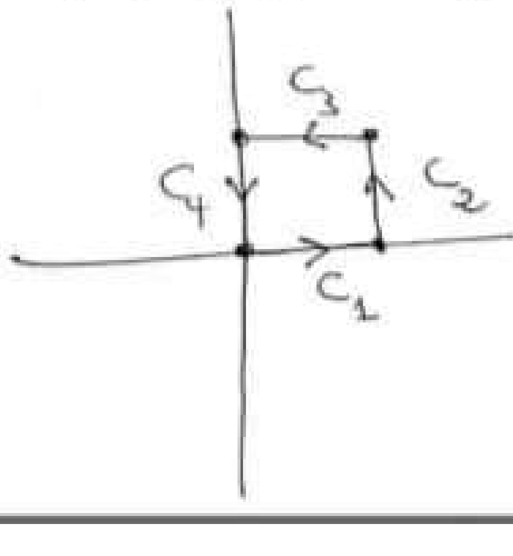
$$= i\int_0^{2\pi} (1+e^{-it})^2 (e^{it})\,dt = i\int_0^{2\pi} (1+2e^{-it}+e^{-2it})(e^{it})\,dt$$

$$= i\int_0^{2\pi} (e^{it}+2+e^{-it})\,dt = i\left(\frac{1}{i}e^{it}+2t-\frac{1}{i}e^{-it}\right)\Big|_0^{2\pi}$$

$$= 4\pi i$$

③ $\oint |z|^2\,dz$, carré de sommets : $(0,0) \cdot (1,0) \cdot (1,1) \cdot (0,1)$

$C = C_1 + C_2 + C_3 + C_4$



$C_1 = \{(x,y) \in \mathbb{R}^2,\ y = 0,\ 0 \leqslant x \leqslant 1\}$

$C_2 = \{(x,y) \in \mathbb{R}^2,\ x = 1\ 0 \leqslant y \leqslant 1\}$

$C_3 = \{(x,y) \in \mathbb{R}^2,\ y = 1\ 0 \leqslant x \leqslant 1$

$C_4 = \{(x,y) \in \mathbb{R}^2,\ x = 0,\ 0 \leqslant y \leqslant 1\}$

$$\int_{C_1} + \oint_{C_2} + \oint_{C_3} + \oint_{C_4}$$

$$= \oint_{C_1} (x^2+y^2)(dx+i\,dy) = \int_0^1 x^2\,dx = \frac{1}{3}x^3\Big|_0^1 = \frac{1}{3}$$

$$= \oint_{C_2} (x^2+y^2)(dx+i\,dy) = \int_0^1 (1+y^2)\,i\,dy = i\left(y+\frac{1}{3}y^3\right)\Big|_0^1 = \frac{4}{3}i$$

$$I_3 = \oint_{C_3} (x^2+y^2)(dx+i\,dy) = \int_1^0 (1+x^2)\,dx = \left(x+\frac{1}{3}x^3\right)\Big|_1^0 = -\frac{4}{3}$$

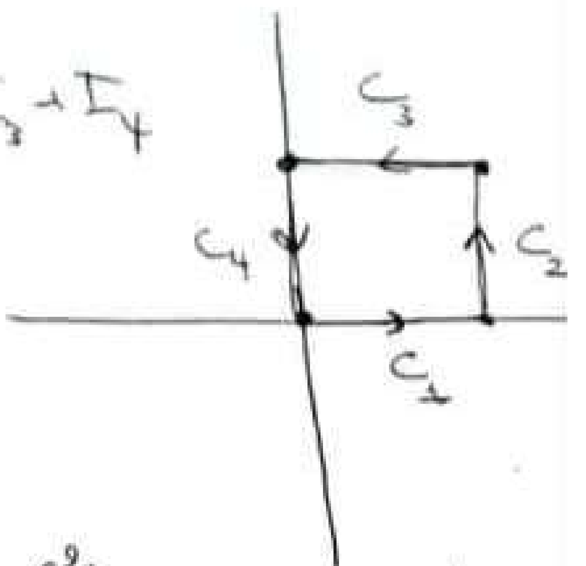
$$I_4 = \oint_{C_4} (x^2+y^2)(dx+i\,dy) = \int_1^0 i y^2\,dy = \frac{i}{3}y^3\Big|_1^0 = -\frac{i}{3}$$

$$I = I_1+I_2+I_3+I_4 = \frac{1}{3} + \frac{4}{3}i + -\frac{4}{3} - \frac{i}{3} = \frac{-3}{3} + \frac{3}{3}i = -1+i$$

<u>Ex3</u> :

1) $I = \oint_C (x^2 - 2xy)\,dx + (y^2 - x^3 y)\,dy$ où C est un carré de sommets $(0,0)\cdot(2,0)\cdot(2,2)\cdot(0,2)$

$C = C_1 + C_2 + C_3 + C_4 \Rightarrow I = I_1 + I_2 + I_3 + I_4$



$C_1 = \{(x,y) \in \mathbb{R}^2,\ y = 0,\ 0 \leq x \leq 2\}$

$$I_1 = \int_0^2 x^2\,dx = \frac{1}{3}x^3\Big|_0^2 = \frac{8}{3}$$

$C_2 = \{(x,y) \in \mathbb{R}^2,\ x = 2,\ 0 \leq y \leq 2\}$

$$I_2 = \int_0^2 (4-4y)\,d(2) + (y^2 - 8y)\,dy = \int_0^2 (y^2 - 8y)\,dy = \frac{-40}{3}$$

$0 \leq x \leq 2\}$

$I_3 = \int_2^0 (x^2 - 4x)\,dx = \frac{1}{3}x^3 - \frac{4}{2}x^2 \Big|_2^0 = -\frac{8}{3} + \frac{16}{2} = \frac{16}{3}$

$C_4 = \{(x,y) \in \mathbb{R}^2,\ x = 0,\ 0 \leq y \leq 2\}$

$I_4 = \int_2^0 y^2\,dy = \frac{1}{3}y^3 \Big|_2^0 = -\frac{8}{3}$

$I = I_1 + I_2 + I_3 + I_4 = \cancel{\frac{8}{3}} - \frac{40}{3} + \frac{16}{3} - \cancel{\frac{8}{3}} = \frac{-24}{3} = -8$

$I = \oint_C P\,dx + Q\,dy.$ $P(x,y) = x^2 - 2xy \Rightarrow \frac{\partial P}{\partial y} = -2x$

$Q(x,y) = y^2 - x^3 y \Rightarrow \frac{\partial Q}{\partial x} = -3x^2 y$

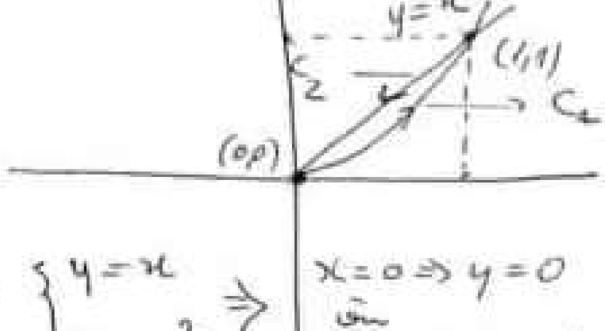
$I = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy = \iint_D (-3x^2 y + 2x)\,dx\,dy$

D et l'intérieur de carré de sommets $(0,0).(2,0).(2,2).(0,2$

$I = \int_0^2 \left[\int_0^2 (-3x^2 y + 2x)\,dx\right] dy = \int_0^2 -\frac{3}{3}x^3 y + \frac{2}{2}x^2 \Big|_0^2 dy$

$= \int_0^2 (-8y + 4)\,dy = -\frac{8}{2}y^2 + 4y \Big|_0^2 = -\frac{8}{2}\cdot 4 + 8 = -16 + 8 = -8$

b) $I = \oint_C (xy + y^2)\,dx + x^2\,dy$



$C = C_1 + C_2 \Rightarrow I = I_1 + I_2$

$C_1 = \{(x,y) \in \mathbb{R}^2,\ y = x^2,\ 0 \leq x \leq 1\}$

$\gamma(t) = \begin{cases} x = t \\ y = t^2 \end{cases} 0 \leq t \leq 1 \Rightarrow \gamma'(t) = \begin{cases} dx = dt \\ dy = 2t\,dt \end{cases}$ $\begin{cases} y = x \\ y = x^2 \end{cases} \Rightarrow$ $x = 0 \Rightarrow y = 0$ ou $x = 1 \Rightarrow y = 1$

$$\dots = \int_0^1 (3t^3 + t^4)\,dt$$

$$= \frac{3}{4}t^4 + \frac{1}{5}t^5 \Big|_0^1 = \frac{3}{4} + \frac{1}{5} = \frac{19}{20}$$

$$C_2 = \{(x,y) \in \mathbb{R}^2 ;\ y = x ,\ 0 < x < 1\}$$

$$C_2(t) = \begin{cases} x = t \\ y = t \end{cases} \quad 0 < t < 1 \Rightarrow C_2(t) \begin{cases} dx = dt \\ dy = dt \end{cases}$$

$$I_2 = \int_1^0 (t \cdot t + t^2)\,dt + t^2\,dt = \int_1^0 (3t^2)\,dt = \frac{3}{3}t^3 \Big|_0^1 = -1$$

$$I = I_1 + I_2 = \frac{19}{20} - 1 = \frac{19-20}{20} = \frac{-1}{20}$$

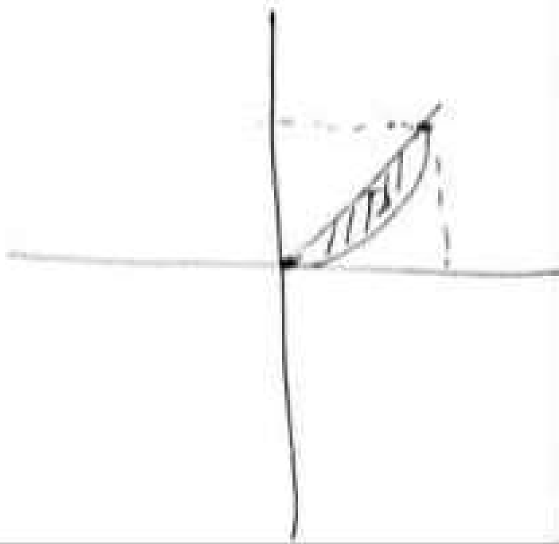
$$I = \oint_C P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$$

$$P(x,y) = xy + y^2 \Rightarrow \frac{\partial P}{\partial y} = x + 2y$$

$$Q(x,y) = x^2 \Rightarrow \frac{\partial Q}{\partial x} = 2x$$

$$I = \iint_D (2x - x - 2y)\,dx\,dy = \iint_D (x - 2y)\,dx\,dy$$

$$D = \{(x,y) \in \mathbb{R}^2 ,\ y > x^2 \text{ et } y < x\}$$



$$I = \int_0^1 \left(\int_{x^2}^{x} (x - 2y)\,dy \right) dx$$

$$\int_0 \int_{x^2} \dots 2y)\,dy\,|dx = \int_0 (xy - y^2)\Big|_{x^2}^{x}$$

$$= \int_0^1 (x^2 - x^2 - x^3 + x^4)\,dx = -\frac{1}{4}x^4 + \frac{1}{5}x^5\Big|_0^1 = -\frac{1}{20}$$

Ex V :

① $\oint_{|z|=1} z\,Re(z^2)\,dz$.

$z(t) = z_0 + re^{it} = e^{it}$. $0 \leq t \leq 2\pi$

$z'(t) = ie^{it}$, $z^2 = e^{2it} = \cos 2t + i\sin 2t$

$$\Rightarrow \int_0^{2\pi} e^{it}(\cos 2t)\, i e^{it}\,dt = i\int_0^{2\pi} e^{2it}\left(\frac{e^{2it} + e^{-2it}}{2}\right)dt$$

$$= \frac{i}{2}\int_0^{2\pi}(e^{4it} + 1)\,dt = \frac{\cancel{i}}{2}\cdot\frac{1}{4\cancel{i}}e^{4it} + \frac{i}{2}t\Big|_0^{2\pi}$$

$$= \frac{1}{8}e^{4it} + \frac{i}{2}t\Big|_0^{2\pi} = \frac{1}{8}e^{8i\pi} + \frac{i}{2}2\pi - \frac{1}{8}e^0$$

$$= \frac{1}{8} + i\pi - \frac{1}{8} = i\pi$$

② $\oint_{|z-a|=r} \frac{dz}{z-a}$,

formule de Cauchy : $f(z_0) = \frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z - z_0} \Rightarrow$

$\int_\gamma \frac{f(z)}{z - z_0} = 2\pi i f(z_0)$ si z_0 intérieur à Ω

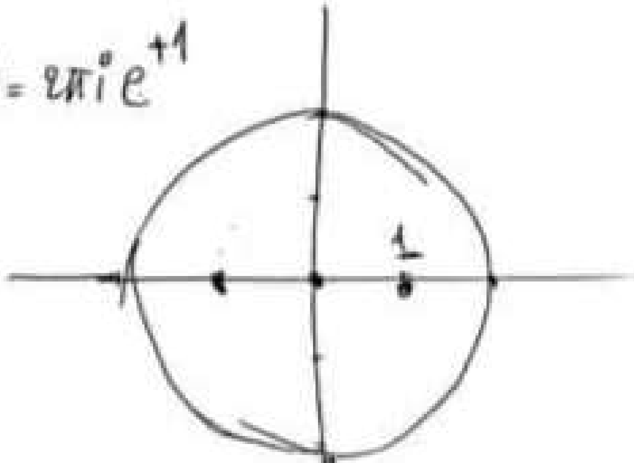
alors : $\int_{|z-a|=r} \frac{1\cdot dz}{z-a} = 2\pi i \cdot 1(a) = 2\pi i$

$\int_{|z|=2} \frac{\ldots}{z-1} \ldots$, e^z fonction analy[...]
disque de centre 0 et rayon 2
γ est le bord de ce disque.

la formule de Cauchy: $\displaystyle\int_\gamma \frac{f(z)}{z-z_0}\,dz = \begin{cases} 2\pi i\, f(z_0) & \text{si } z_0 \in D(0,2) \\ 0 & \text{si } z_0 \notin D(0,2) \end{cases}$

$z_0 = +1 \in D(0,2)$ alors:

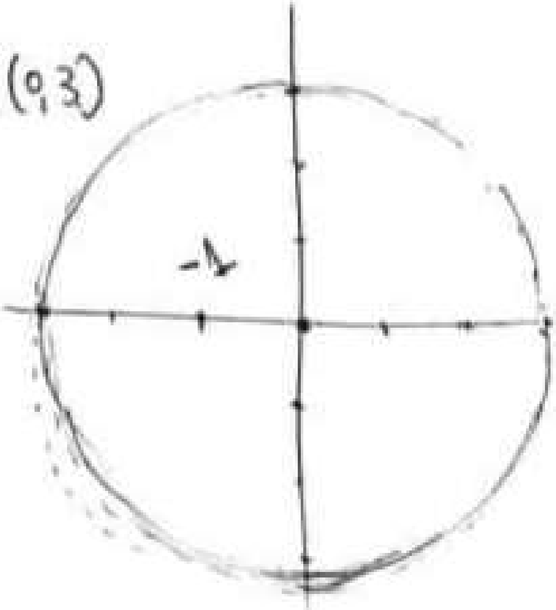
$$\oint_{|z|=2} \frac{f(z)}{z-1}\,dz = \oint_{|z|=2} \frac{e^z}{z-1}\,dz = 2\pi i\, f(z_0) = 2\pi i\, e^{+1}$$



④ $\displaystyle\oint_{|z|=3} \frac{e^{2z}}{(z+1)^4}\,dz$. la formule de Cauchy:

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \int_\gamma \frac{f(z)}{(z-z_0)^{n+1}}\,dz \;,\; f(z) = e^{2z} \;,\; n+1 = 4 \Rightarrow n = 3$$

$z_0 = -1$, $f(z)$ analytique dans $D(0,3)$
γ est le bord de $D(0,3)$.
$z_0 = -1 \in D(0,3)$. alors



$$f^{(3)}(-1) = \frac{3!}{2\pi i} \int_\gamma \frac{e^{2z}}{(z+1)^4}\,dz$$

$$\Rightarrow \oint_\gamma \frac{e^{2z}}{(z+1)^4}\,dz = \frac{2\pi i}{6} f^{(3)}(-1)$$

$f(z) = e^{2z} \Rightarrow f'(z) = 2e^{2z}, f''(z) = 4$

(a) $(-1) = 8e^{-2}$

$$\frac{e^{2z}}{(z+1)^4}dz = \frac{2\pi i}{6} \cdot 8e^{-2} = \frac{8}{3}\pi i e^{-2}$$

$$I = \oint_{|z|=3} \frac{\sin(\pi z^2) + \cos(\pi z^2)}{(z-1)(z-2)}, \quad \frac{1}{(z-1)(z-2)} = \frac{-1}{z-1} + \frac{1}{z-2}$$

$$I = \oint_{|z|=3} \frac{-(\sin(\pi z^2) + \cos(\pi z^2))}{z-1}dz + \oint_{|z|=3} \frac{\sin(\pi z^2) + \cos(\pi z^2)}{z-2}dz$$

$= I_1 + I_2$

$I_1 = -\oint_{|z|=3} \frac{f(z)}{z-1}dz$, $f(z) = \sin \pi z^2 + \cos \pi z^2$ est holomorph dans $\mathbb{C} \Rightarrow$ holomorphe dans $\overline{D(0,3)}$

Alors d'après la formule de Cauchy, on a :

$$\int_\gamma \frac{f(z)}{z-z_0}dz = \begin{cases} 2\pi i f(z_0) & \text{si } z_0 \in D(0,3) \\ 0 & \text{si } z_0 \notin D(0,3) \end{cases}$$

• $z_0 = 1 \in D(0,3) \Rightarrow I_1 = -2\pi i f(1)$

$= -2\pi i(\sin \pi(1)^2 + \cos \pi(1)^2)$

$= 2\pi i$

• $z_0 = 2 \in D(0,3)$

$$I_2 = \int_{|z|=3} \frac{f(z)}{z-2}dz = 2\pi i f(2) = 2\pi i(\sin(\pi 4) + \cos(4\pi))$$

$= 2\pi i$

$I = I_1 + I_2 = 2\pi i + 2\pi i = 4\pi i$