Position Vector and Force Vector

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Objectives :

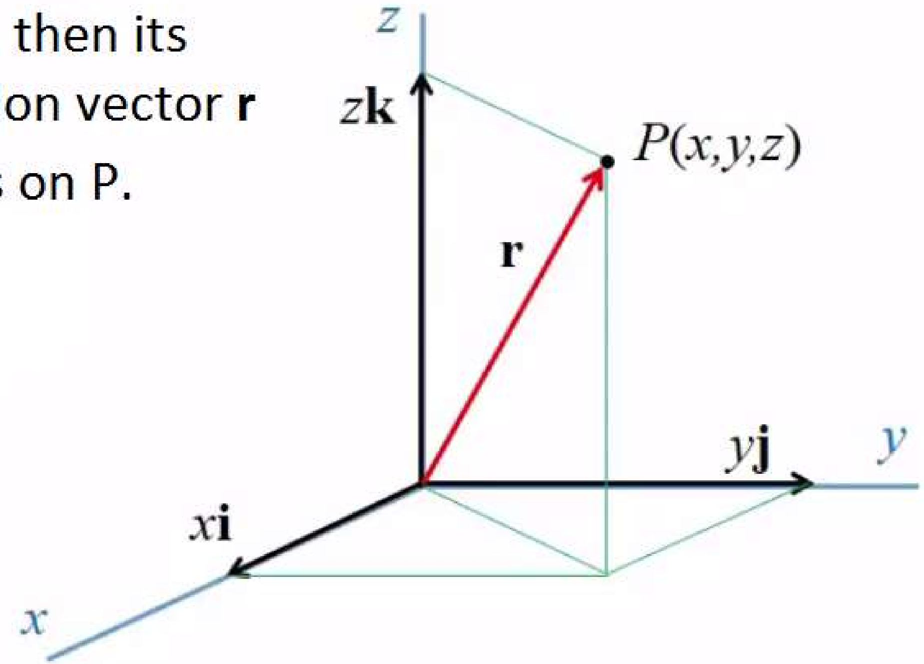
- To introduce the concept of position vector.
- To demonstrate the general strategy of writing the force vector from the corresponding position vector.

Question 1: You have learned about the unit vector. Can different physical quantities such as position, velocity, acceleration or force have the same unit vector? Why or why not?

Position vector

If a point P has coordinates x, y and z, then its position can be expressed by its position vector **r** which starts from the **origin** and ends on P.

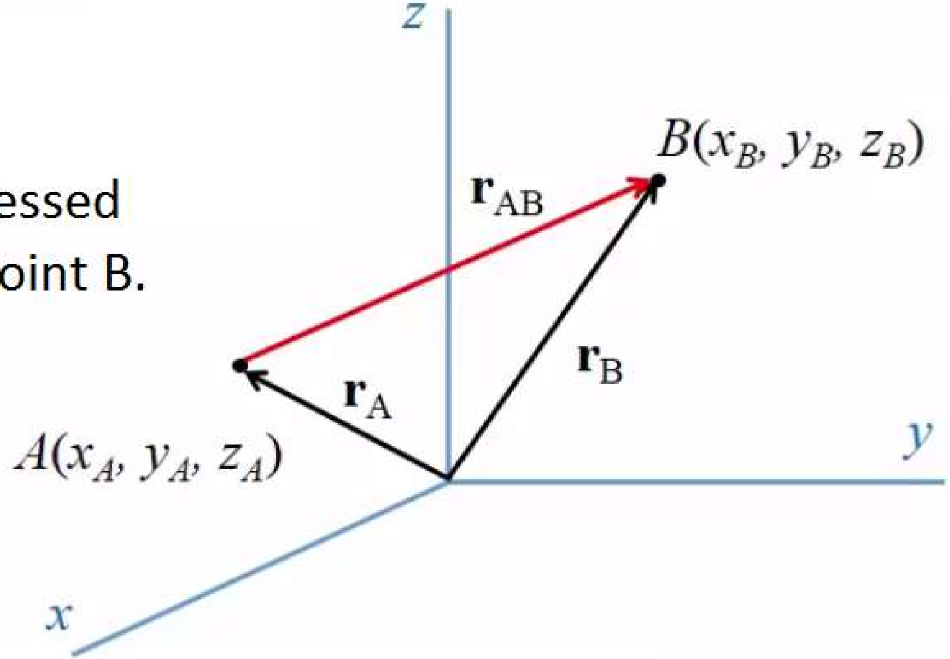
$$\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$$



Position vector

If you want to find the relative position of ponit B relative to point A, this relative position can be expressed by a vector **r**AB that starts from point A and end on point B.

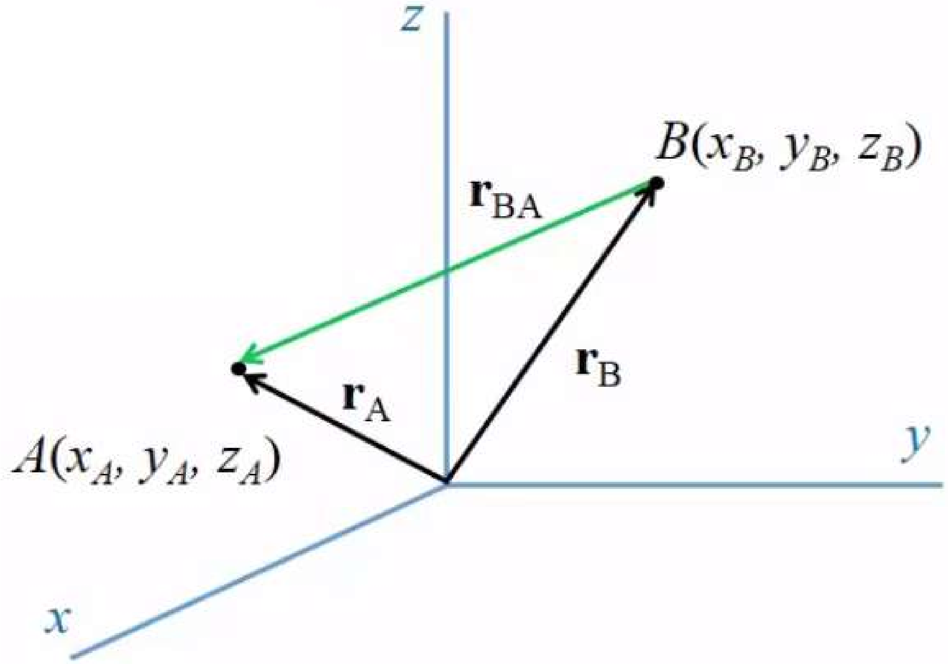
$$\mathbf{r}_{AB} = \mathbf{r}_B - \mathbf{r}_A$$
$$= (x_B - x_A)\mathbf{i} + (y_B - y_A)\mathbf{j} + (z_B - z_A)\mathbf{k}$$



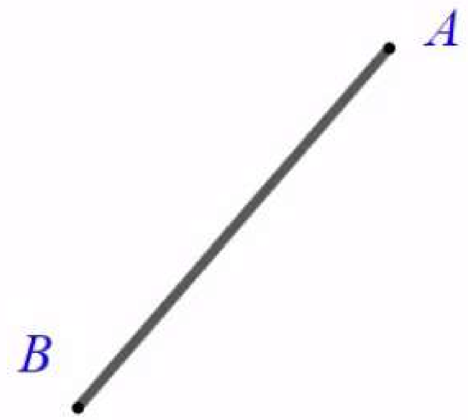
Position vector

The relative position of point A relative to poit B is expressed by the **opposite** vector $\mathbf{r}_{BA}$ that starts from point B and ends on point A.

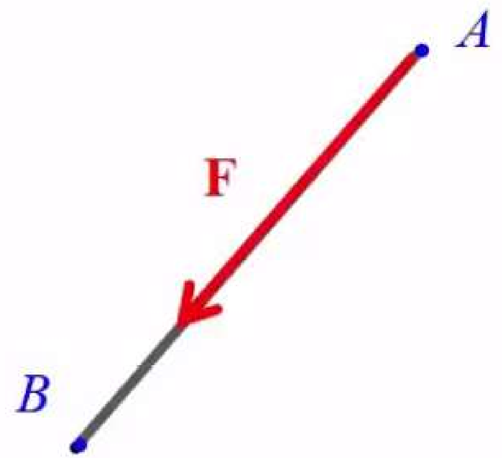
$$\begin{aligned}\mathbf{r}_{BA} &= \mathbf{r}_A - \mathbf{r}_B \\ &= (x_A - x_B)\mathbf{i} + (y_A - y_B)\mathbf{j} + (z_A - z_B) \\ &= -\mathbf{r}_{AB}\end{aligned}$$



Force vector

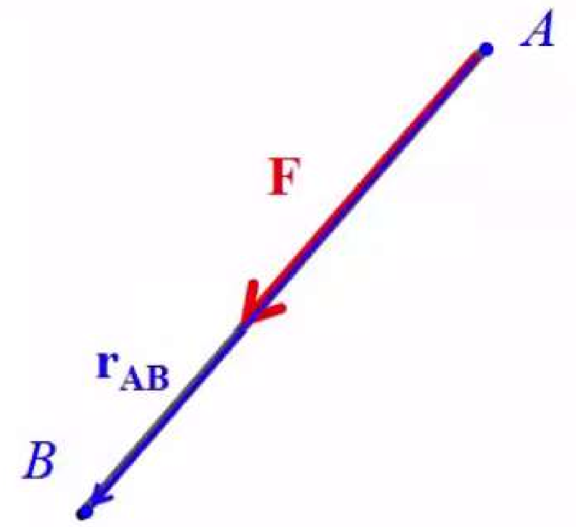


we can also express a force vectors as **Cartesian vectors**. For example, for the tension force **F** in the cable directed from point A to point B, we know that we can express it as **its magnitude multiplied by a unit vector** that describes its direction.



How do we find this unit vector ?

Force vector

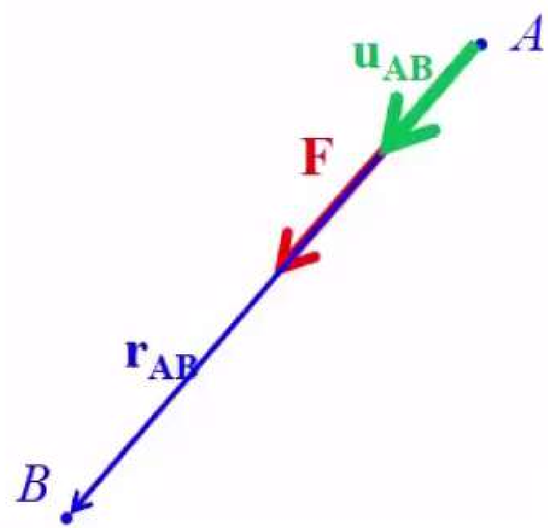


Since the **position vector** from A to B has the **same direction** as the force, we can use the position vector $\mathbf{r}_{AB}$ to find the **unit vector $\mathbf{u}_{AB}$** which is given by this equation:

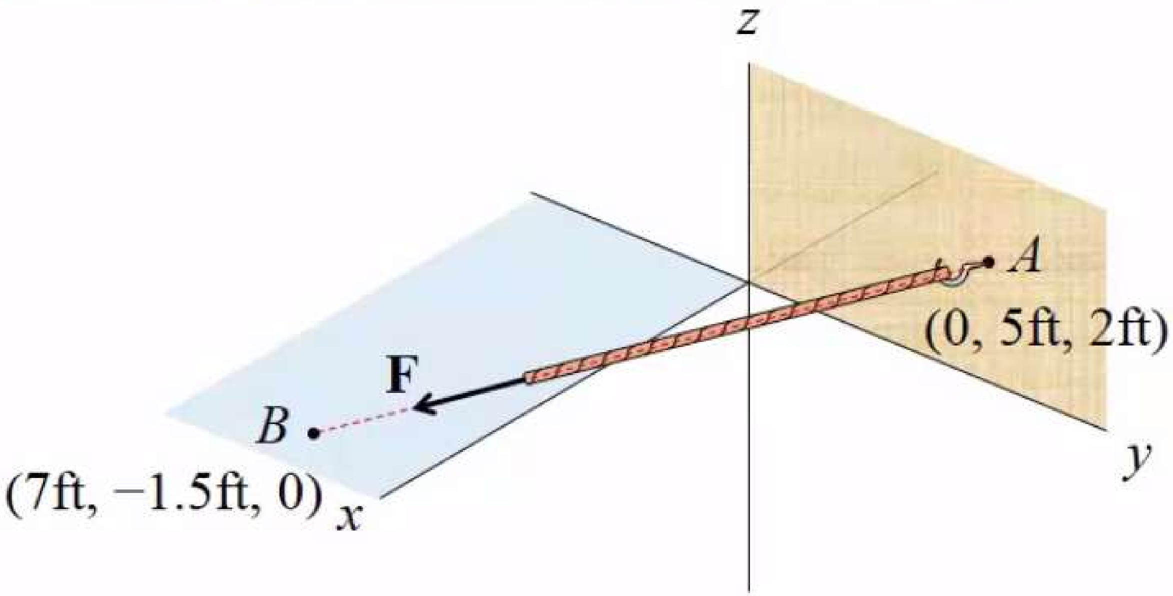
$$\mathbf{u}_{AB} = \frac{\mathbf{r}_{AB}}{r_{AB}}$$

$$= \frac{(x_B - x_A)\mathbf{i} + (y_B - y_A)\mathbf{j} + (z_B - z_A)\mathbf{k}}{\sqrt{(x_B - x_A)^2 + (y_B - y_A)^2 + (z_B - z_A)^2}}$$

$$\mathbf{F} = F\mathbf{u}_{AB}$$

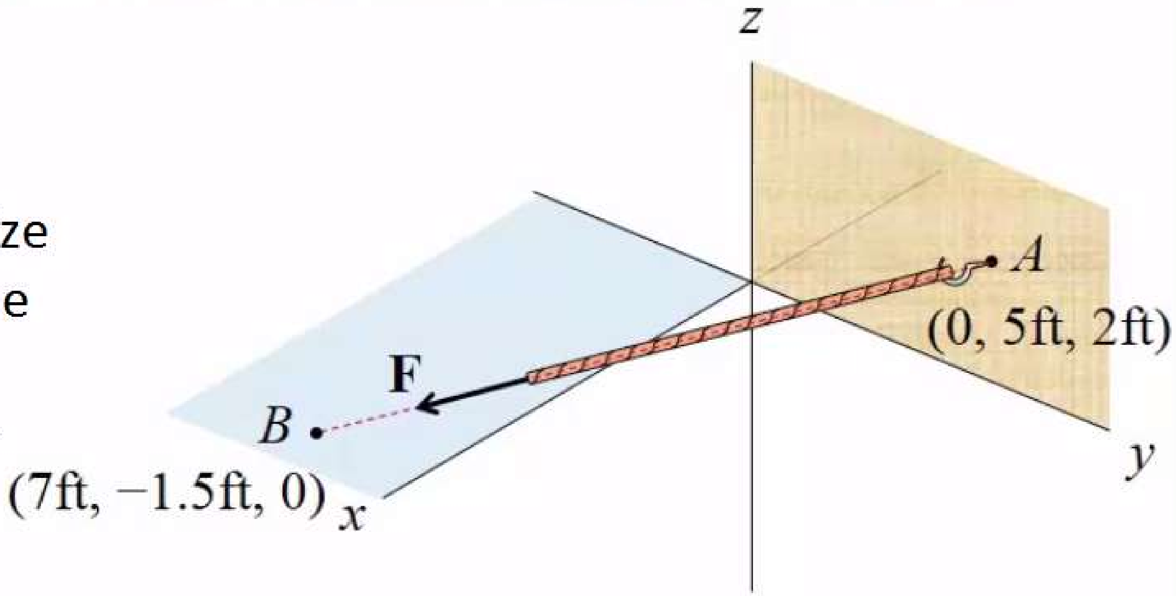


Example: The line of action of force $\mathbf{F}$ directs from point A to point B. If the magnitude of the force is 120 lb, express the force in Cartesian vector form.



Example: The line of action of force $\mathbf{F}$ directs from point A to point B. If the magnitude of the force is 120 lb, express the force in Cartesian vector form.

The key of solving this problem is to recognize that **this force has the same direction** as the **position vector** from A to B, therefore they have **the same unit vector** since unit vector only indicates **the direction**.



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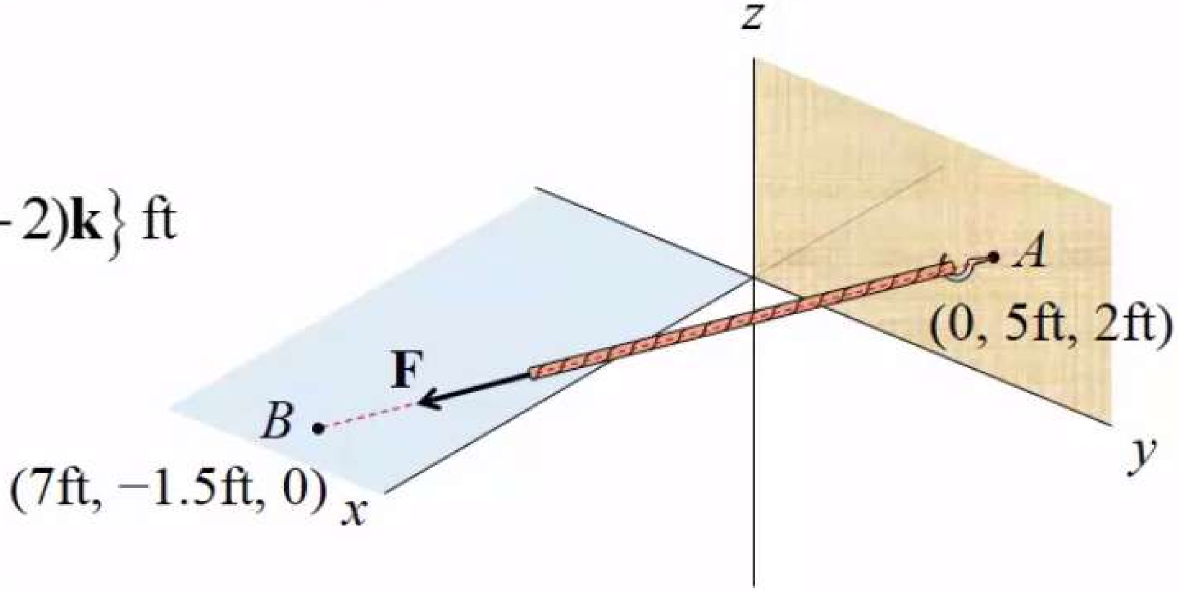


Position vector:

$$\mathbf{r}_{AB} = \mathbf{r}_B - \mathbf{r}_A = \{(7-0)\mathbf{i} + (-1.5-5)\mathbf{j} + (0-2)\mathbf{k}\}\ \text{ft}$$
$$= \{7\mathbf{i} - 6.5\mathbf{j} - 2\mathbf{k}\}\ \text{ft}$$

Unit vector:

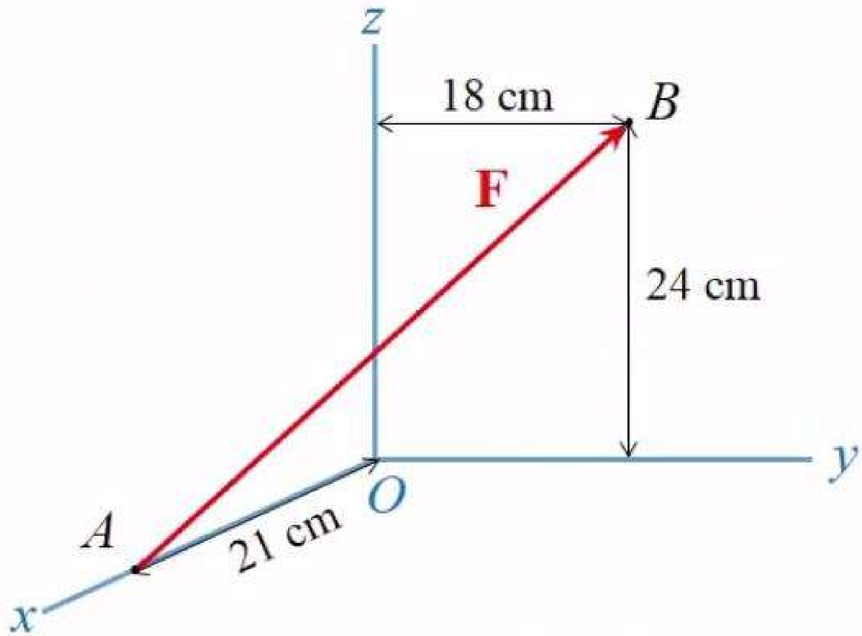
$$\mathbf{u}_{AB} = \frac{\mathbf{r}_{AB}}{r_{AB}} = \frac{\{7\mathbf{i} - 6.5\mathbf{j} - 2\mathbf{k}\}\ \text{ft}}{\sqrt{7^2 + (-6.5)^2 + (-2)^2}\ \text{ft}}$$
$$= 0.717\mathbf{i} - 0.666\mathbf{j} - 0.205\mathbf{k}$$

Force vector:

$$\mathbf{F} = F \cdot \mathbf{u}_{AB} = 120\ \text{lb} \cdot \{0.717\mathbf{i} - 0.666\mathbf{j} - 0.205\mathbf{k}\}$$
$$= \{86.1\mathbf{i} - 79.9\mathbf{j} - 24.6\mathbf{k}\}\ \text{lb}$$

Ans.

Question 2: If force **F** has magnitude of 450 N and is directed from point *A* to *B* as shown, determine the force in Cartesian vector form.



(a) $\{-150\mathbf{i}+129\mathbf{j}+171\mathbf{k}\}$cm

(b) $\{-258\mathbf{i}+221\mathbf{j}+295\mathbf{k}\}$N

(c) $\{-150\mathbf{i}+129\mathbf{j}+171\mathbf{k}\}$N

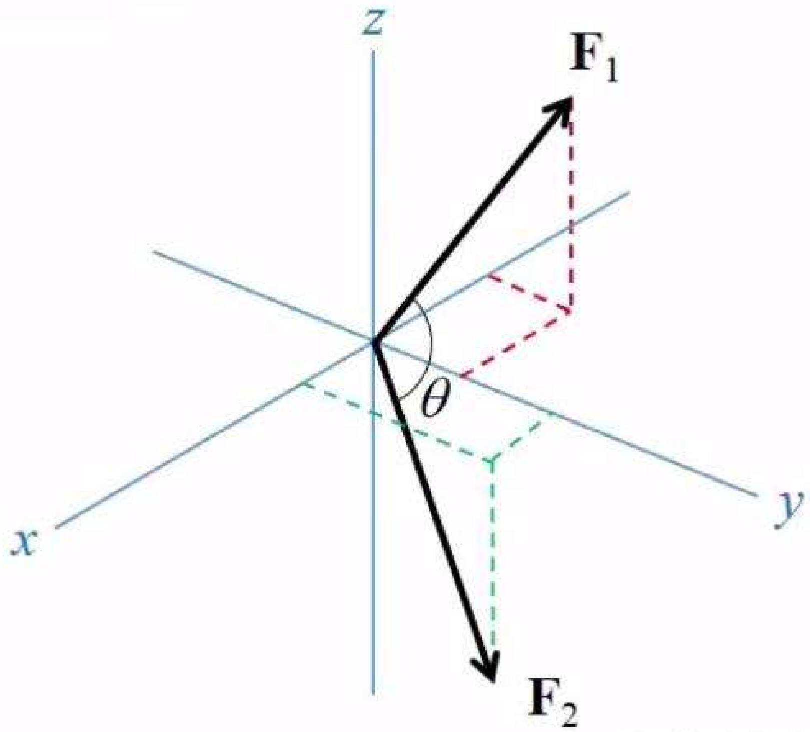
(d) $\{-21\mathbf{i}+18\mathbf{j}+24\mathbf{k}\}$ cm

Dot Product of Cartesian Vectors

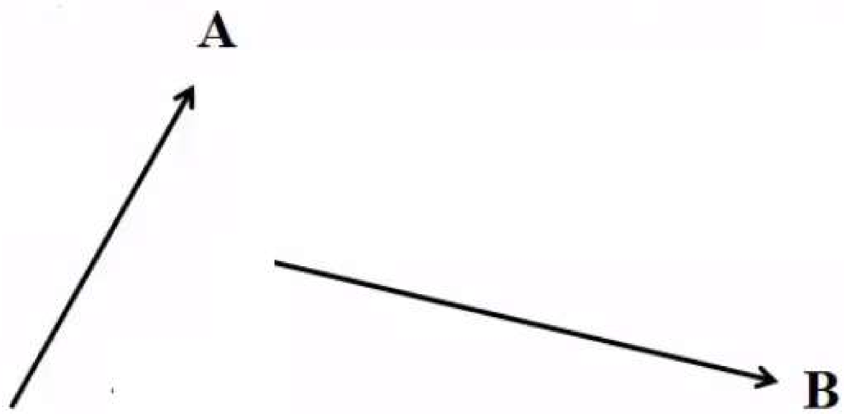
Objectives:

- To revisit the concept of dot product.
- To determine the angle between two vectors using their dot product.
- To calculate the projection of a force along a specified axis using dot product.

Question 1: If you are to use trigonometry to determine the angle between the two forces $\mathbf{F}_1 = \{-4.2\mathrm{i} + 2.8\mathrm{j} + 5.4\mathrm{k}\}$ kip and $\mathbf{F}_2 = \{2.5\mathrm{i} + 5.8\mathrm{j} - 7.1\mathrm{k}\}$ kip, how do you plan to do it? Do you think there's a better way?



Dot product



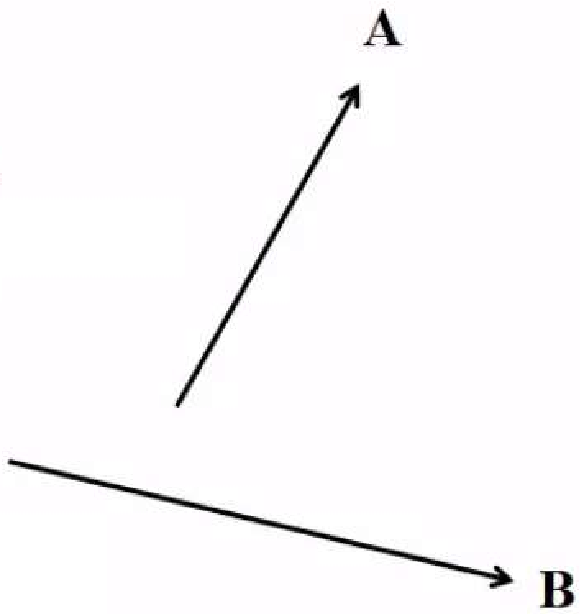
$$\mathbf{A} = A_x\mathbf{i} + A_y\mathbf{j} + A_z\mathbf{k}$$

$$\mathbf{B} = B_x\mathbf{i} + B_y\mathbf{j} + B_z\mathbf{k}$$

For two arbitrary vectors **A** and **B** expressed as Cartesian vectors, their **dot product** is a **scalar** and is defined **algebraically** as:

$$\mathbf{A} \cdot \mathbf{B} = A_x B_x + A_y B_y + A_z B_z$$

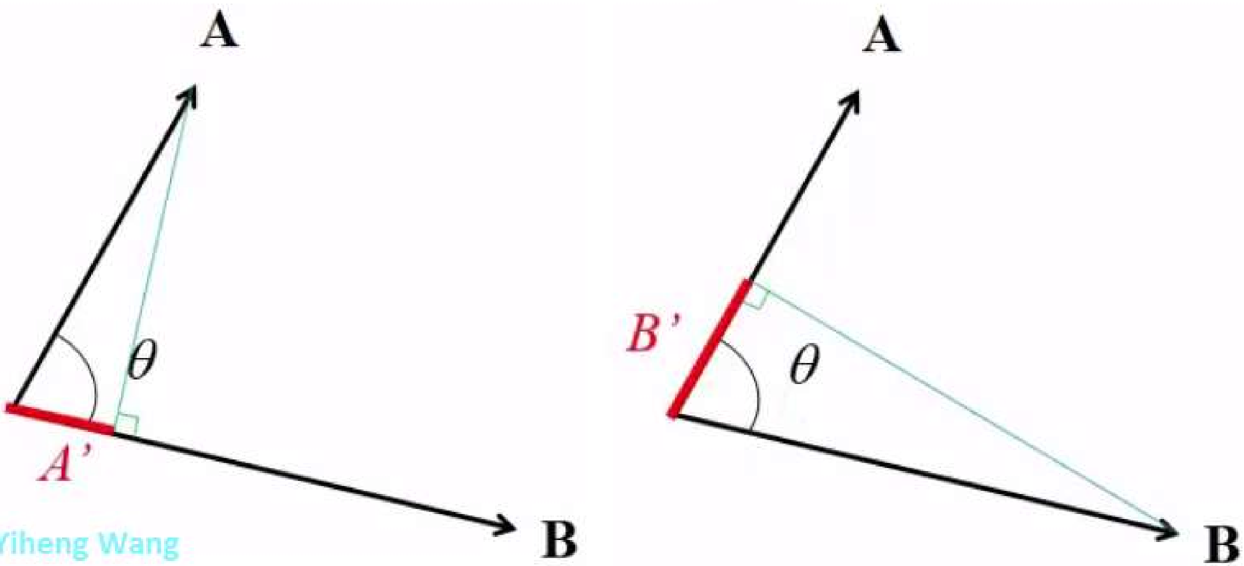
Dot product



Geometrically, even if the two vectors are not on the same plane, they can be parallel transported to be **concurrent**.

They form an angle θ

Their **dot product** equals to: $\mathbf{A} \cdot \mathbf{B} = AB\cos\theta$



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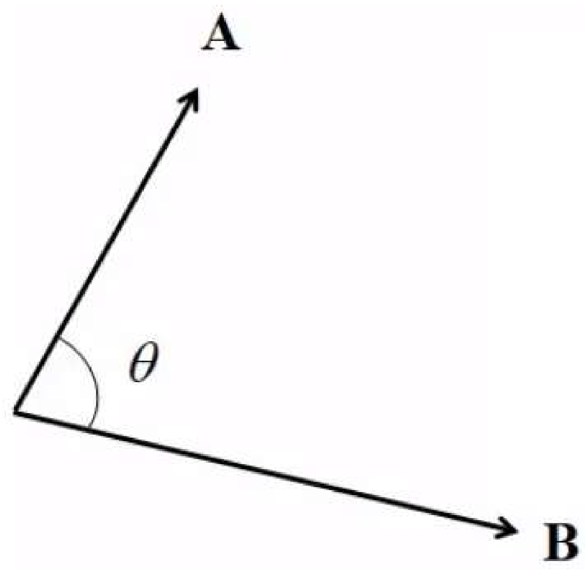
Application of dot product

Because of the **algebraic** and **geometric** definitions of dot product, dot product can now be used **to find the angle** between the two vectors **A** and **B**.

$$\theta = \cos^{-1}\left(\frac{\mathbf{A}\cdot\mathbf{B}}{AB}\right)$$

$$= \cos^{-1}\left(\frac{A_xB_x + A_yB_y + A_zB_z}{AB}\right)$$

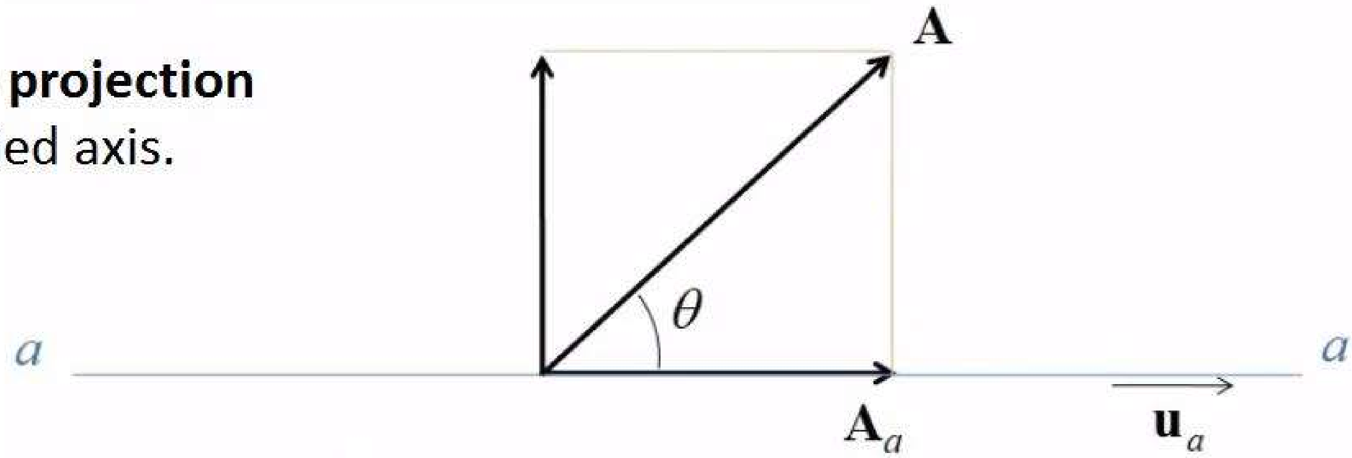
$$(0^\circ \le \theta \le 180^\circ)$$



Application of dot product

We can use dot product to find the **projection vector** of any vector along a specified axis.

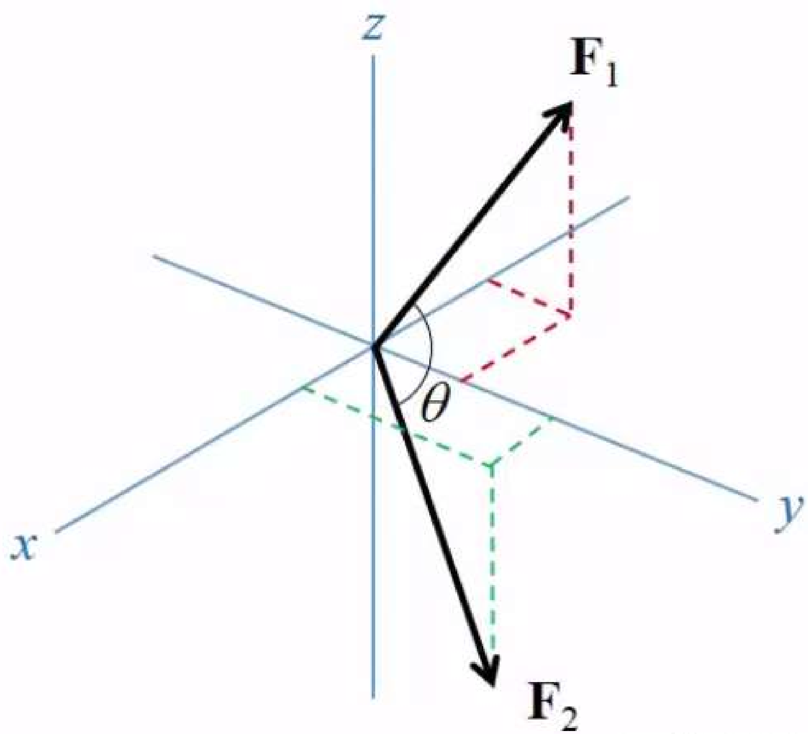
$\mathbf{u}_a$ is the unit vector along the axis.



$$A_a = \mathbf{A} \cdot \mathbf{u}_a$$

$$\mathbf{A}_a = A_a \mathbf{u}_a = A \cos\theta \, \mathbf{u}_a$$

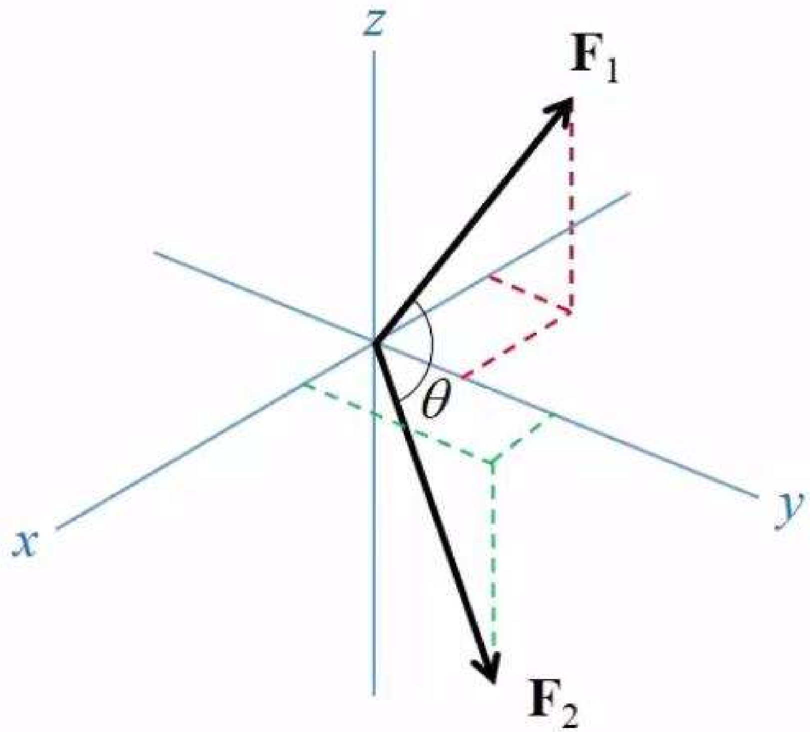
Example: For two forces $\mathbf{F}_1 = \{-4.2\mathbf{i} + 2.8\mathbf{j} + 5.4\mathbf{k}\}$ kip and $\mathbf{F}_2 = \{2.5\mathbf{i} + 5.8\mathbf{j} - 7.1\mathbf{k}\}$ kip, determine the angle between them and the magnitude of the projection of $\mathbf{F}_1$ along the line of action of $\mathbf{F}_2$.



Magnitude:

$$F_1 = \sqrt{(-4.2)^2 + 2.8^2 + 5.4^2} = 7.4\,(\text{kip})$$

$$F_2 = \sqrt{2.5^2 + 5.8^2 + (-7.1)^2} = 9.5\,(\text{kip})$$



Dot product:

$$\mathbf{F}_1 \cdot \mathbf{F}_2 = (-4.2)\cdot 2.5 + 2.8\cdot 5.8 + 5.4\cdot(-7.1)$$

$$= -32.6\,(\text{kip}^2)$$

Angle θ:

$$\theta = \cos^{-1}\left(\frac{\mathbf{F}_1 \cdot \mathbf{F}_2}{F_1 F_2}\right) = \cos^{-1}\left(\frac{-32.6}{7.4\cdot 9.5}\right) = 118°$$

Projection:

$$\left|F_{1\,\text{on}\,2}\right| = \left|F_1 \cdot \cos\theta\right| = \left|7.4\cdot\cos 118°\right| = 3.4\,(\text{kip})$$

Magnitude: $F_1 = 7.4\,(\text{kip}) \qquad F_2 = 9.5\,(\text{kip})$

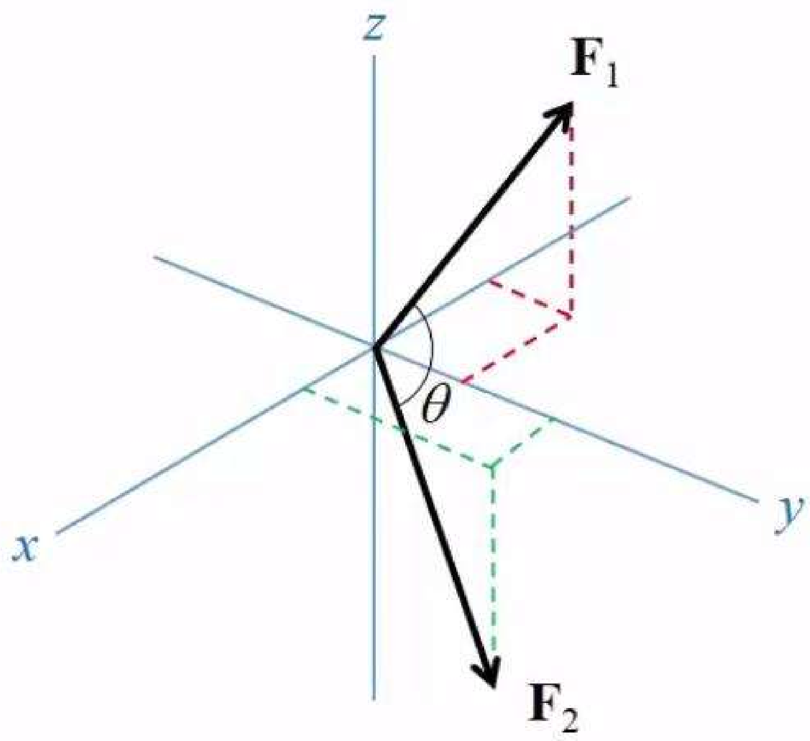
Alternatively:

Unit vectors:

$$\mathbf{u}_{F_1} = \frac{\mathbf{F}_1}{F_1} = -0.568\mathbf{i} + 0.378\mathbf{j} + 0.730\mathbf{k}$$

$$\mathbf{u}_{F_2} = \frac{\mathbf{F}_2}{F_2} = 0.263\mathbf{i} + 0.611\mathbf{j} - 0.747\mathbf{k}$$

Angle θ: $\theta = \cos^{-1}\left(\mathbf{u}_{F_1} \cdot \mathbf{u}_{F_2}\right) = 118^\circ$



Projection:

$$\left|F_{1\,\text{on}\,2}\right| = \left|\mathbf{F}_1 \cdot \mathbf{u}_{F_2}\right| = 3.4\ \text{kip}$$ Ans.

Particle Equilibrium

Objectives :

- To apply Newton's first law to solve 2D and 3D particle equilibrium problems.

Recall:

Newton's second law: $\mathbf{F} = m\mathbf{a}$

Newton's first law:

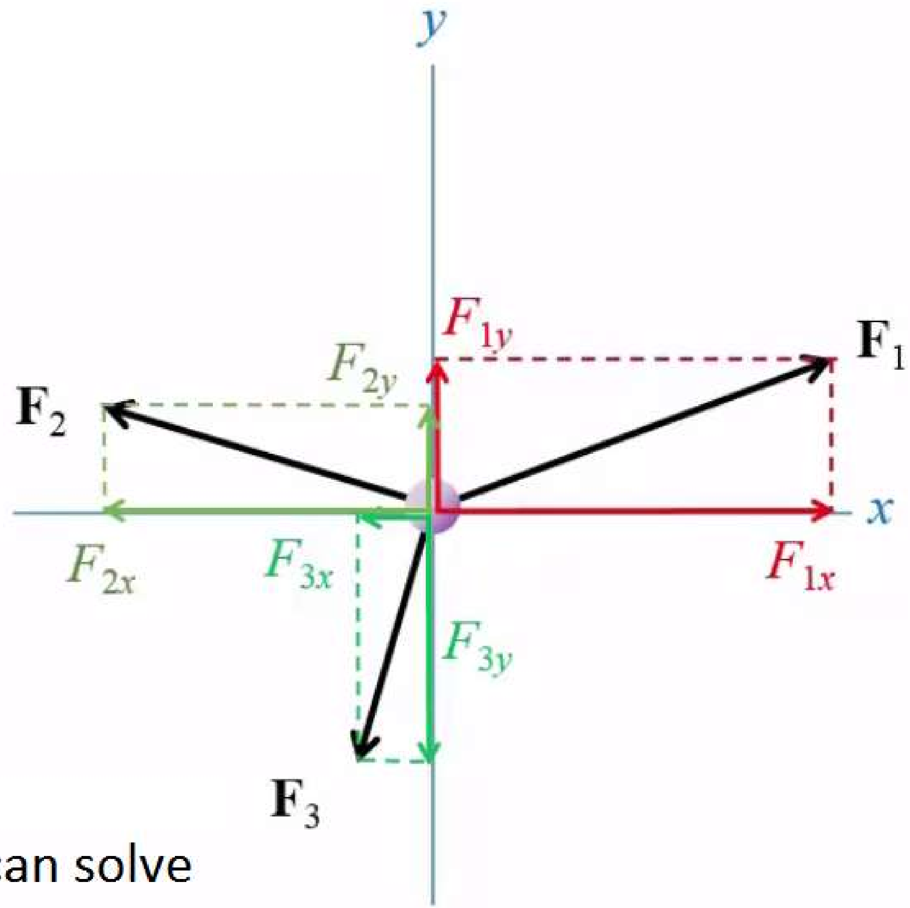
$$\mathbf{F}_R = \mathbf{0} \quad \Rightarrow \quad \mathbf{a} = \frac{\mathbf{F}_R}{m} = \mathbf{0}$$

The condition for **particle equilibrium** is simply described by Newton's first law, that is the resultant force must be **Zero**.

$$\mathbf{F}_R = \sum \mathbf{F} = \mathbf{0}$$

2-D Particle equilibrium

$$\begin{cases} \sum F_x = 0 \\ \sum F_y = 0 \end{cases}$$



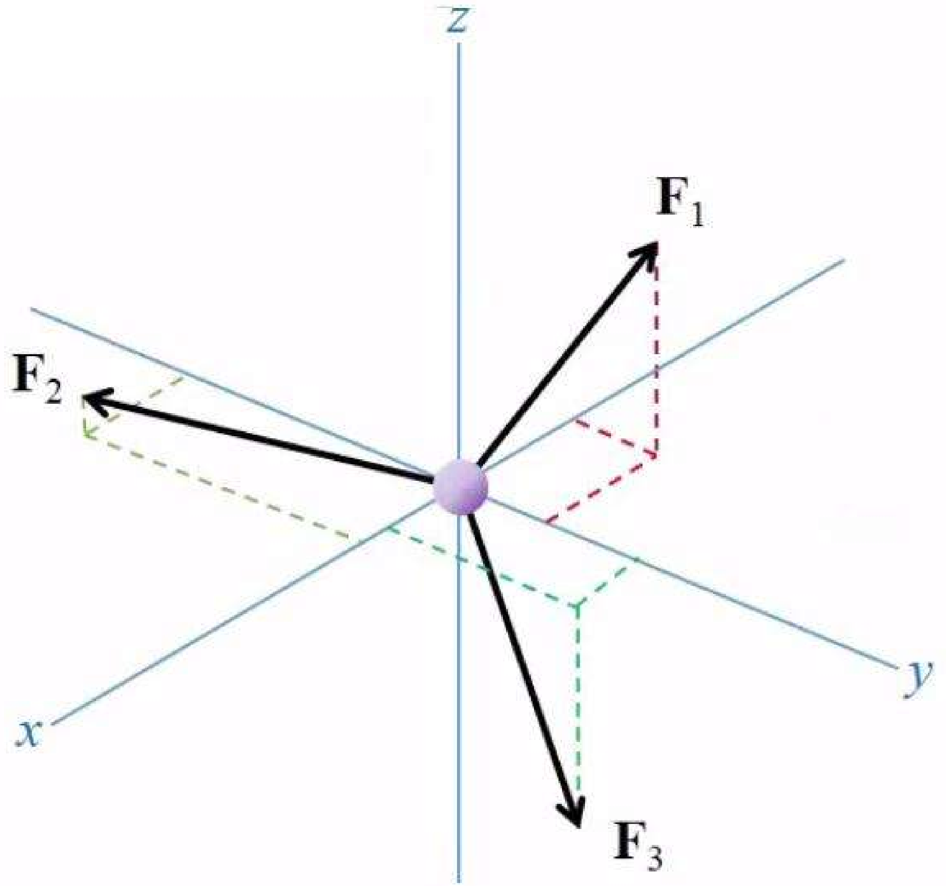
We know that with **two independent equations** we can solve for **a maximum of two unknowns**.

3-D Particle equilibrium

$$\mathbf{F}_R = \sum \mathbf{F} = \mathbf{0}$$

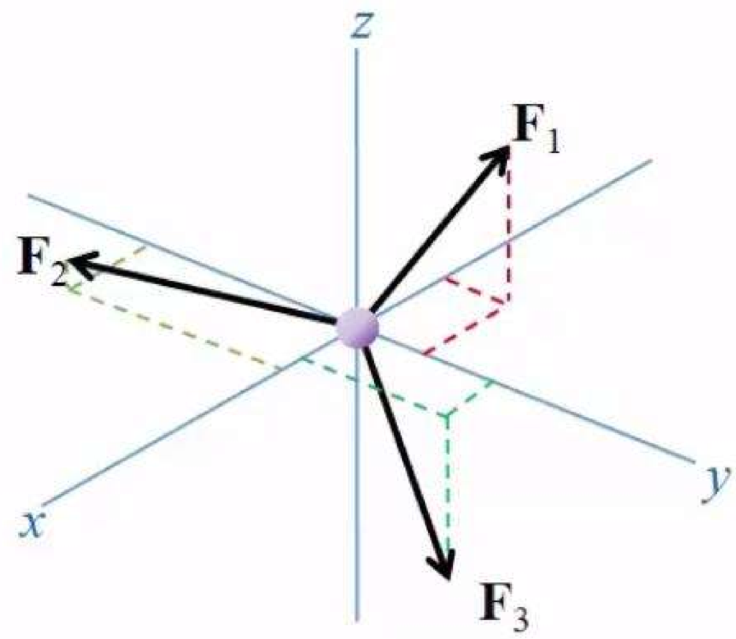
For 3D problem, since each force that acts on a particle can now be resolved into three components along x, y and z directions respectively, the same vector equation can now be rewritten as:

$$\begin{cases} \sum F_x = 0 \\ \sum F_y = 0 \\ \sum F_z = 0 \end{cases}$$



Enabling us to solve for a maximum of 3 unknowns

Example: If the particle is subjected to the three forces and is in equilibrium, $\mathbf{F}_1 = \{-40\mathbf{i} + 30\mathbf{j} + 45\mathbf{k}\}\ N$ and $\mathbf{F}_2 = \{35\mathbf{i} - 65\mathbf{j} + 10\mathbf{k}\}\ N$, determine $\mathbf{F}_3$.



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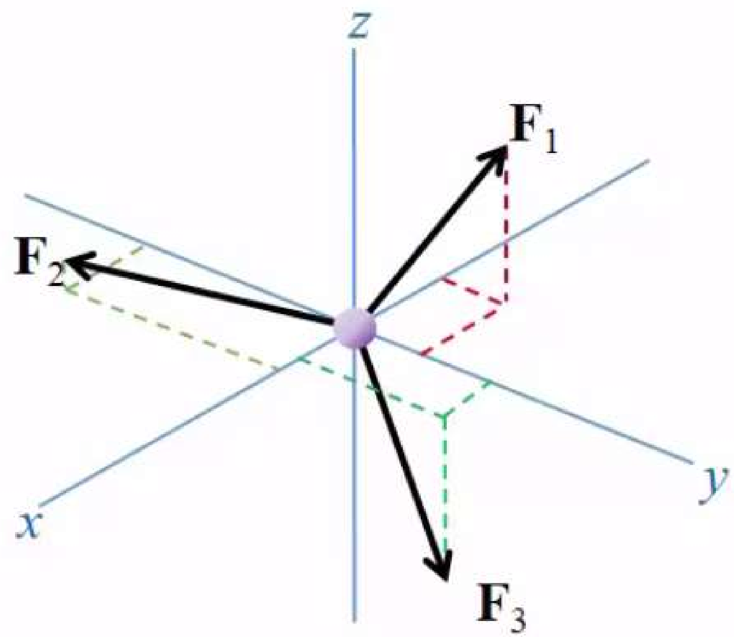
$$\begin{cases} \sum F_x = 0 \\ \sum F_y = 0 \\ \sum F_z = 0 \end{cases}$$



$$\begin{cases} \sum F_x = -40 + 35 + F_{3x} = 0 \\ \sum F_y = 30 - 65 + F_{3y} = 0 \\ \sum F_z = 45 + 10 + F_{3z} = 0 \end{cases} \quad \therefore \quad \begin{cases} F_{3x} = 5\ N \\ F_{3y} = 35\ N \\ F_{3z} = -55\ N \end{cases} \quad \therefore \quad \mathbf{F}_3 = \{5\mathbf{i} + 35\mathbf{j} - 55\mathbf{k}\}\ N$$

Moment of a Force

Objectives :

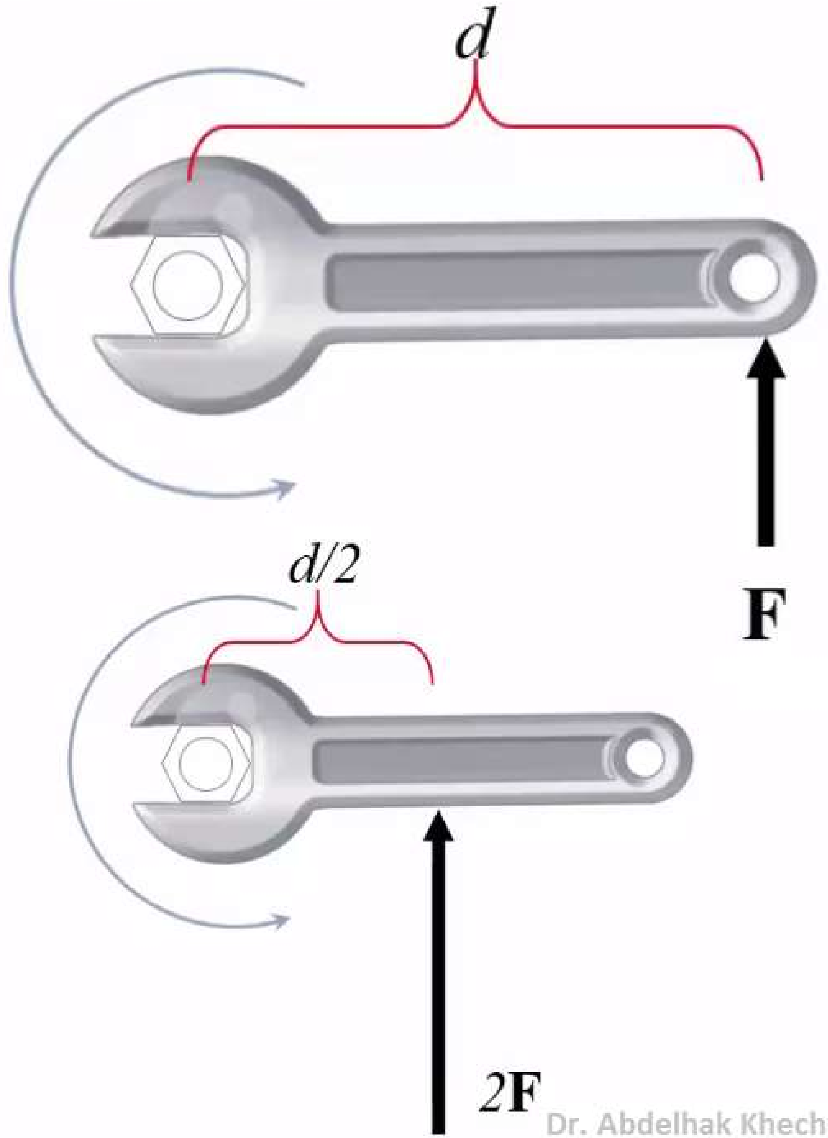
- To define the moment of a force.
- To visually represent the moment of a force in 2D and 3D views.

We know that forces can cause not only **translational motion** but also **Rotational Motion**.

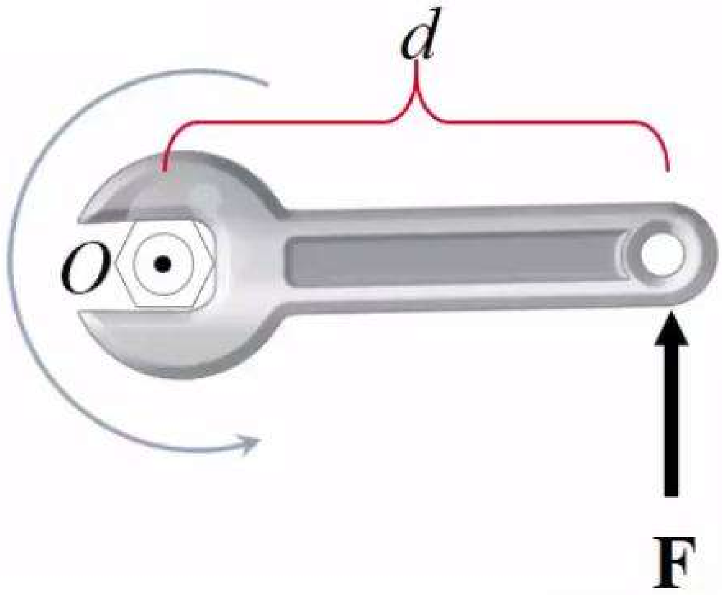
When we apply a force on a handle, we can cause the screw **to rotate**.

We also know almost intuitively to apply the force at **the edge**, creating **a maximum distance** from the screw. Why is that ?

If we shorten the distance by **half**, here we need to **double** that force to get the **same rotational effect** on the screw.



Moment is a physical quantity that describes the *rotational* effect (or rotational tendency) about an *axis* produced by a *force*.

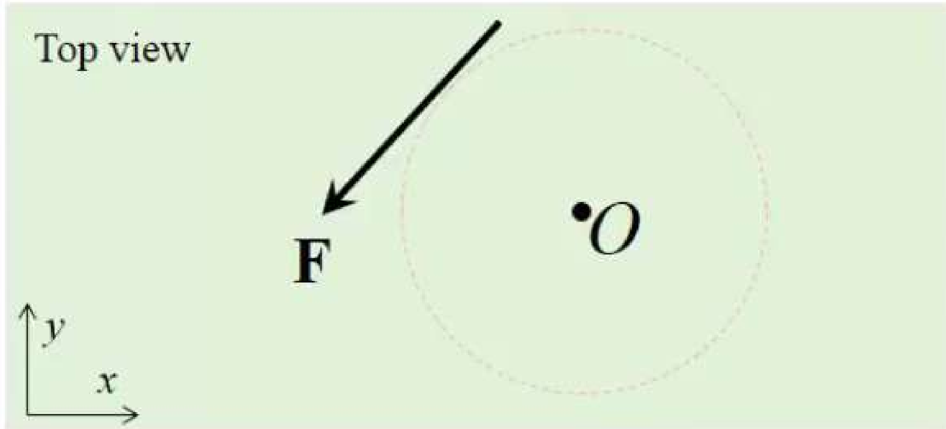


In this example, the axis is **perpendicular** to the plane.

Sometimes a moment is also called a **Torque**.

Just like force, moment is a ***vector***. It follows all vector calculation rules.

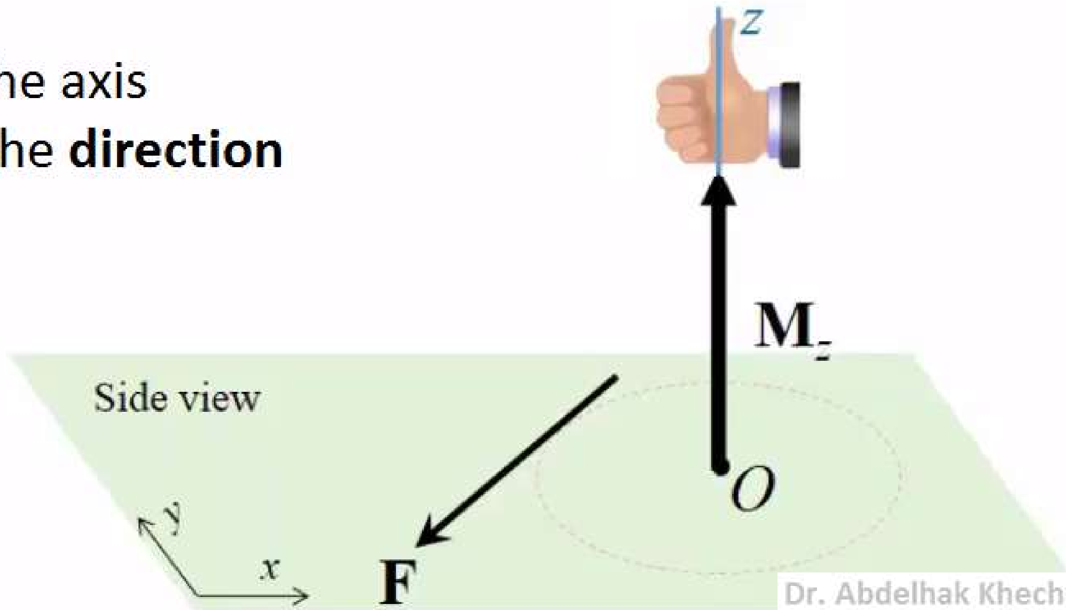
We want to find the **moment** caused by the forece **F** about an axis *z* which is perpendicular to the *xy* plane.



This axis intercepts the plane at point *O*.

The **rotational effect** caused by the force can be determined by the **Right-hand** rule.

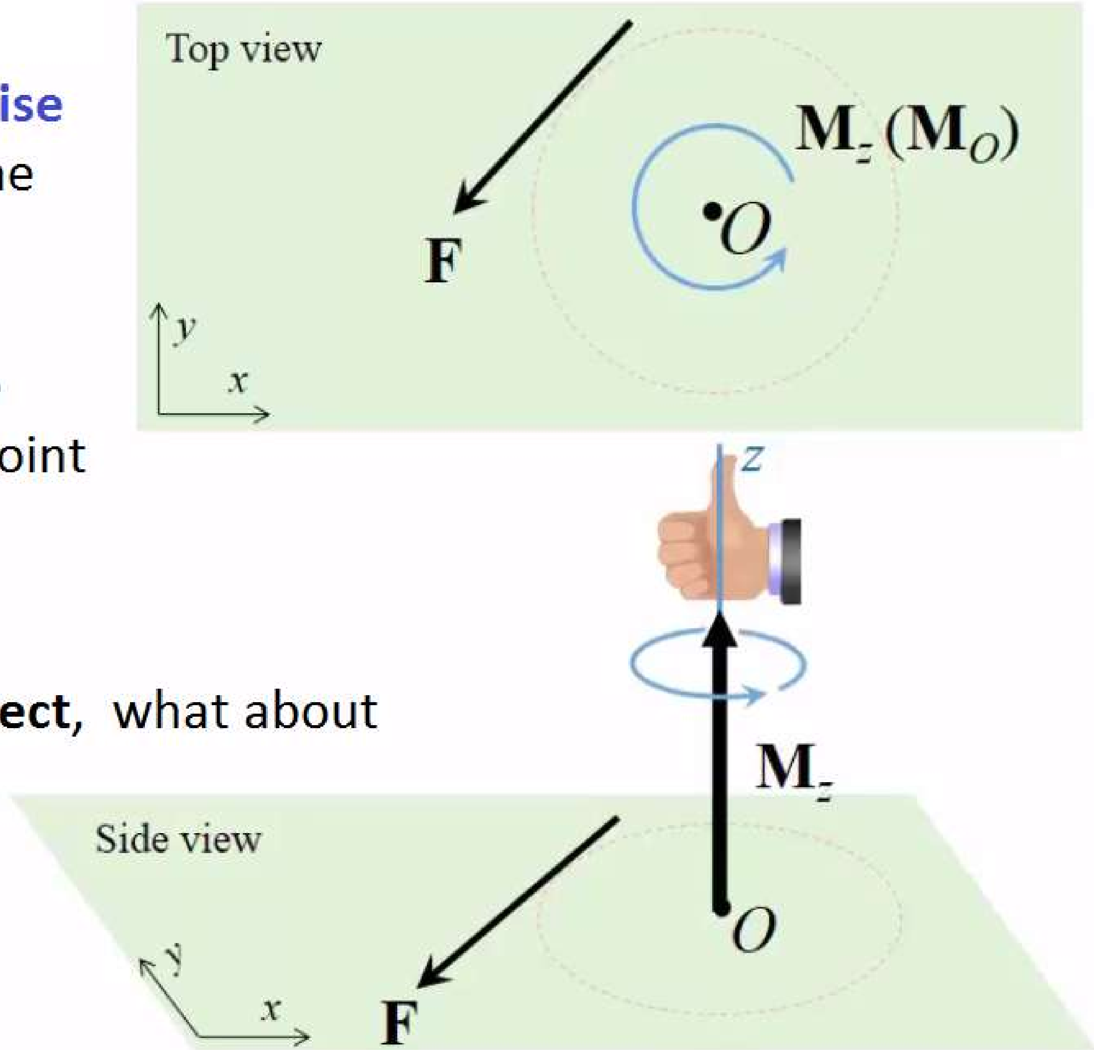
If you extend the four right-hand fingers from the axis towards the force, and then **roll** the fingers to the **direction of the force**, your **thumb** will point towards the direction of the **moment vector** noted by **Mz**.



The rotational effect is always counterclockwise about the **moment vector**, also agrees with the rolling of your four right-hand fingers.

For a 2D problem, the rotational effect can be considered to be within the plane about the point *O*. Therefore, the moment **Mz** can also be expressed as **Mo**.

We know the **direction** and the **rotational effect**, what about the magnitude?



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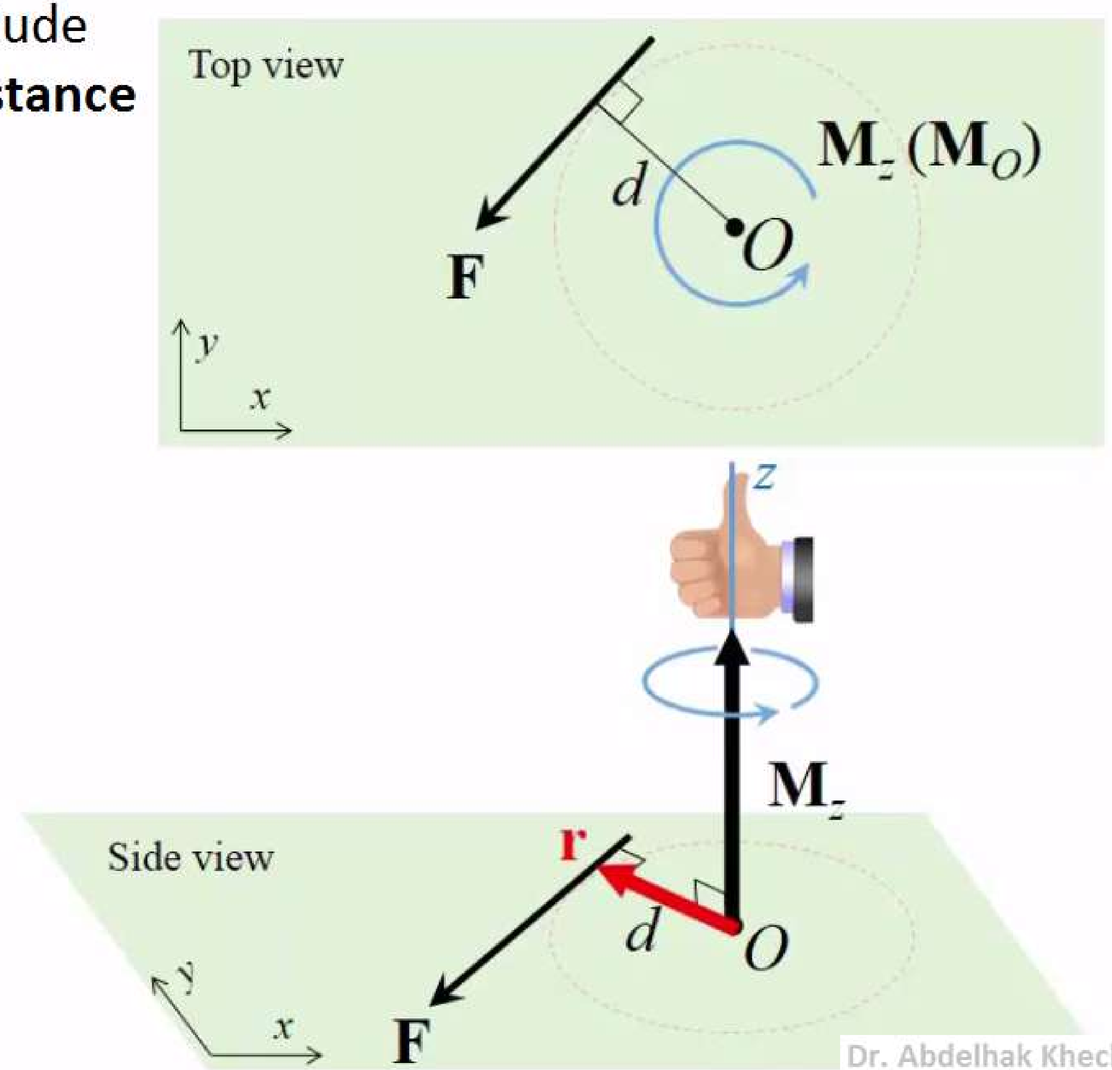
The **magnitude** is determined by the magnitude of the force as well as the **perpendicular distance** between the axis and the force *d* known as the **moment arm**.



In the scalar form:

$$M_O = Fd$$

In the vector form:

$$\mathbf{M} = \mathbf{r} \times \mathbf{F}$$

(Cross vector product)

r is the position vector from point *O* to **F**.

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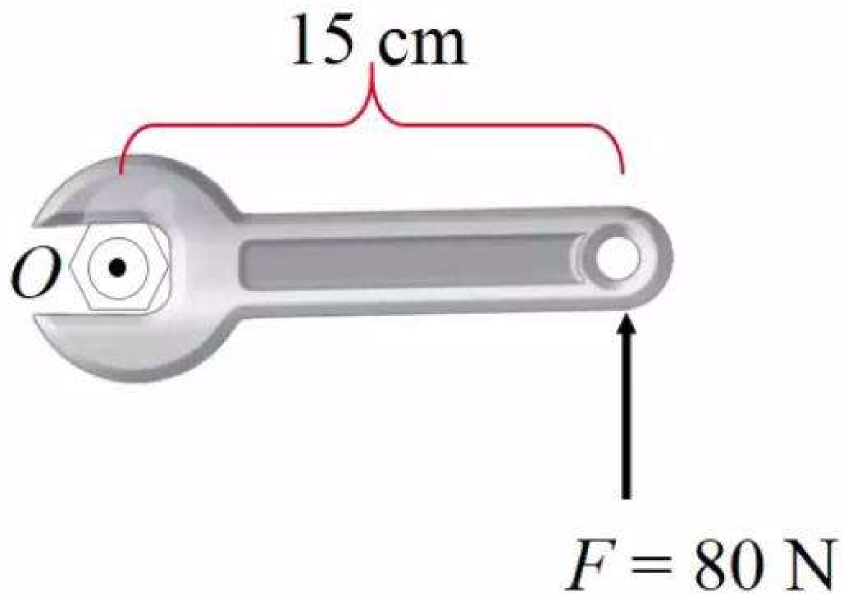
Question 3: Determine the moment about point O caused by force F.



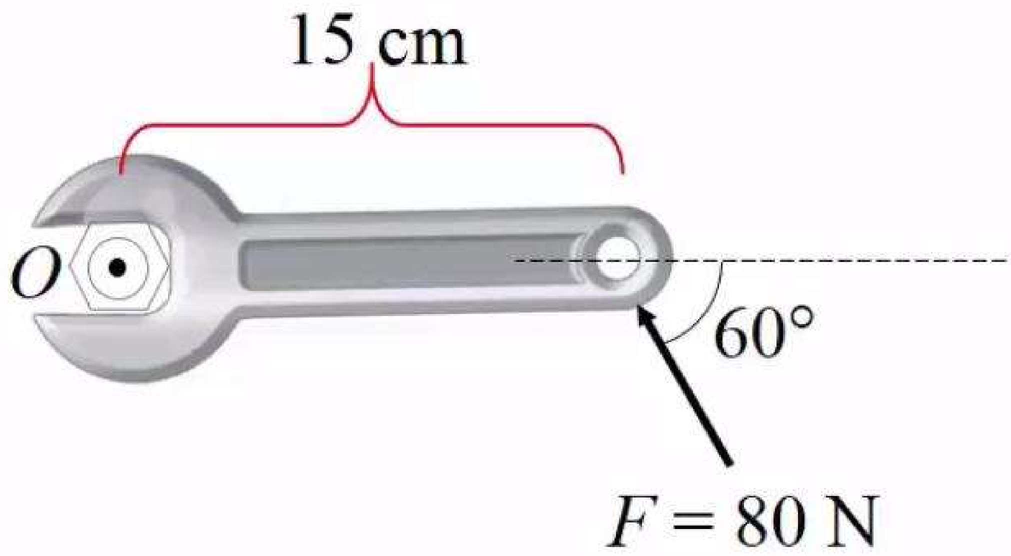
(a) 12 Pa

(b) $12\ N \cdot cm$

(c) $12\ N \cdot m$

(d) $1200\ N \cdot m$

Question 4: Determine the moment about point O caused by force F.



(a) $20.8\ \mathrm{N \cdot m}$ (b) $10.4\ \mathrm{N \cdot m}$

(c) $12\ \mathrm{N \cdot m}$ (d) $6\ \mathrm{N \cdot m}$

Moment Calculation

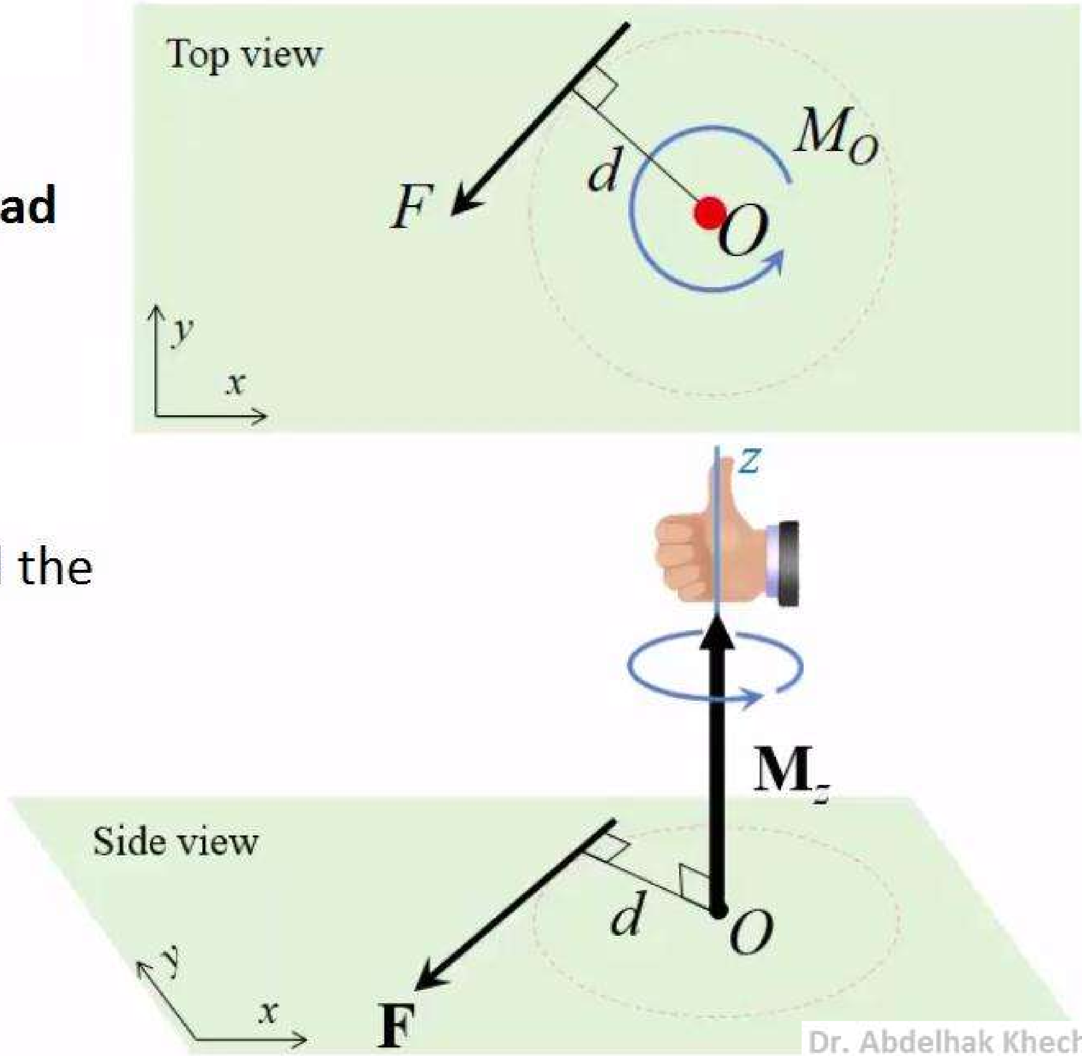
Scalar Formulation

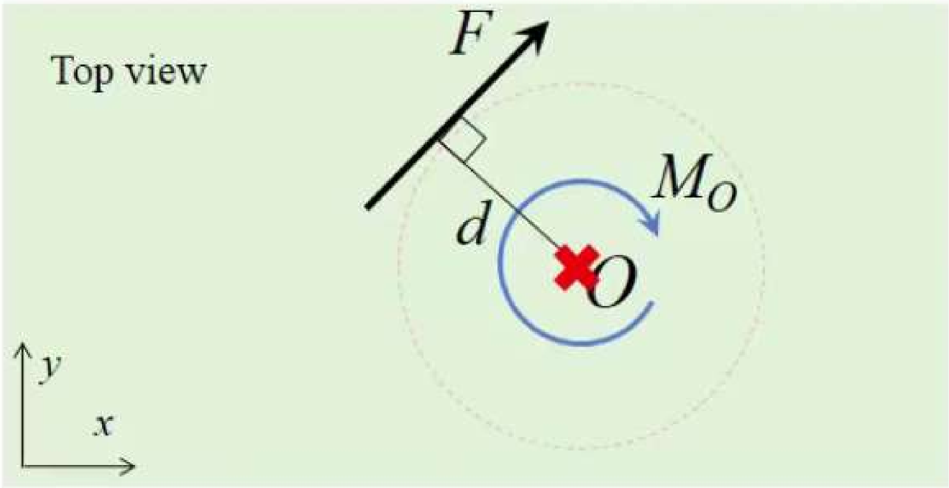
Objectives :

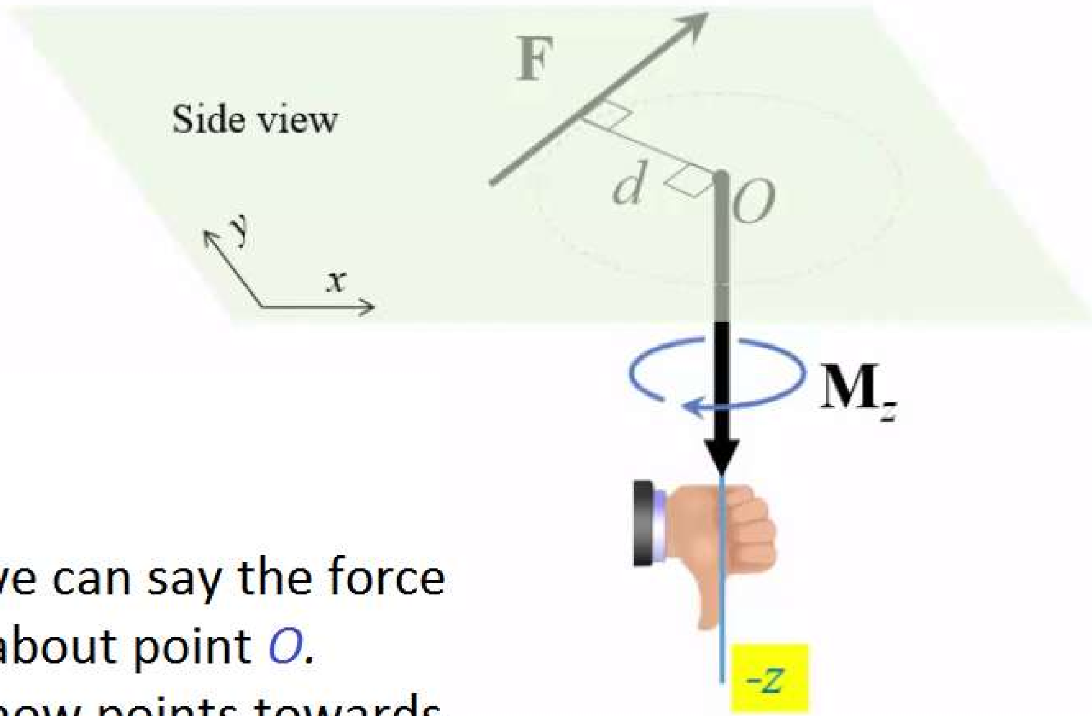
- To illustrate the moment of a force as viewed on a 2D plane.
- To calculate the moment of a force about a point in scalar formulation.

In a **2D** plane the moment vector cannot be visualized but you can imagine it to be **the head** of an arrow **shooting out** of the plane represented by a **dot**.

The rotational effect is **counterclockwise** and the magnitude of the moment is **positive**.



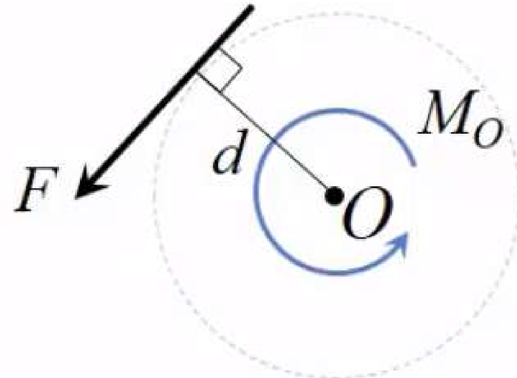




If we reverse the direction of the force, then we can say the force is now creating a **clockwise rotational effect** about point *O*. However, the moment that the force creates now points towards the -z direction still following the **right-hand rule**.

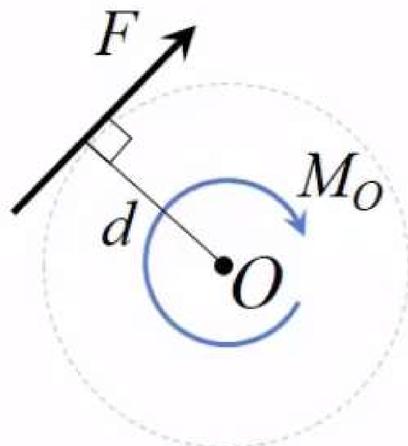
In a 2D plane, you should imagine the moment vector as an arrow **shooting into** the plane and you can only see **the tail** of the arrow.

When we calculate **the moment** caused by a force F about a point *O* in a 2D plane, if the force creates **a counterclockwise rotational effect** about *O* the moment Mo:



$$M_O = Fd$$

If the force creates **a clockwise effect** about point *O*, the moment Mo equals to:



$$M_O = -Fd$$

d is the moment arm.

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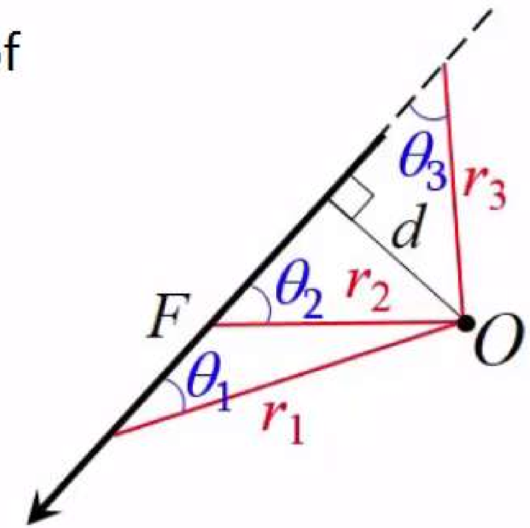
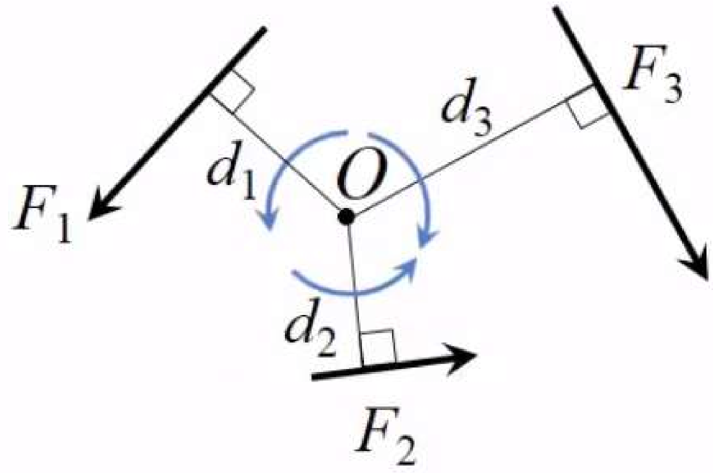
We can just draw a line from point O to anywhere on the line of action of the force F, r_1 or r_2 or r_3 and determine the angle between each of these three lines and the force.



The moment can be determined to be:

$$M_O = Fd = F \cdot r_1 \cdot \sin\theta_1 = F \cdot r_2 \cdot \sin\theta_2 = F \cdot r_3 \cdot \sin\theta_3$$

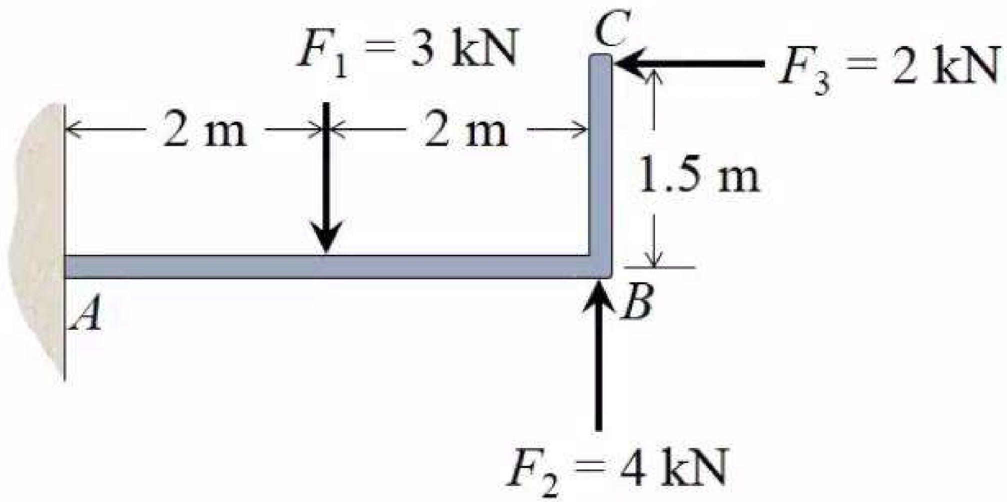
The **resultant moment** caused by **multiple forces** can be determined by simply **adding up** the individual moment caused by each force about the same point.



$$(M_R)_O = \sum Fd = F_1 d_1 + F_2 d_2 - F_3 d_3$$

F_3 : is creating **a clockwise rotational effect** about point *O*.

Question 1: Determine the total moment about **point *A*** caused by the three forces F_1, F_2 and F_3. Take counterclockwise as positive.



(a) $25\,\text{kN}\cdot\text{m}$ (b) $13\,\text{kN}\cdot\text{m}$

(c) $7\,\text{kN}\cdot\text{m}$ (d) $11\,\text{kN}\cdot\text{m}$

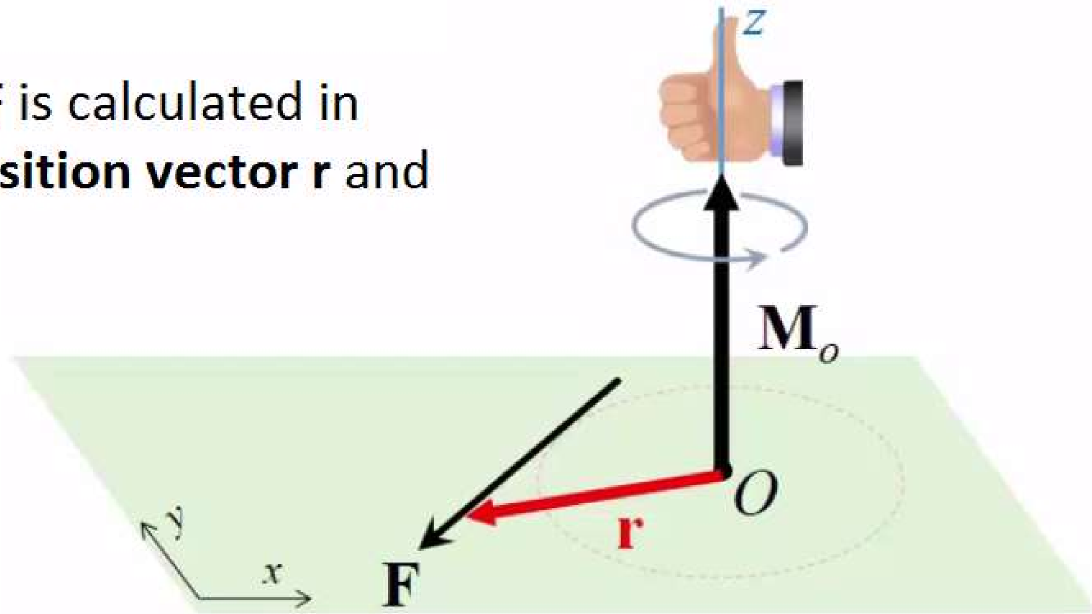
Moment Calculation

Vector Formulation

Objectives :

- To review the cross product of two vectors.
- To calculate the moment of a force about a point in vector formulation.

The moment about a point O caused by force **F** is calculated in **vector form** simply as the **Cross-Product** of **position vector r** and the force vector **F**.



$$\mathbf{M}_O = \mathbf{r} \times \mathbf{F}$$

Note that **r** could be any vector as long as it starts from point *O* and ends anywhere on **the line of action** of the force.

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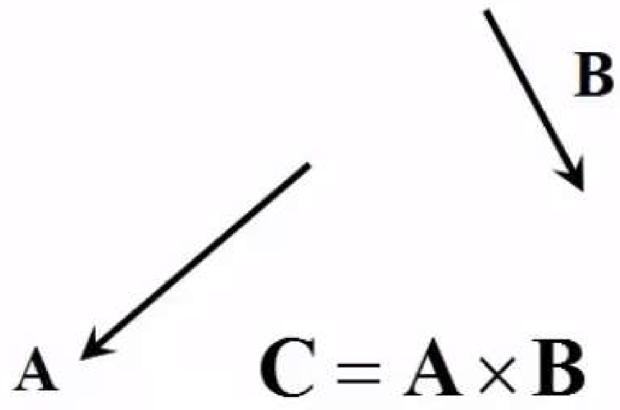
Cross product

We **join the tails** of the two vectors together and then determine the **angle** between them.

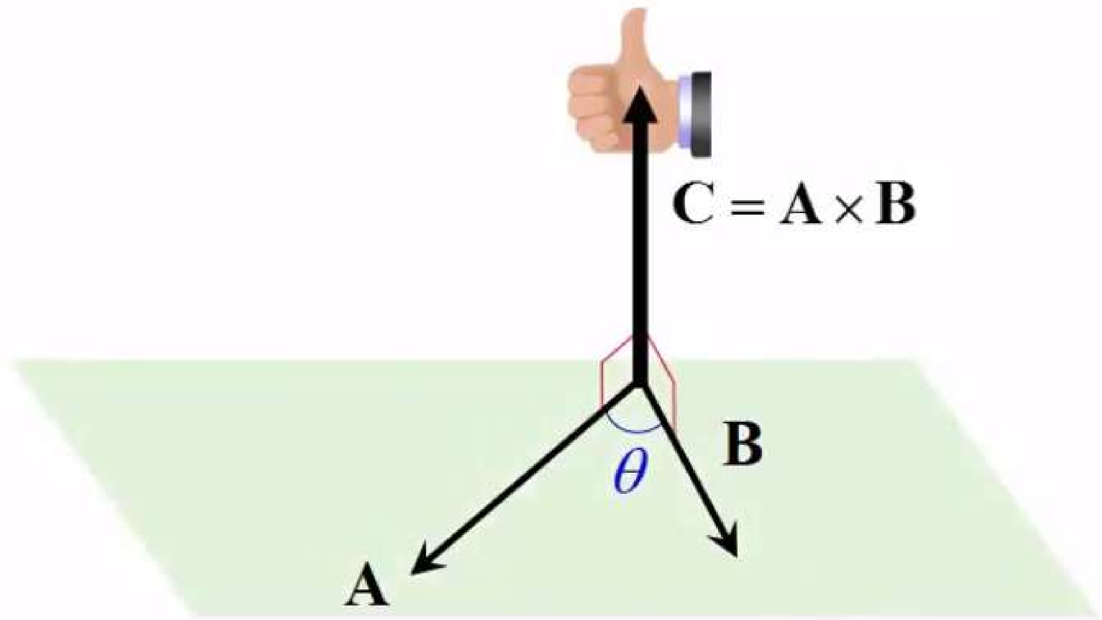


The magnitude of vector **C** is determined as:

$$C = AB\sin\theta$$

The **direction** is determined by the **right-hand rule**. When you roll your four right-hand fingers from vector **A** towards vector **B**, your thumb points to vector **C**'s direction.

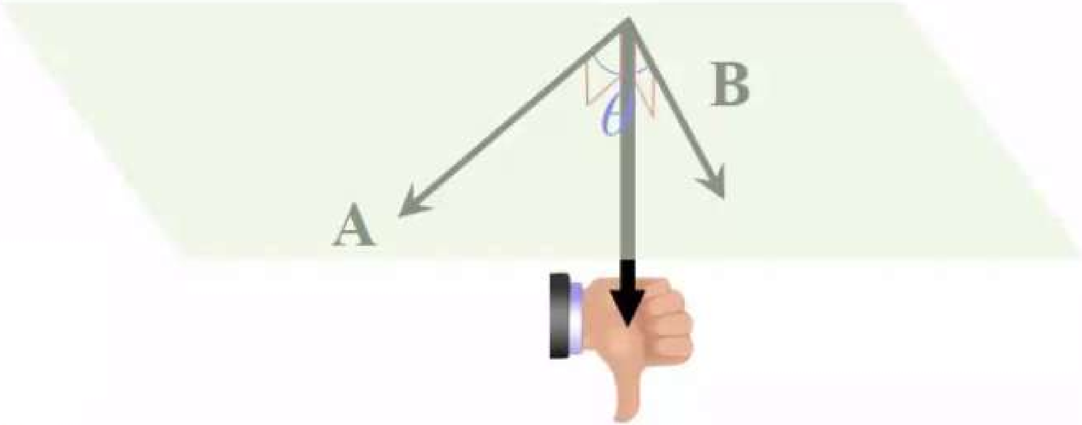
Vector **C** is perpendicular to the plane made by **A** and **B**.



Cross product

A cross **B** is not the same as **B** cross **A**.

$$\mathbf{A} \times \mathbf{B} \neq \mathbf{B} \times \mathbf{A}$$



B cross **A** represents another vector **C'** that is in the opposite direction as vector **C**.

$$\mathbf{C}' = \mathbf{B} \times \mathbf{A} = -\mathbf{C}$$

Cross product

If the vectors **A** and **B** are given in the **Cartesian forms**, then we can use **a matrix** to determine the **Cartesian form** of the cross product of **A** and **B**.

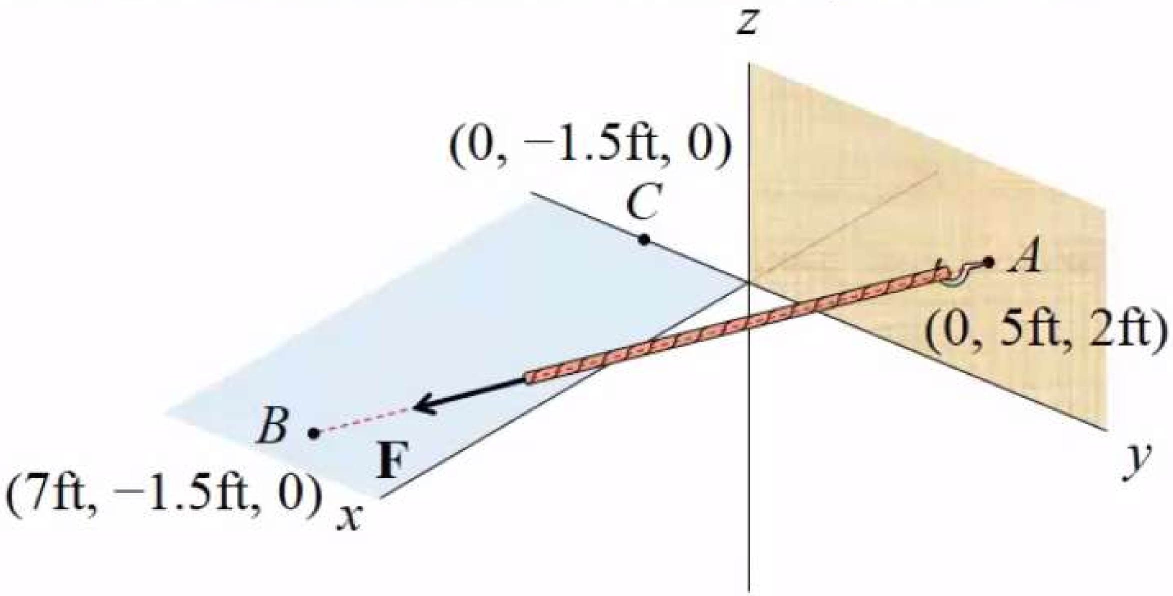
$$\mathbf{A} = A_x\mathbf{i} + A_y\mathbf{j} + A_z\mathbf{k}$$

$$\mathbf{B} = B_x\mathbf{i} + B_y\mathbf{j} + B_z\mathbf{k}$$

$$\mathbf{A}\times\mathbf{B} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

$$= \left(A_yB_z - A_zB_y\right)\mathbf{i} - \left(A_xB_z - A_zB_x\right)\mathbf{j} + \left(A_xB_y - A_yB_x\right)\mathbf{k}$$

Example: The line of action of force **F** directs from point A to point B. If the magnitude of the force is 120 lb, determine the moment of **F** about point C in Cartesian vector form.



Force vector:

$$\mathbf{F} = \{86.1\mathbf{i} - 79.9\mathbf{j} - 24.6\mathbf{k}\}\ \text{lb}$$

Position vector:

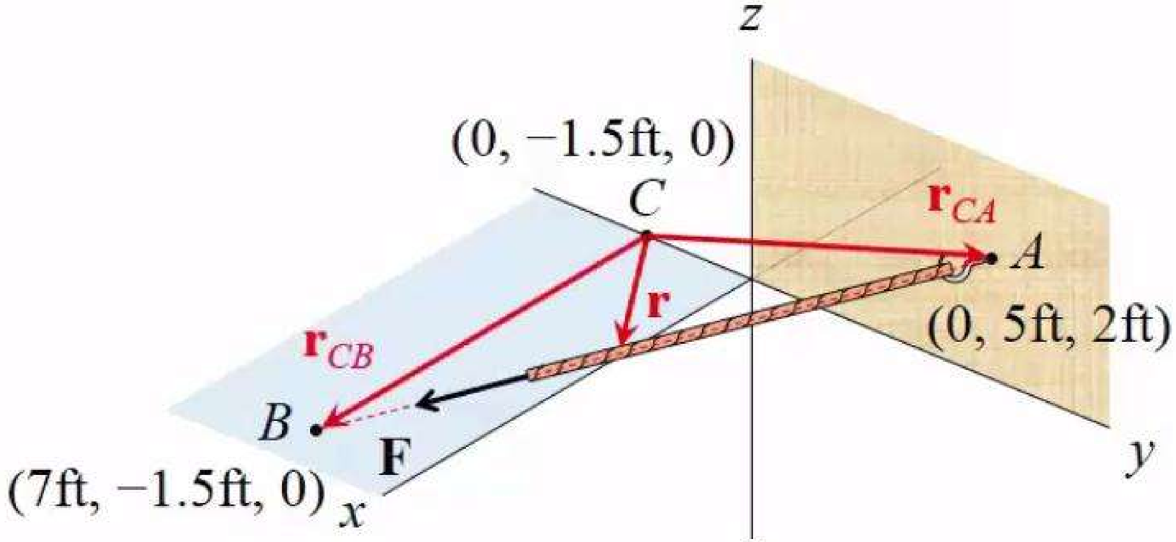
$$\mathbf{r}_{CB} = 7\mathbf{i}\ \text{ft}$$

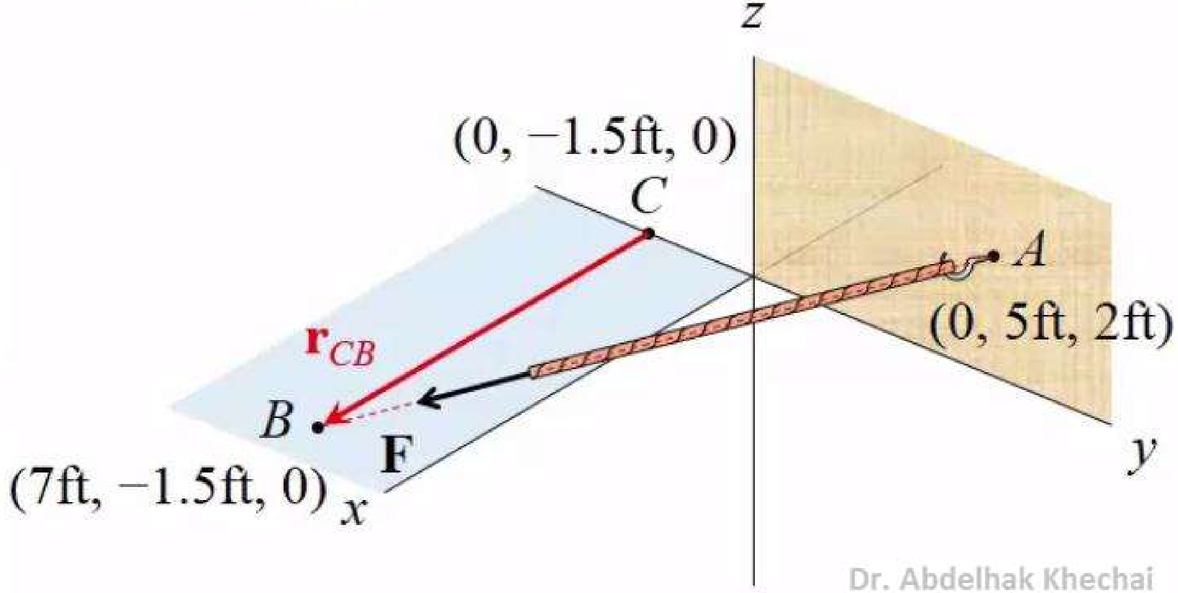
Moment vector:

$$\mathbf{M} = \mathbf{r}_{CB} \times \mathbf{F}$$

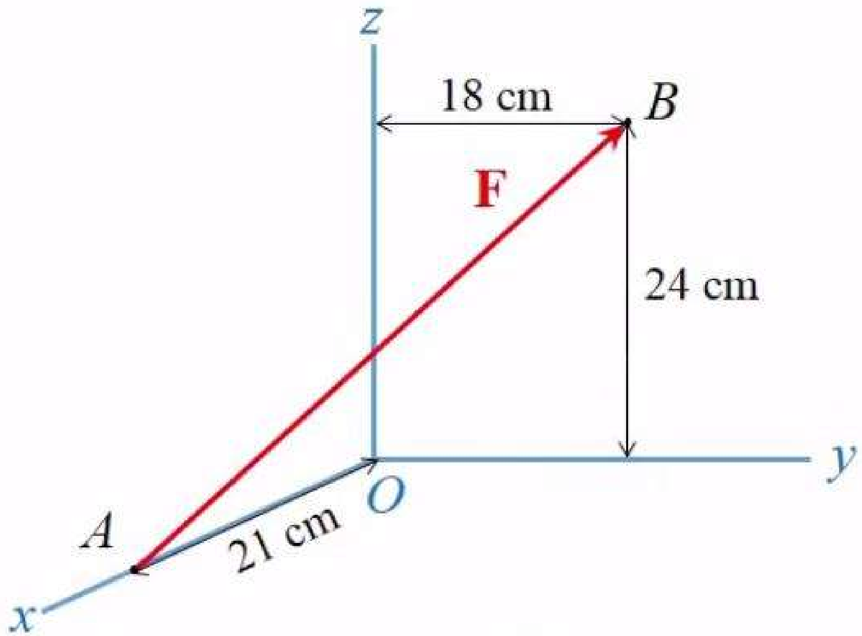
$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 7 & 0 & 0 \\ 86.1 & -79.9 & -24.6 \end{vmatrix}$$

$$= \{172\mathbf{j} - 559\mathbf{k}\}\ \text{lb}\cdot\text{ft}$$ Ans.





Question 2: If force **F** has magnitude of 450 N and is directed from point *A* to *B* as shown, determine the moment (Cartesian vector form) caused by **F** about point *O*.



(a) $\{62.0\mathbf{j} - 46.4\mathbf{k}\}\,\text{N}\cdot\text{m}$

(b) $\{-258\mathbf{i} + 295\mathbf{k}\}\,\text{N}\cdot\text{m}$

(c) $\{-15.0\mathbf{i} + 12.9\mathbf{j} + 17.1\mathbf{k}\}\,\text{N}\cdot\text{m}$

(d) $\{-62.0\mathbf{j} + 46.4\mathbf{k}\}\,\text{N}\cdot\text{m}$

Principle of Moments

Dr. Yiheng Wang

Objectives :

To explain the application of the principle of moments to simplify moment calculation.

Used **vector formulation**, the moment caused by **F** about *O* is equal to:

$$\mathbf{M}_O = \mathbf{r} \times \mathbf{F}$$

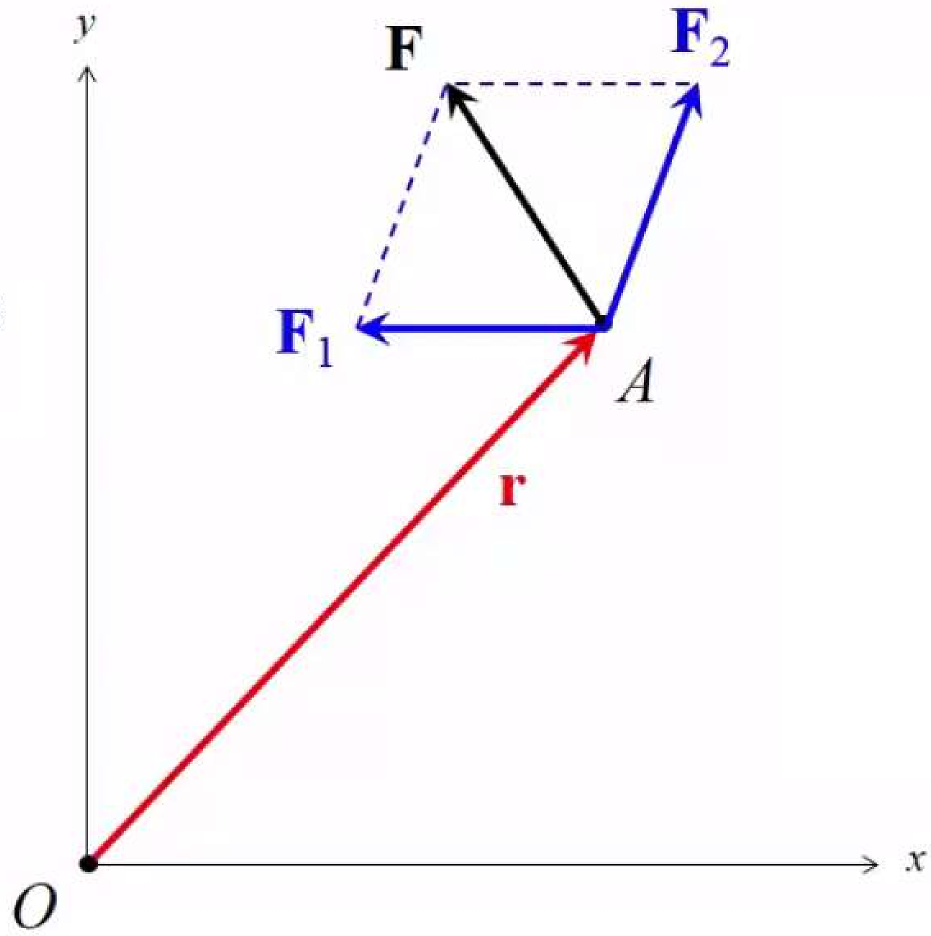
Since **F** is a vector, it can **be resolved** into two or more components following the **parallellogram law**.

$$\mathbf{M}_O = \mathbf{r} \times (\mathbf{F}_1 + \mathbf{F}_2)$$

Following the distributive law:

$$= \mathbf{r} \times \mathbf{F}_1 + \mathbf{r} \times \mathbf{F}_2$$

$$= \mathbf{M}_{O,1} + \mathbf{M}_{O,2}$$



The moment caused by a force can be calculated by summarizing the moments caused by its component forces about the same point and this is the Principle of Moments.

Why we care about the **principle of moments?**

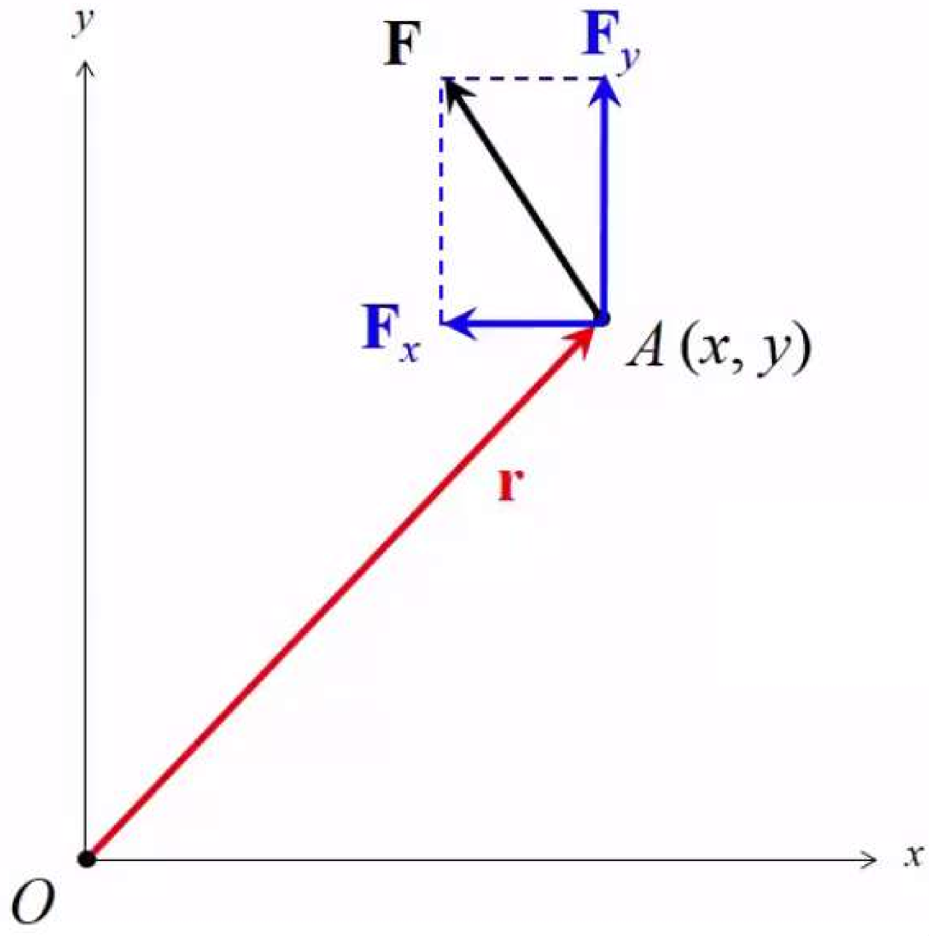
We want to use it to help us simplify the calculation of moment.

If we know the point of action of the force *A*(x,y), we would resolve the force vector into **Fx** and **Fy.**

The moment arms of these two component forces are *x* and *y*, respectively.

The calculation of **the magnitude** of the moment can be easily achieved to be:

$$M_O = F_x \cdot y + F_y \cdot x$$

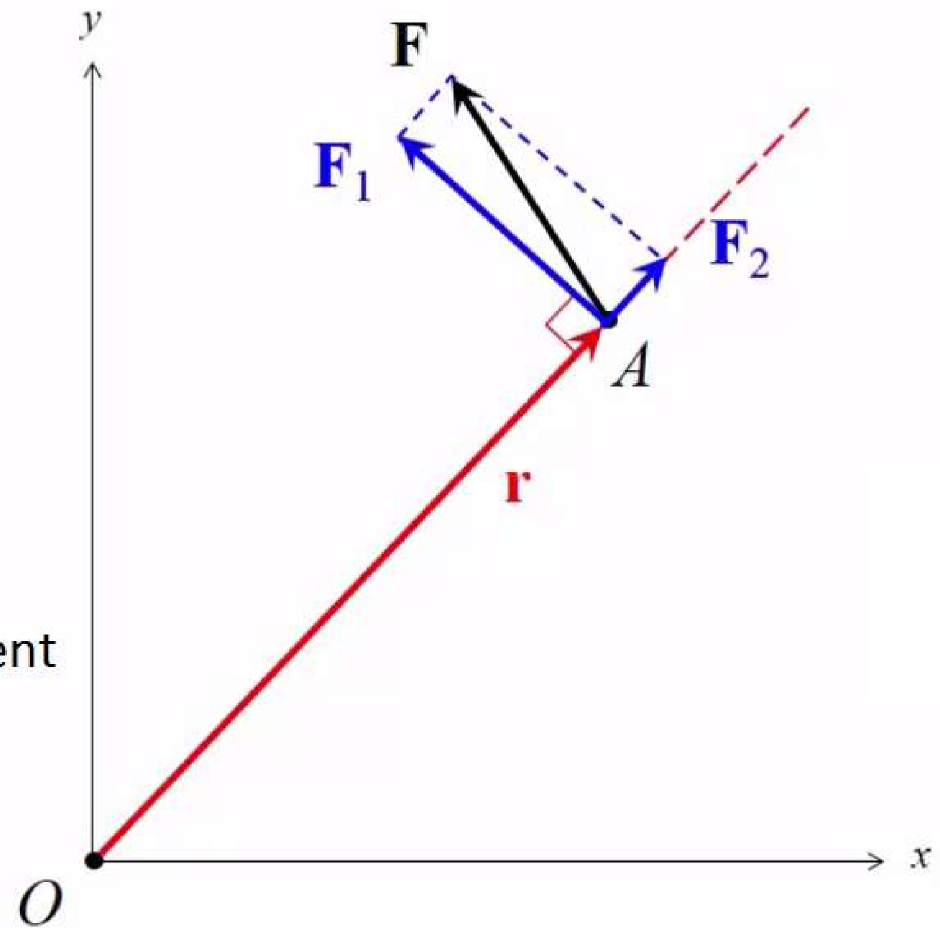


Sometimes, we can conveniently resolve the force into one component that is **along** the direction of **r** pointing from *O* towards *A*, and another component that is **perpendicular** to **r**.

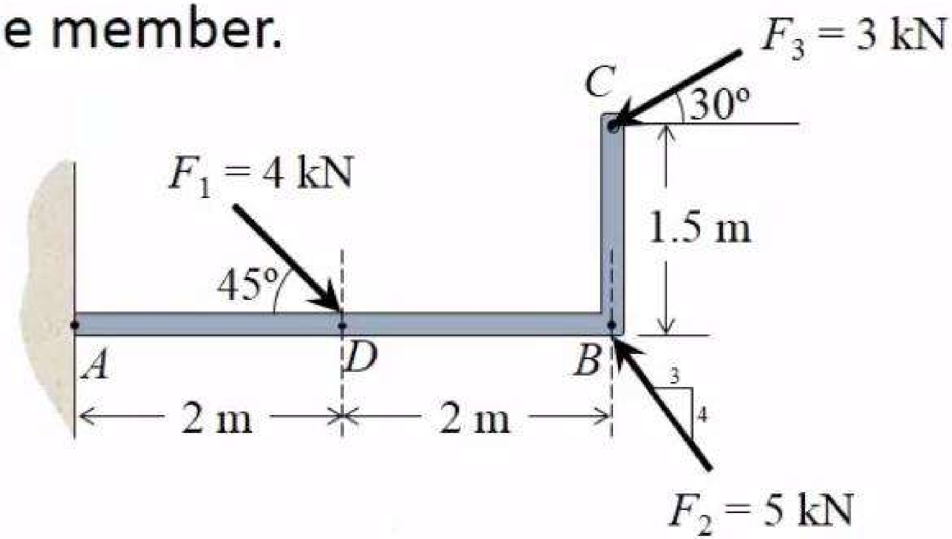
The advantage of this way is the **moment arm** of **F1** is the magnitude of **r** or in other words, the distance from *O* to *A*.

The moment arm of **F2** is simply **Zero**. (The component **F2 does not create** any moment about *O*.)

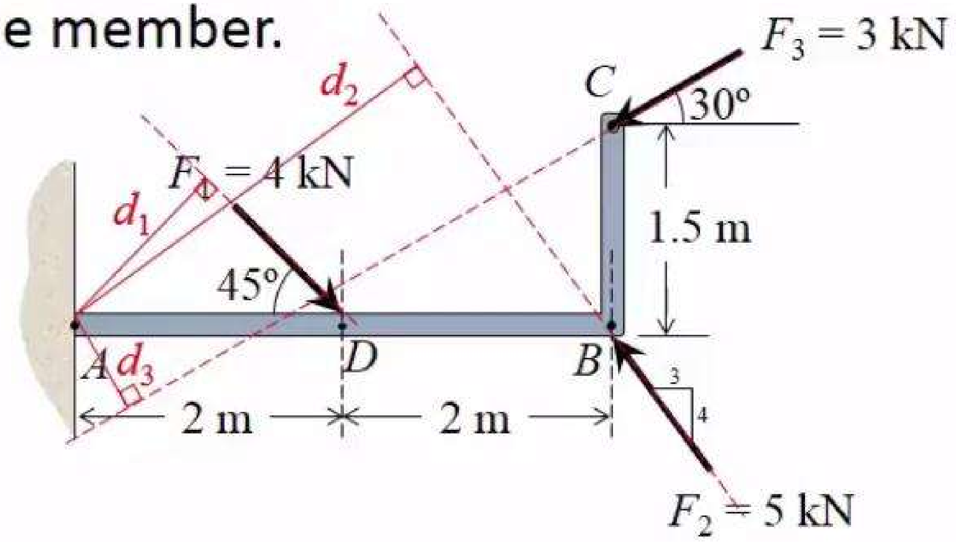
$$M_O = F_1 \cdot r$$



Example: Determine the total moment about **point *A*** caused by the three forces F_1, F_2 and F_3. Take counterclockwise as positive. Neglect the thickness of the member.

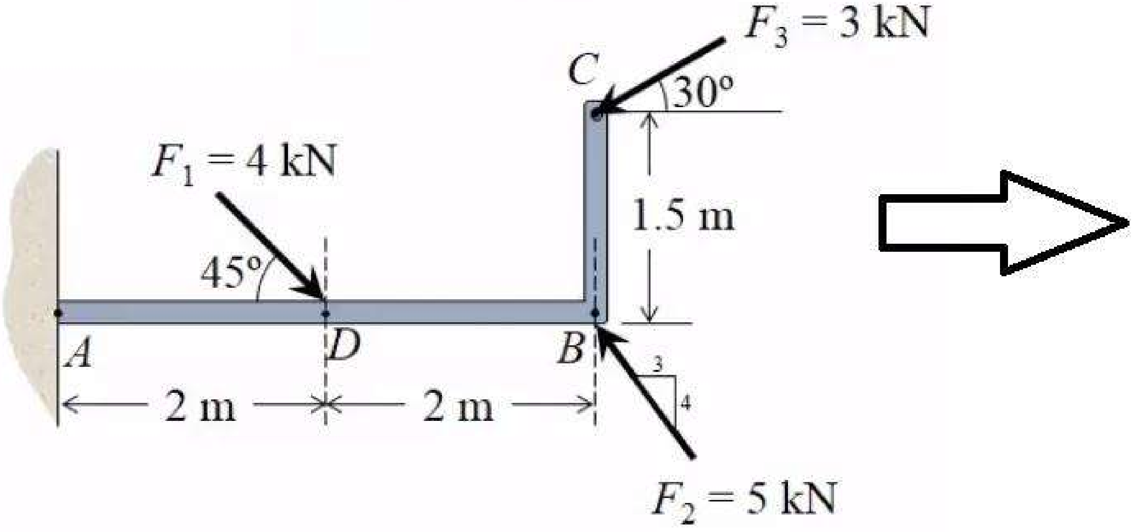


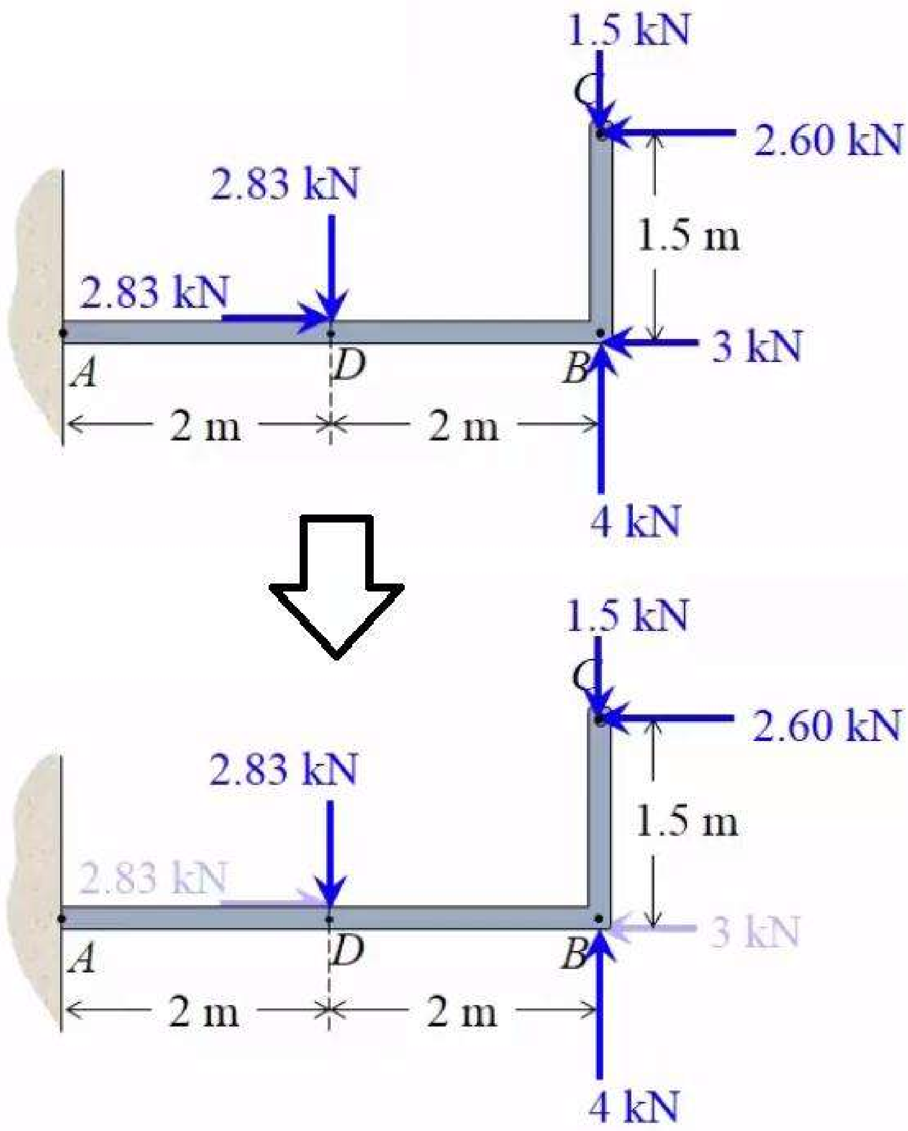
Example: Determine the total moment about **point *A*** caused by the three forces F_1, F_2 and F_3. Take counterclockwise as positive. Neglect the thickness of the member.



$$M_A = -F_1 \cdot d_1 + F_2 \cdot d_2 - F_3 \cdot d_3$$

Principle of Moments

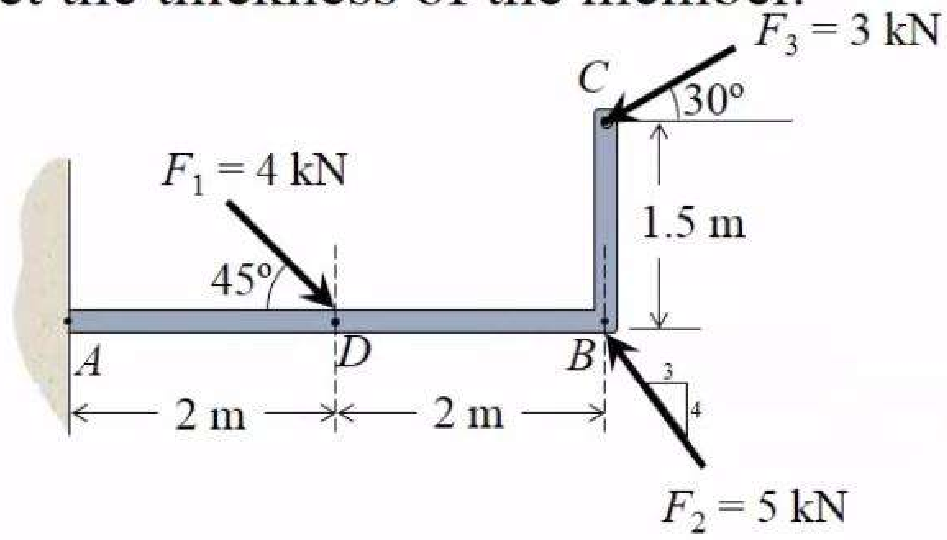




$$M_A = -2.83 \cdot 2 + 4 \cdot 4 - 1.5 \cdot 4 + 2.60 \cdot 1.5$$

$$= 8.24\,(\text{kN} \cdot \text{m})$$ Ans.

Question 1: Determine the total moment about **point *B*** caused by the three forces F_1, F_2 and F_3. Take counterclockwise as positive. Neglect the thickness of the member.



Question 2: Determine the total moment about **point** ***C*** caused by the three forces F_1, F_2 and F_3. Take counterclockwise as positive. Neglect the thickness of the member.