Vector Operation:

Parallelogram Law and Triangle Rule

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Objectives :

- To revisit the concepts of scalar and vector.
- To show how to properly represent a force vector.
- To explain the parallelogram law and the triangle rule for vector addition and subtraction.

Question 1: In your own words, what is a scalar and what is a vector? List at least three examples of scalars and three examples of vectors.

Scalars and vectors

Scalar: a physical/mathematical quantity that can be completely specified by its ***magnitude***.

length (*l*)
mass (*m*)
time (*t*)
volume (*V*)

Scalars and vectors

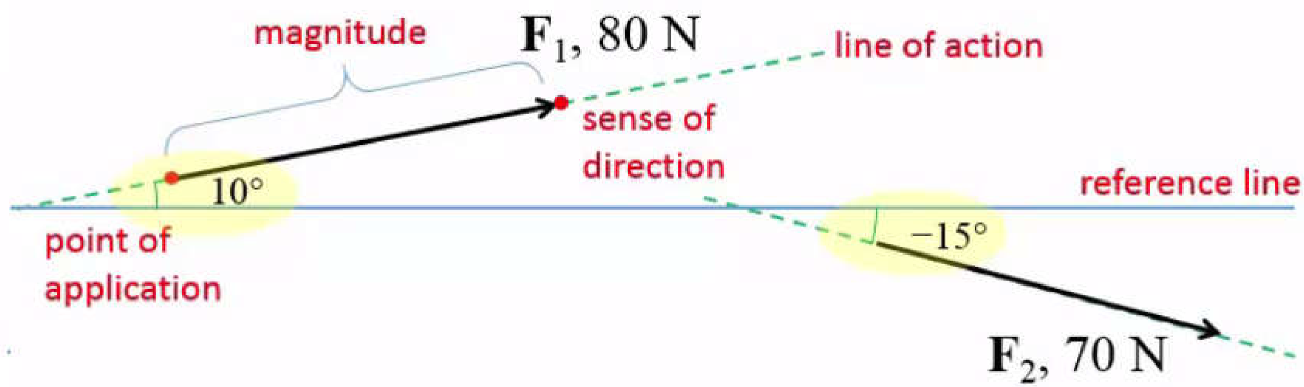
Vector: a physical/mathematical quantity that requires both a ***magnitude*** and ***direction*** for its complete description.

force (F)
velocity (v)
acceleration (a)
moment (M)

$\mathbf{F}$ $\quad \vec{F}$ $\quad \vec{F}$

A force can be represented by an arrow

It can be fully characterized by:
- Point of application,
- Sense of direction,
- Magnitude.



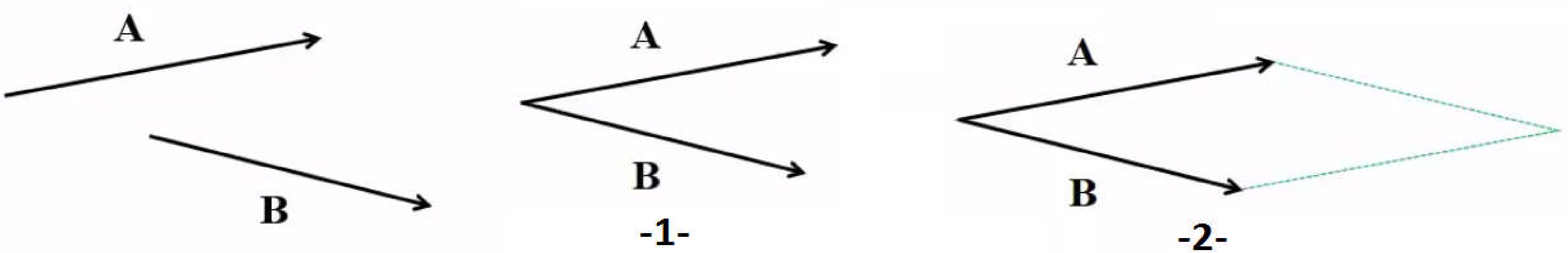
The **direction** of a force can be descibed by the **angle** made by its **line of action** and a **reference line**.

Sometimes, you might see **negative** angles, this is because by **sign convention**, **positive** angle represents **counterclockwise** rotation.

Negative angle represents clockwise rotation from the **reference line.**

Vector addition

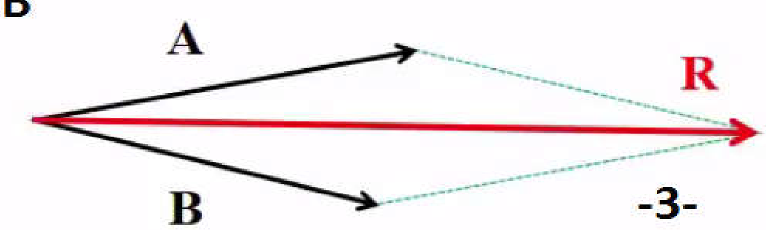
To perform vector addition, we need to follow the ***Parallelogram law***.



First step, we need to join the tails of the two vectors so that they are Concurrent.

Then, we construct a Parallelogram using A and B as the two sides.

Then, we draw an arrow that starts from the tails of A and B and points to the other end.

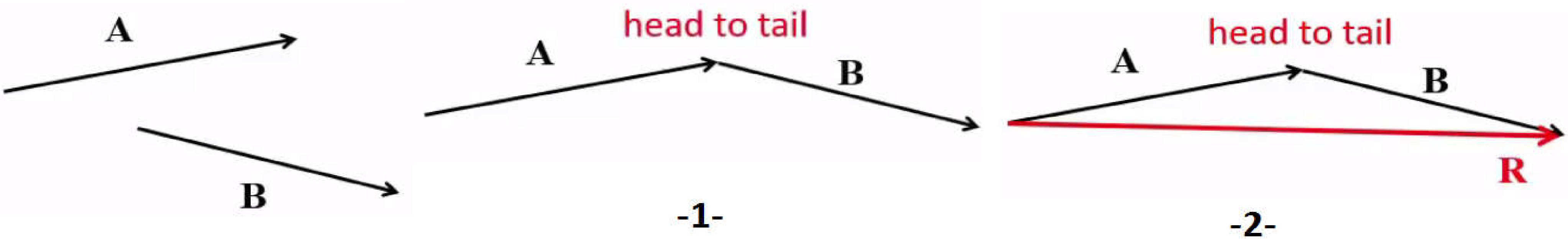


$$\mathbf{R} = \mathbf{A} + \mathbf{B}$$

R: is the resultant vector.

Vector addition

As a simplification to the parallelogram law, we can use the ***Triangle rule***



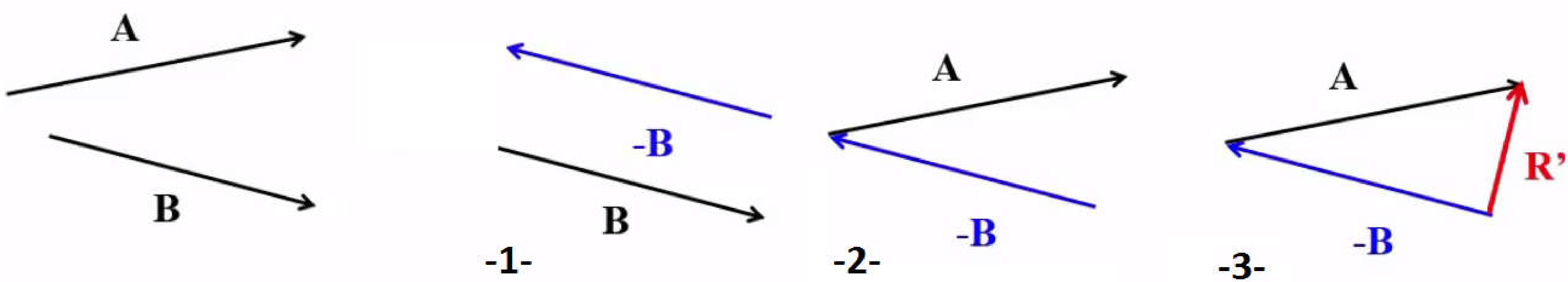
We join the vectors A and B in a head to tail fashion.

The resultant vector R can simply be represented by an arrow that starts from the tail of vector A to the head of vector B.

$$\mathbf{R} = \mathbf{A} + \mathbf{B}$$

Vector subtraction

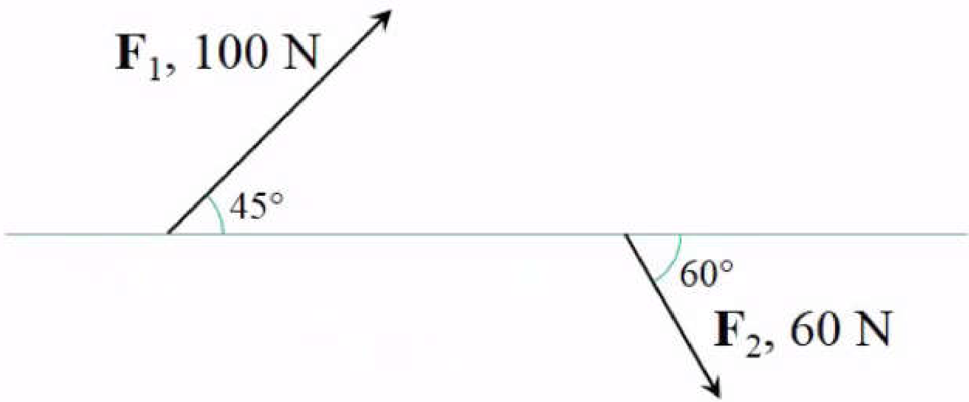
Since we know subtraction can be considered as addition with a negative quantity:
First, we can define the vector (-B), which has the same magnitude but opposite direction.



Then, we can simply add vector A to (-B) together using either Parallelogram Low or Triangle Rule.

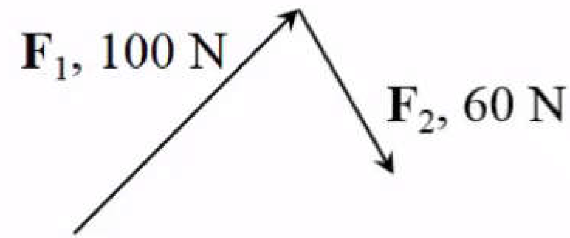
$$R' = A - B$$

Example 1: What this the magnitude and direction (with respect to the horizontal reference line) of the resultant force of $\mathbf{F}_1$ and $\mathbf{F}_2$?



Example 1: What this the magnitude and direction (with respect to the horizontal reference line) of the resultant force of $\mathbf{F}_1$ and $\mathbf{F}_2$?

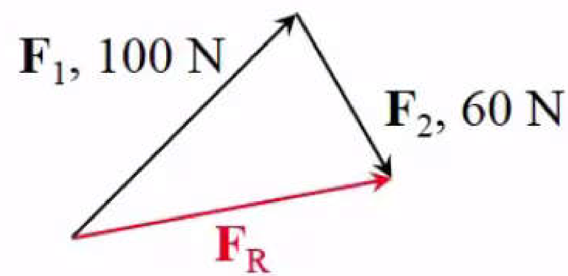
Triangle Rule



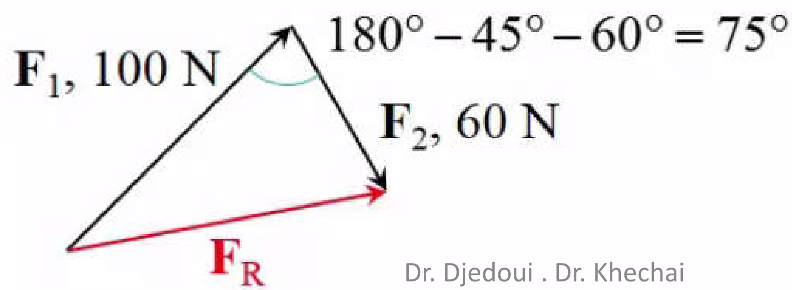
We join the two forces in a head to tail fashion.

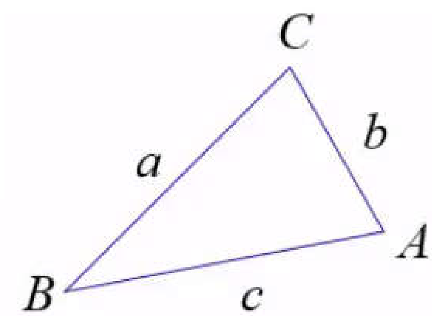
F_R is the resultant vector.

The three forces form a triangle.



Based on the geometry given in the problem statement, the angle between these two vectors is **75°**.





In a Triangle, the relation between the sides a, b, c, and the angles A, B and C, we have:

Law of sines: $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$

Law of cosines:

$$a^2 = b^2 + c^2 - 2bc\cos A$$

$$b^2 = a^2 + c^2 - 2ac\cos B$$

$$c^2 = a^2 + b^2 - 2ab\cos C$$

Based on the information we have:

$$a = 100$$

$$b = 60$$

$$C = 75°$$

we can start with the equation:

$$c^2 = a^2 + b^2 - 2ab\cos C$$

Then, we use this equation to calculate the angle B

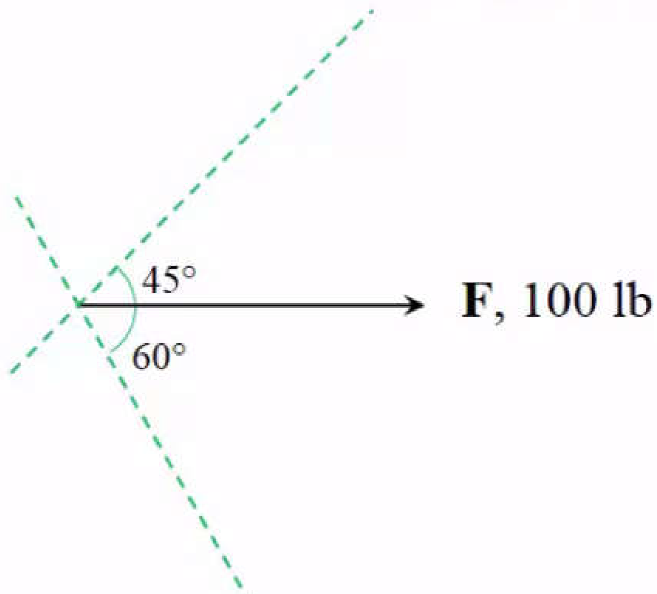
$$\frac{b}{\sin B} = \frac{c}{\sin C}$$

Here are the results:

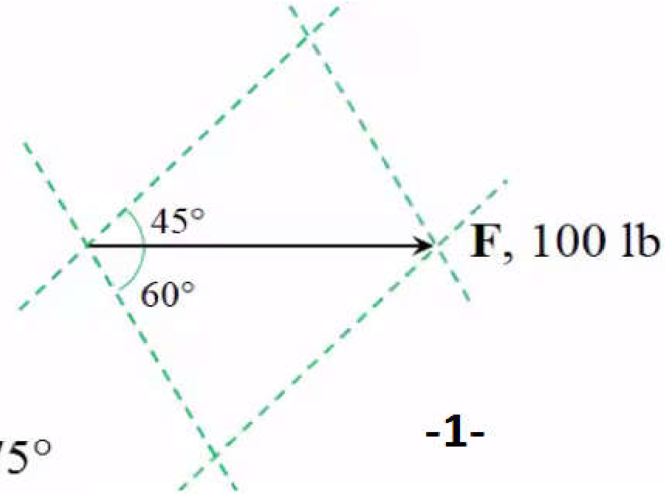
$$c = 102$$

$$B = 34.5°$$

Example 2: The given force **F** is the resultant force of forces $\mathbf{F}_1$ and $\mathbf{F}_2$, for which the lines of action are given. Determine the magnitudes of these two forces.



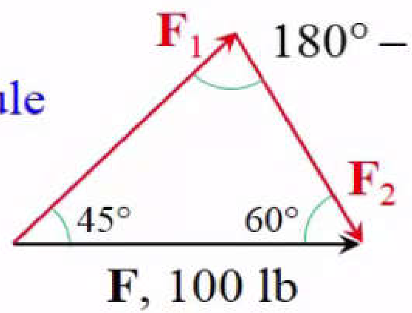
we are going to apply the Parallelogram Law



we can visualize the two component forces **F_1** and **F_2**

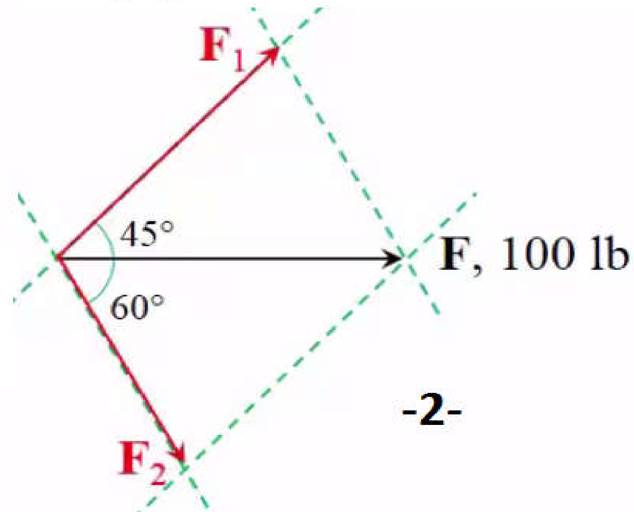
Let's make the triangle, and calculate the third angle.

Triangle Rule



$$180° - 45° - 60° = 75°$$

-3-



Then, apply law of sines directly to calculate the magnitude of these two forces

$$\frac{100}{\sin 75°} = \frac{F_1}{\sin 60°} = \frac{F_2}{\sin 45°} \qquad \therefore \begin{cases} F_1 = 89.7 \text{ lb} \\ F_2 = 73.2 \text{ lb} \end{cases}$$

Ans.

Question 3: Which of the following is NOT a vector?

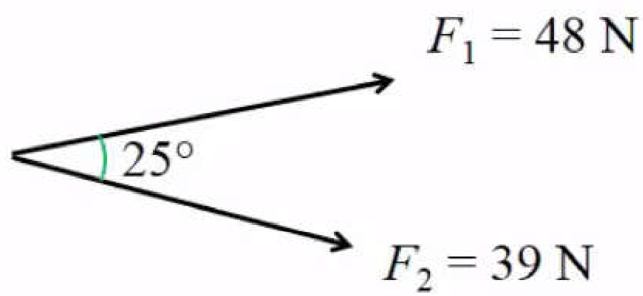
(a) Force

(b) Pressure

(c) Acceleration

(d) Energy

Question 4: What is the magnitude of the resultant force of forces F_1 and F_2?



(a) 85 N

(b) 87 N

(c) 21 N

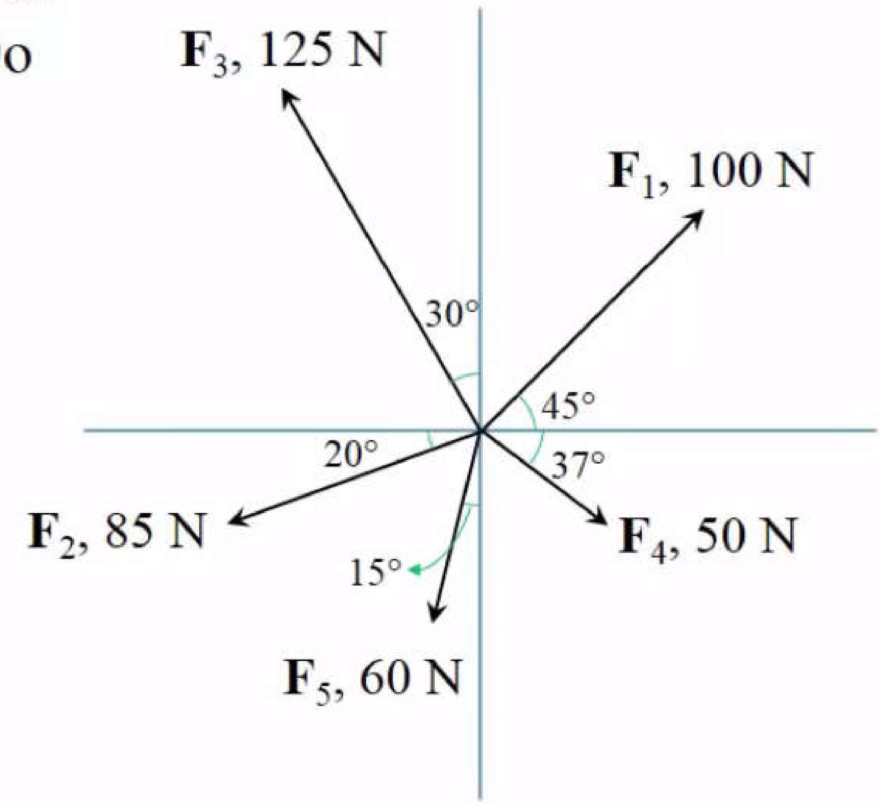
(d) 11 N

Cartesian Vectors and Operation

Objectives :

- To express a vector in a rectangular coordinate system and in the Cartesian vector form.
- To introduce key concepts of component vectors and unit vector.
- To determine the magnitude of a Cartesian vector and express its direction using coordinate direction angles.
- To perform vector addition of Cartesian vectors.

Question 1: If you are to use the parallelogram law or triangle rule to find the resultant force of these five forces, how do you plan to do it? Do you think there's a better way?



Right-handed Rectangular Coordinate System

(Cartesian Coordinate System)

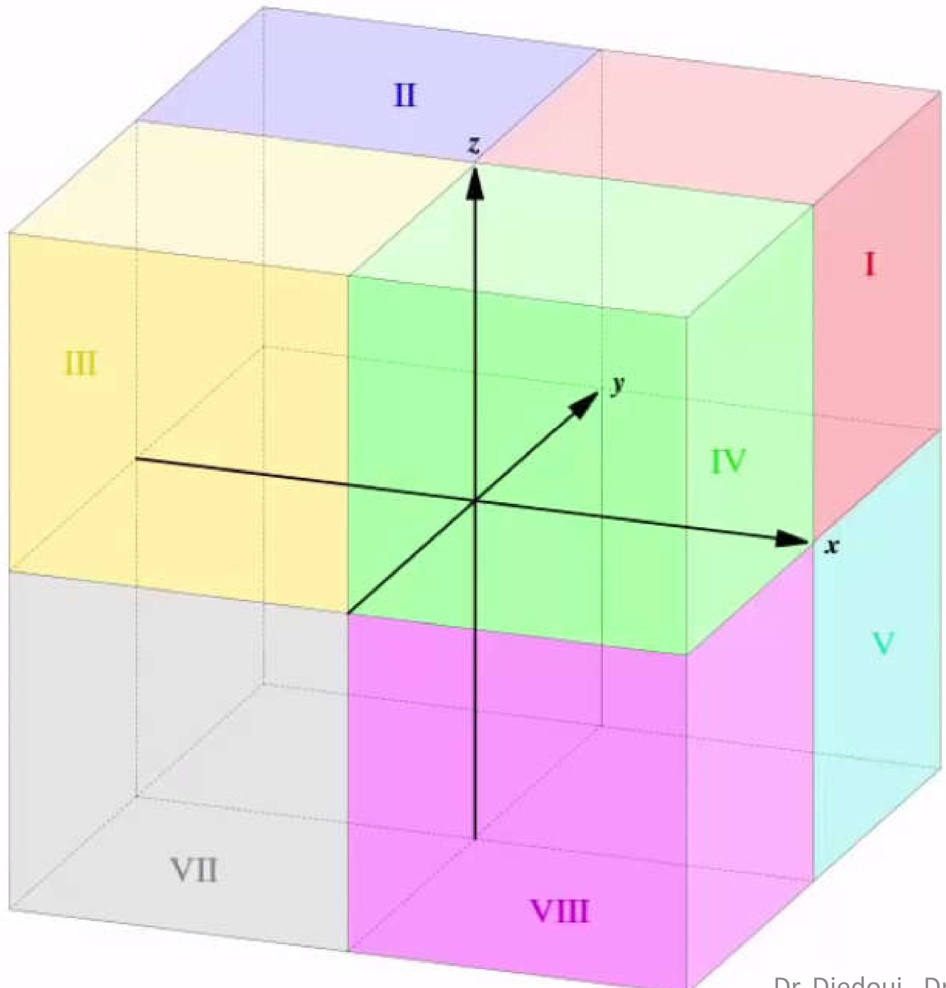
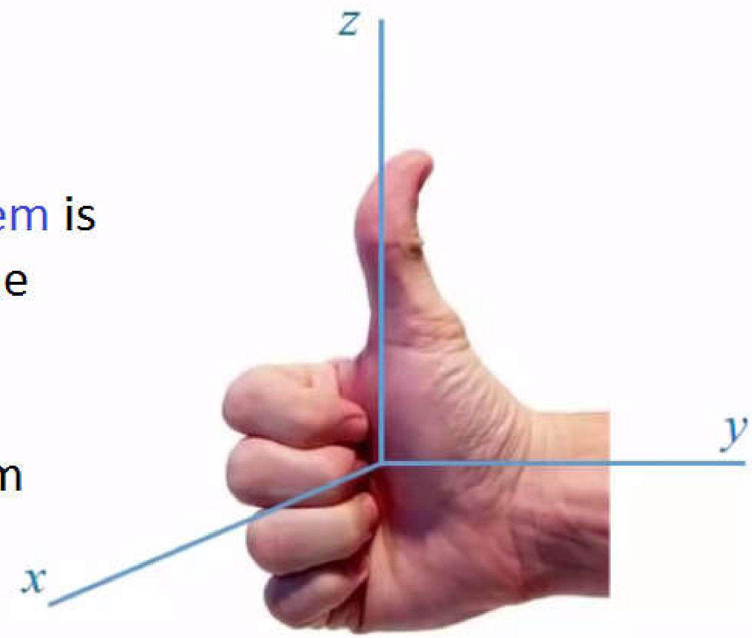


Image from wikipedia.org.

Right-handed Rectangular Coordinate System:

The reason why this is called Right-handed coordinate system is because the **POSITIVE** directions of the three axes follow the right-hand rule.



This means that if you roll the four fingers in your hand from the positive 'x' direction towards the positive 'y' direction as shown in this image, your thumb will point towards the positive 'z' direction.

Cartesian vectors

Rectangular components of a vector:

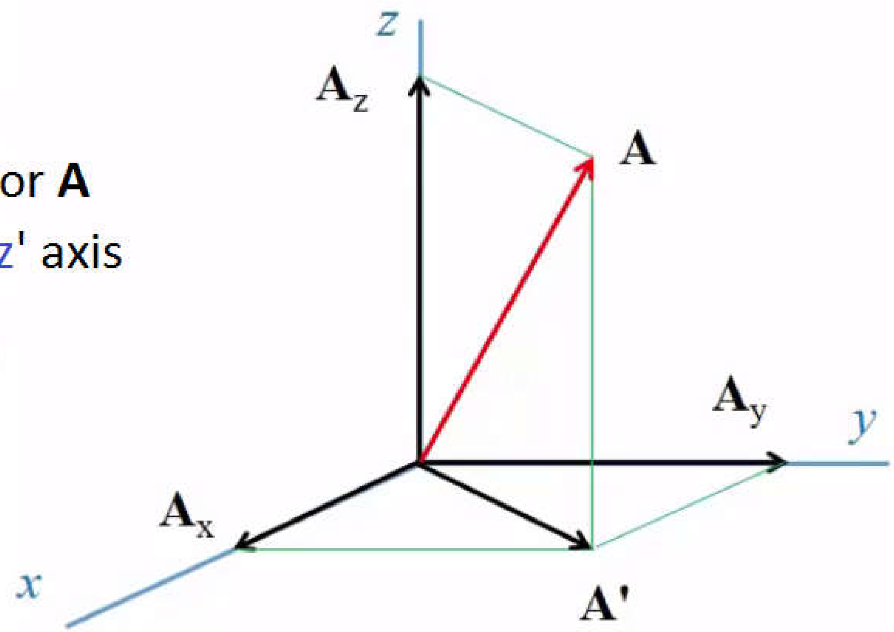
First, we apply the parallelogram law to resolve vector **A** into two component vectors **Az** that falls along the 'z' axis and **A'** that falls within the 'xy' plane. $\mathbf{A} = \mathbf{A}' + \mathbf{A}_z$

Then, we can apply the parallelogram law again to resolve **A'**.

$$\mathbf{A}' = \mathbf{A}_x + \mathbf{A}_y$$

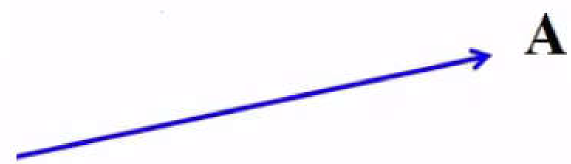
Finally

$$\therefore \quad \mathbf{A} = \mathbf{A}_x + \mathbf{A}_y + \mathbf{A}_z$$



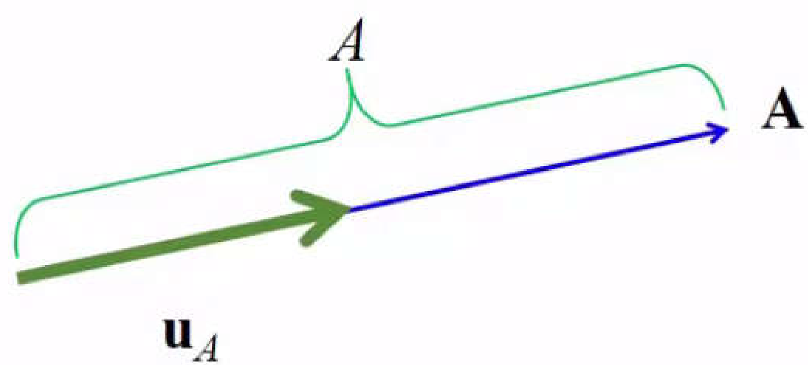
Cartesian unit vectors

Since a vector needs to be described by two parts its **magnitude** and its **direction**, we can separate these two parts by defining **unit vectors.**



For any arbitrary vector **A**, its unit vector $\mathbf{u}_a$ has the same **direction** but a **magnitude** of unit length 1.

Vector **A** can be expressed by its magnitude A, which is a scalar multiplied by its unit vector $\mathbf{u}_a$

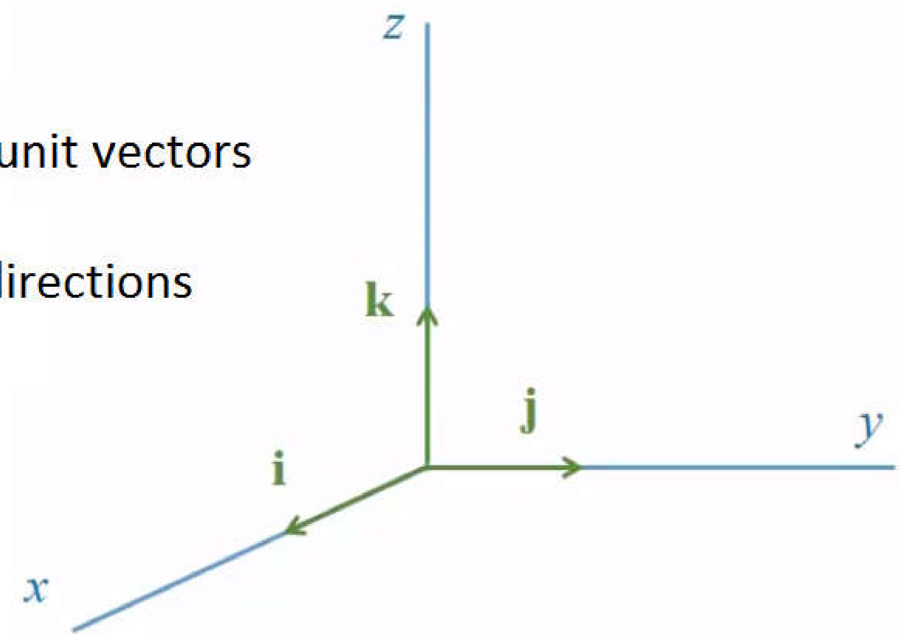


$$\mathbf{A} = A \cdot \mathbf{u}_A$$

Cartesian unit vectors

Cartesian Unit Vectors:

In a Cartesian coordinate system, there are 3 special unit vectors **i**, **j** and **k**.
They are special because they are designated to the directions of 'x', 'y' and 'z' axis.



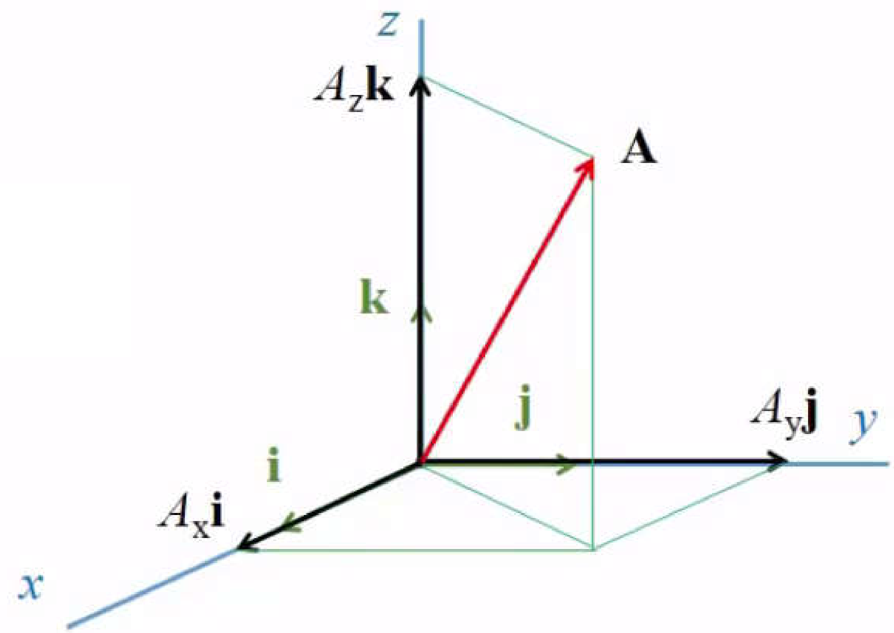
i, **j** and **k** all have magnitude of 1.

Cartesian vectors

Therefore, using the unit vectors **i**, **j** and **k**, the component vectors along the x, y and z axis can now be written as:

$$\mathbf{A} = A_x\mathbf{i} + A_y\mathbf{j} + A_z\mathbf{k}$$

A_x, A_y and A_z being the magnitudes of the component vectors.



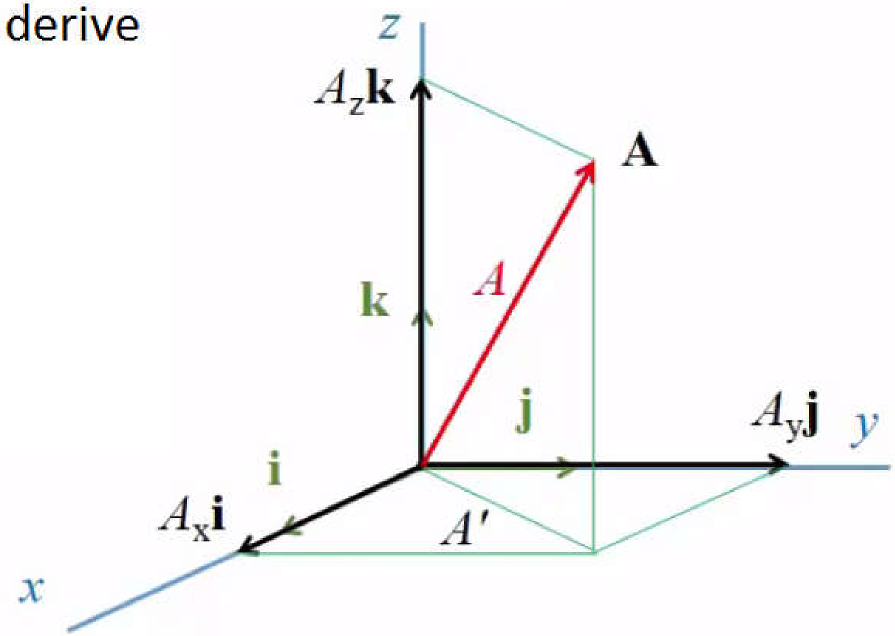
Magnitude of a Cartesian Vector

By applying the **Pythagorean theorem** twice, we can derive the magnitude of vector **A**:

$$A' = \sqrt{A_x^2 + A_y^2}$$

$$A = \sqrt{A'^2 + A_z^2}$$

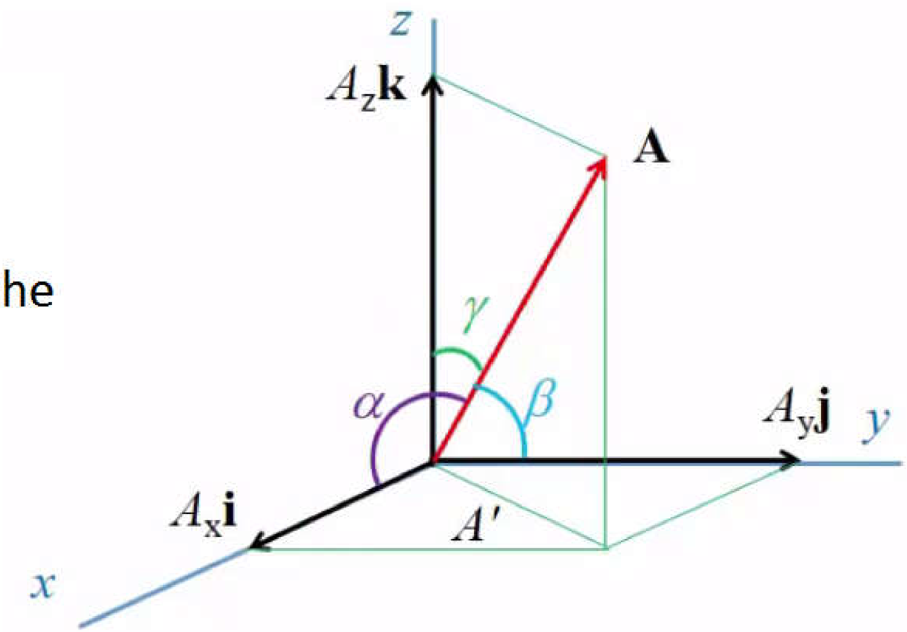
$$\therefore \quad A = \sqrt{A_x^2 + A_y^2 + A_z^2}$$



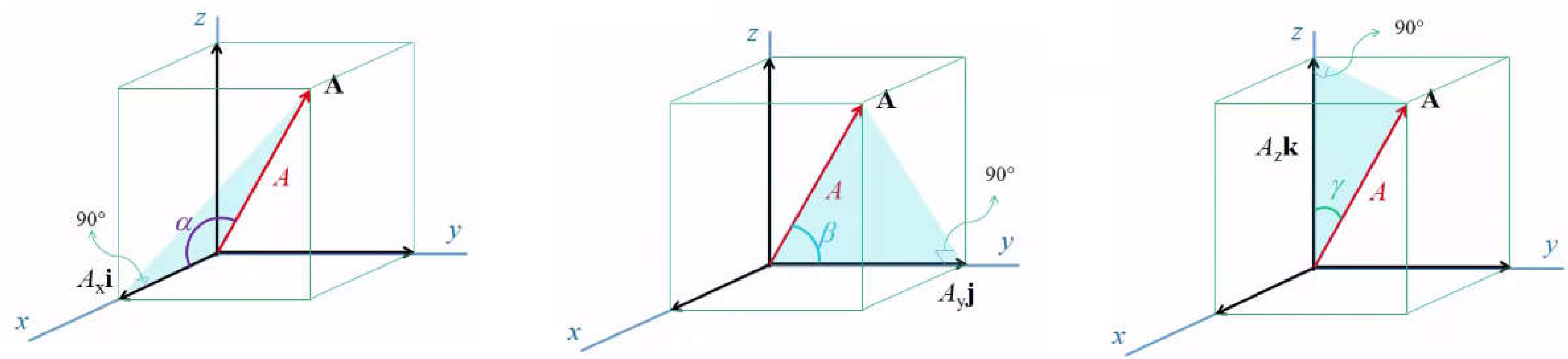
Direction of a Cartesian Vector

Coordinate direction angles α, β and γ.

To describe **the direction** of the vector, we can use the **Coordinate direction angles.**



Direction of a Cartesian Vector



According to trigonometry, we know that:

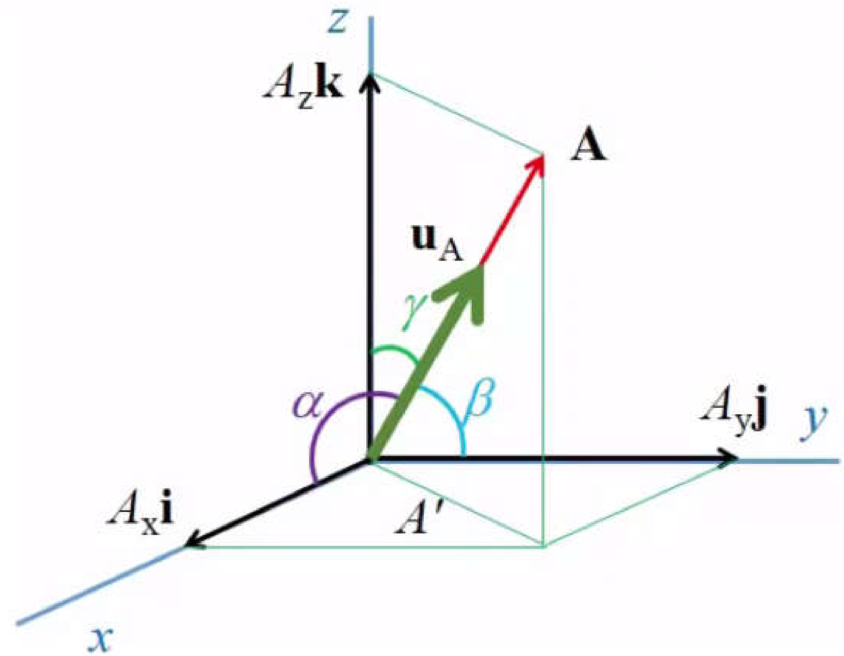
$$\cos\alpha = \frac{A_x}{A} \qquad \cos\beta = \frac{A_y}{A} \qquad \cos\gamma = \frac{A_z}{A}$$

Unit Vector

Because the **unit vector** of vector **A** , **u**A equals to:

$$\mathbf{u}_A = \frac{\mathbf{A}}{A} = \frac{A_x}{A}\mathbf{i} + \frac{A_y}{A}\mathbf{j} + \frac{A_z}{A}\mathbf{k}$$

$$\mathbf{u}_A = \cos\alpha\,\mathbf{i} + \cos\beta\,\mathbf{j} + \cos\gamma\,\mathbf{k}$$



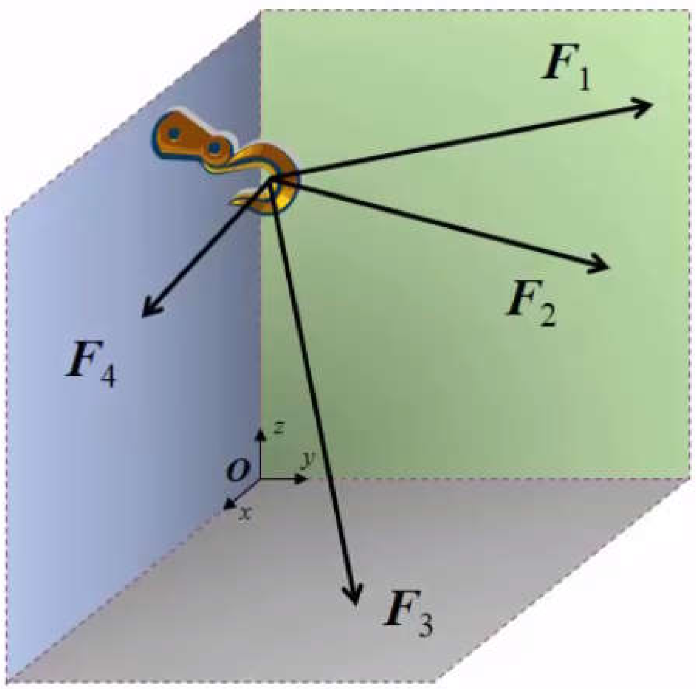
Since the **magnitude** of **unit vector** is always 1, therefore, we come to the conclusion that:

$$\therefore \quad \cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$$

"The cosine squared of the three coordinate direction angles for any Cartesian vector must equal 1."

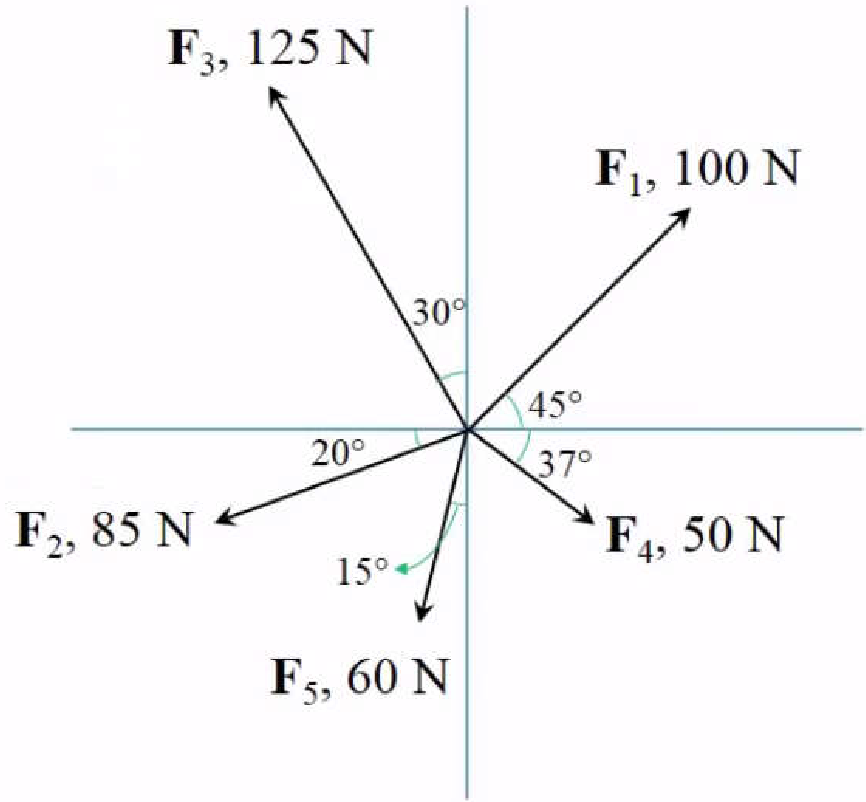
Addition of Cartesian Vectors

$$\mathbf{A}+\mathbf{B}=A_x\mathbf{i}+A_y\mathbf{j}+A_z\mathbf{k}+B_x\mathbf{i}+B_y\mathbf{j}+B_z\mathbf{k}$$
$$=(A_x+B_x)\mathbf{i}+(A_y+B_y)\mathbf{j}+(A_z+B_z)\mathbf{k}$$



$$\mathbf{F}_R=\sum\mathbf{F}=\sum F_x\mathbf{i}+\sum F_y\mathbf{j}+\sum F_z\mathbf{k}$$

Question 2: Determine the magnitude of the resultant force of these five forces.



Example 1: For force vector $\mathbf{F} = \{1.2\mathbf{i} - 1.8\mathbf{j} + 3.6\,\mathbf{k}\}$ kN, determine its magnitude, unit vector and the three coordinate direction angles.

Example 1: For force vector $\mathbf{F} = \{1.2\mathbf{i} - 1.8\mathbf{j} + 3.6\,\mathbf{k}\}$ kN, determine its magnitude, unit vector and the three coordinate direction angles.

Magnitude:

$$F = \sqrt{1.2^2 + (-1.8)^2 + 3.6^2} = 4.2\,(\text{kN})$$

Unit vector:

$$\mathbf{u}_F = \frac{\mathbf{F}}{F} = \frac{\{1.2\mathbf{i} - 1.8\mathbf{j} + 3.6\mathbf{k}\}\ \text{kN}}{4.2\ \text{kN}} = \frac{2}{7}\mathbf{i} - \frac{3}{7}\mathbf{j} + \frac{6}{7}\mathbf{k}$$

Direction angles:

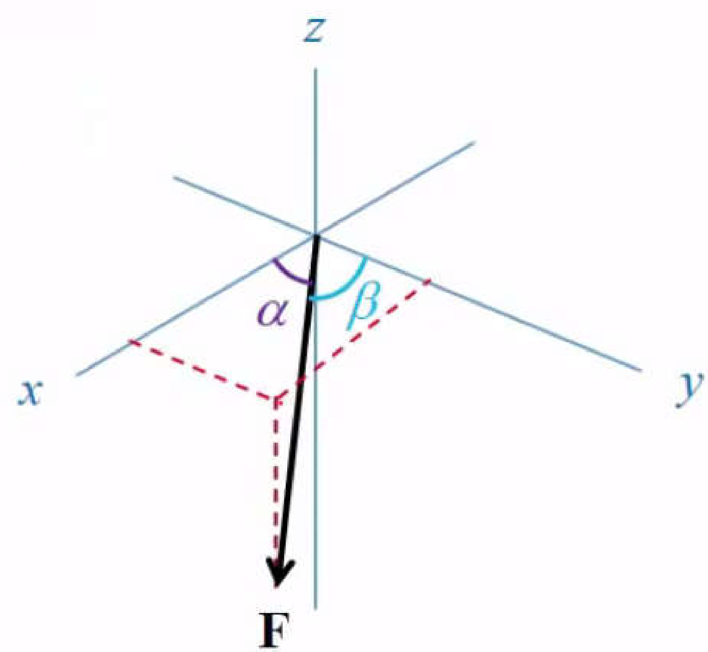
$$\alpha = \cos^{-1}\left(\frac{2}{7}\right) = 73.4°$$

$$\beta = \cos^{-1}\left(-\frac{3}{7}\right) = 115°$$

$$\gamma = \cos^{-1}\left(\frac{6}{7}\right) = 31.0°$$

Ans.

Example 2: If force **F** has a magnitude of 1200 N, and angle α is 60° and β is 45°, express the force in Cartesian vector form and determine its unit vector.



Example 2: If force **F** has a magnitude of 1200 N, and angle α is 60° and β is 45°, express the force in Cartesian vector form and determine its unit vector.

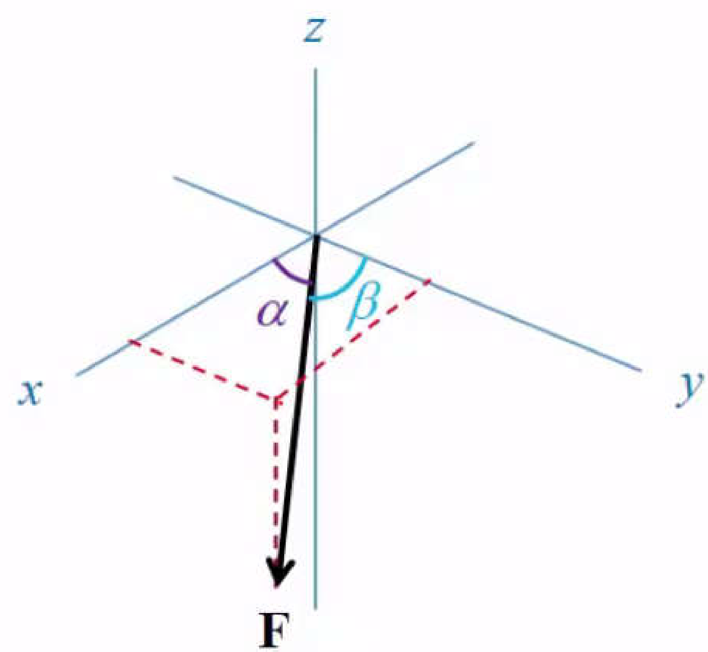
$$\because \quad \cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$$

$$\cos^2\gamma = 1 - \cos^2 45° - \cos^2 60° = 0.25$$

$$\cos\gamma = \pm 0.5 \quad \Rightarrow \quad \gamma = 60° \text{ or } 120°$$

we know that, the direction angle is defined as the angle made by the force with the positve part of the axis.

$$\therefore \quad \gamma = 120°$$



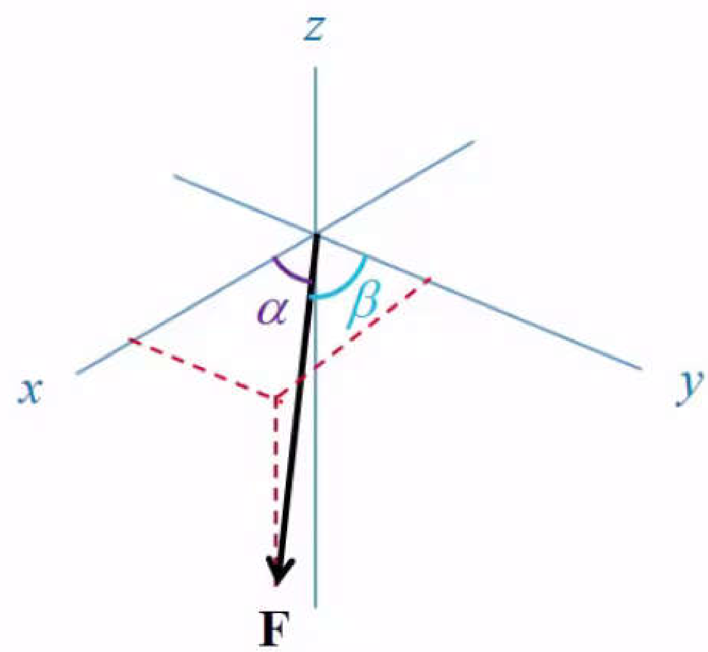
Example 2: If force $\mathbf{F}$ has a magnitude of 1200 N, and angle α is 60° and β is 45°, express the force in Cartesian vector form and determine its unit vector.

$$\mathbf{u}_F = \cos\alpha\mathbf{i} + \cos\beta\mathbf{j} + \cos\gamma\mathbf{k}$$

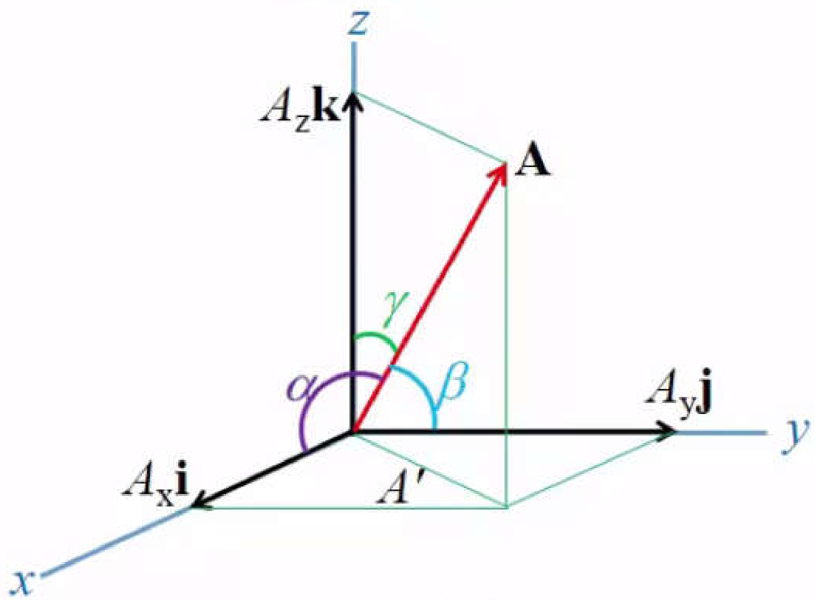
$$= 0.5\mathbf{i} + 0.707\mathbf{j} - 0.5\mathbf{k}$$

$$\mathbf{F} = F\mathbf{u}_F = 1200\text{ N}\cdot\{0.5\,\mathbf{i} + 0.707\,\mathbf{j} - 0.5\mathbf{k}\}$$

$$= \{600\mathbf{i} + 849\mathbf{j} - 600\mathbf{k}\}\text{ N}$$



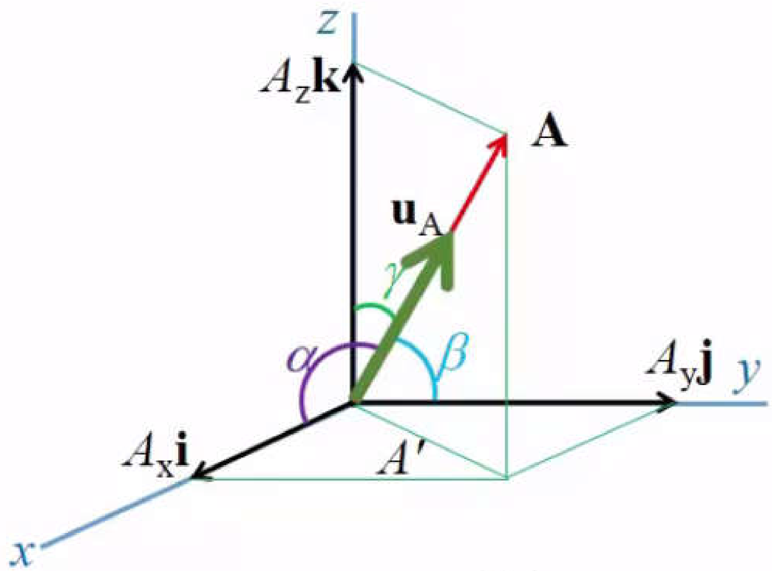
Question 3: If the coordinate direction angles $\alpha = 112°$, $\beta = 75°$ and $A_z = 5.0$ cm, determine the magnitude of vector **A**.



(a) 5.0 cm　　(b) 13 cm

(c) 6.9 cm　　(d) 5.6 cm

Question 4: If the coordinate direction angles $\alpha = 112°$, $\beta = 75°$ and $A_z = 5.0$, determine the unit vector, $\mathbf{u}_A$, of $\mathbf{A}$.



(a) $-0.37\mathbf{i} + 0.26\mathbf{j} + 0.89\mathbf{k}$

(b) $0.37\mathbf{i} + 0.26\mathbf{j} + 0.89\mathbf{k}$

(c) $\{-0.37\mathbf{i} + 0.26\mathbf{j} + 0.89\mathbf{k}\}$cm

(d) $\{0.37\mathbf{i} + 0.26\mathbf{j} + 0.89\mathbf{k}\}$cm

Position Vector and Force Vector

Objectives :

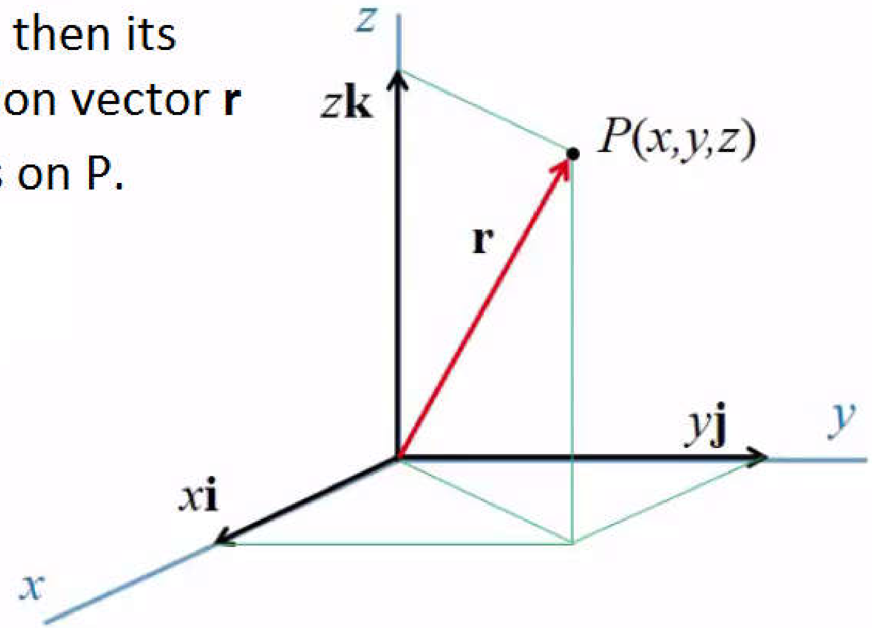
- To introduce the concept of position vector.
- To demonstrate the general strategy of writing the force vector from the corresponding position vector.

Question 1: You have learned about the unit vector. Can different physical quantities such as position, velocity, acceleration or force have the same unit vector? Why or why not?

Position vector

If a point P has coordinates x, y and z, then its position can be expressed by its position vector **r** which starts from the **origin** and ends on P.

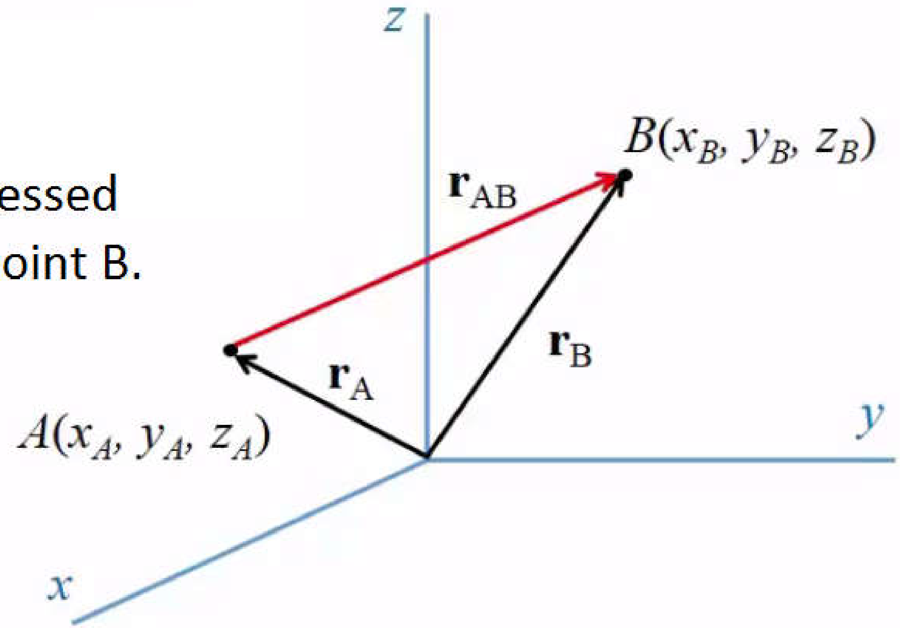
$$\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$$



Position vector

If you want to find the relative position of ponit B relative to point A, this relative position can be expressed by a vector **r**AB that starts from point A and end on point B.

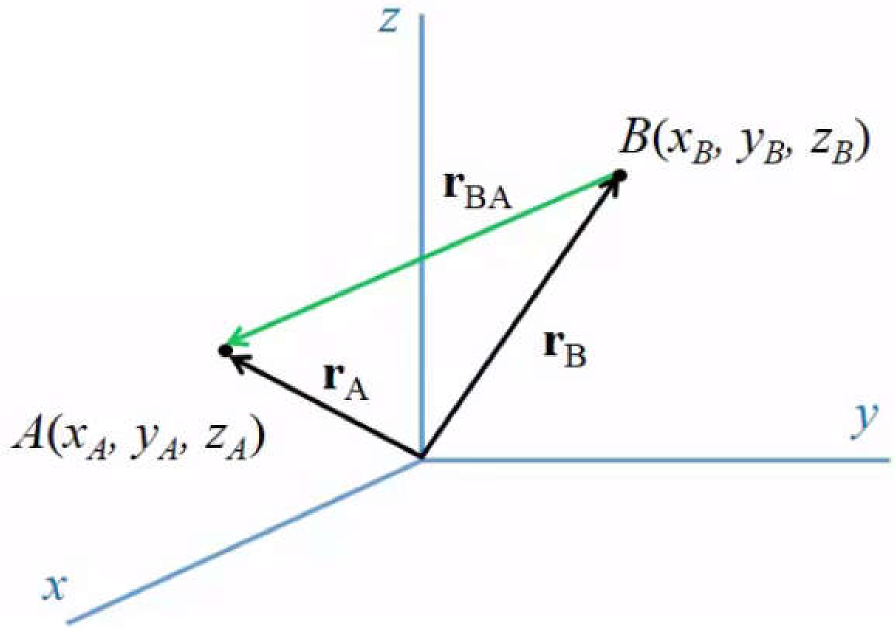
$$\mathbf{r}_{AB} = \mathbf{r}_B - \mathbf{r}_A$$
$$= (x_B - x_A)\mathbf{i} + (y_B - y_A)\mathbf{j} + (z_B - z_A)\mathbf{k}$$



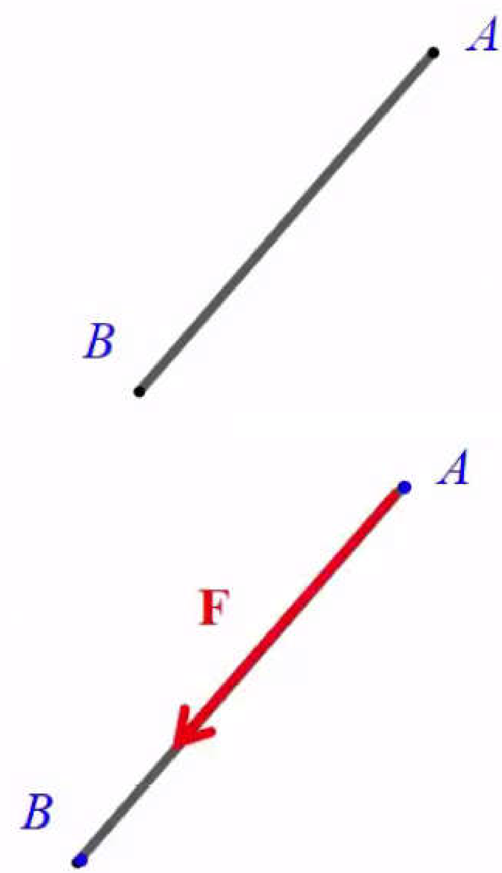
Position vector

The relative position of point A relative to poit B is expressed by the **opposite** vector $\mathbf{r}_{BA}$ that starts from point B and ends on point A.

$$\mathbf{r}_{BA} = \mathbf{r}_A - \mathbf{r}_B$$
$$= (x_A - x_B)\mathbf{i} + (y_A - y_B)\mathbf{j} + (z_A - z_B)$$
$$= -\mathbf{r}_{AB}$$



Force vector

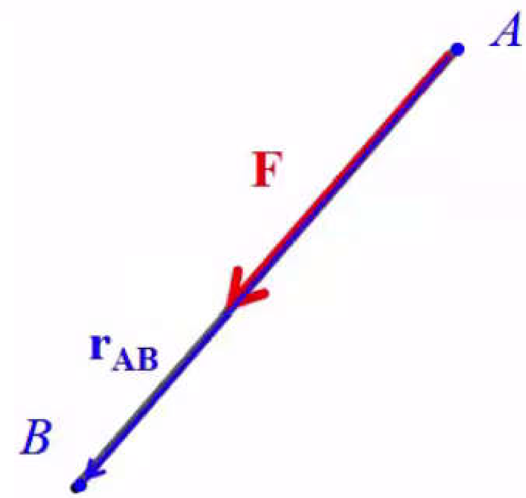


we can also express a force vectors as **Cartesian vectors**. For example, for the tension force **F** in the cable directed from point A to point B, we know that we can express it as **its magnitude multiplied by a unit vector** that describes its direction.

How do we find this unit vector ?

Force vector

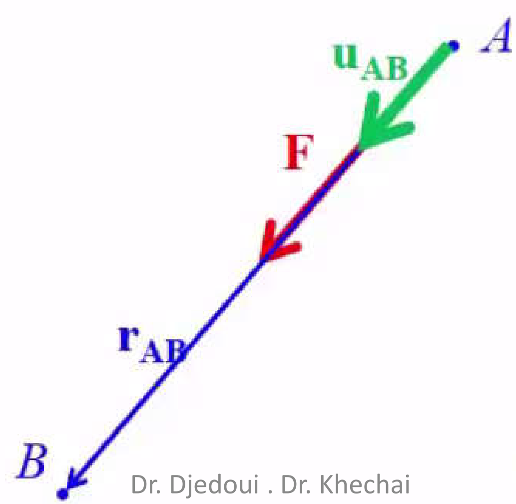
Since the **position vector** from A to B has the **same direction** as the force, we can use the position vector **r**AB to find the **unit vector u**AB which is given by this equation:



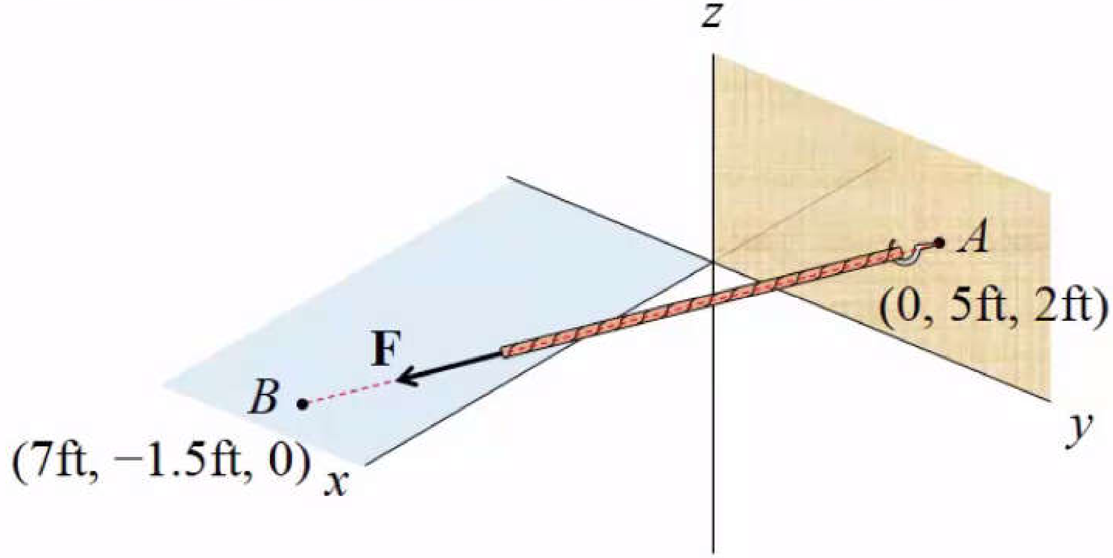
$$\mathbf{u}_{AB} = \frac{\mathbf{r}_{AB}}{r_{AB}}$$

$$= \frac{(x_B - x_A)\mathbf{i} + (y_B - y_A)\mathbf{j} + (z_B - z_A)\mathbf{k}}{\sqrt{(x_B - x_A)^2 + (y_B - y_A)^2 + (z_B - z_A)^2}}$$

$$\mathbf{F} = F\mathbf{u}_{AB}$$

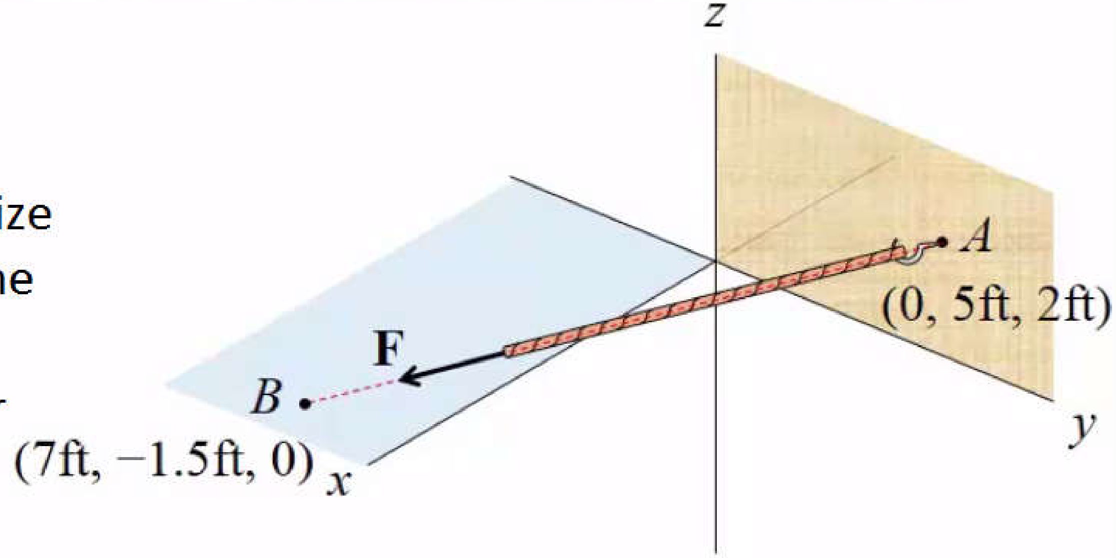


Example: The line of action of force **F** directs from point A to point B. If the magnitude of the force is 120 lb, express the force in Cartesian vector form.



Example: The line of action of force **F** directs from point A to point B. If the magnitude of the force is 120 lb, express the force in Cartesian vector form.

The key of solving this problem is to recognize that **this force has the same direction** as the **position vector** from A to B, therefore they have **the same unit vector** since unit vector only indicates **the direction**.



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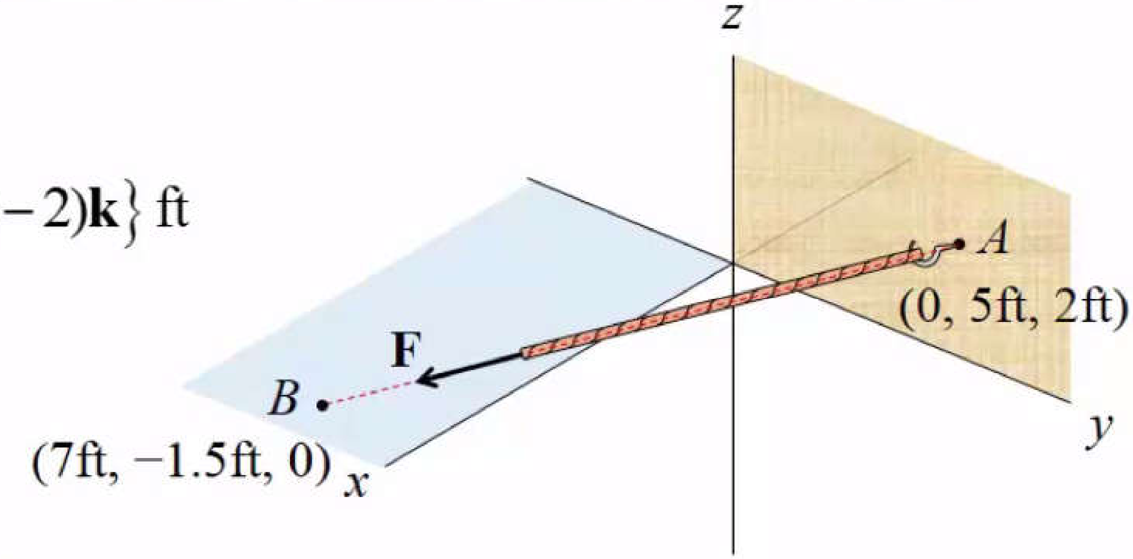


Position vector:

$$\mathbf{r}_{AB} = \mathbf{r}_B - \mathbf{r}_A = \{(7-0)\mathbf{i} + (-1.5-5)\mathbf{j} + (0-2)\mathbf{k}\}\ \text{ft}$$

$$= \{7\mathbf{i} - 6.5\mathbf{j} - 2\mathbf{k}\}\ \text{ft}$$

Unit vector:

$$\mathbf{u}_{AB} = \frac{\mathbf{r}_{AB}}{r_{AB}} = \frac{\{7\mathbf{i} - 6.5\mathbf{j} - 2\mathbf{k}\}\ \text{ft}}{\sqrt{7^2 + (-6.5)^2 + (-2)^2}\ \text{ft}}$$

$$= 0.717\mathbf{i} - 0.666\mathbf{j} - 0.205\mathbf{k}$$

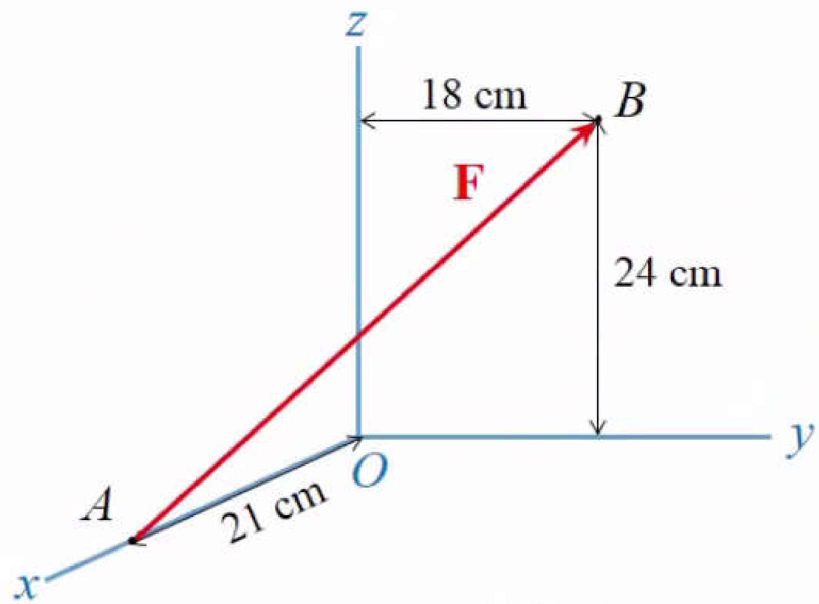
Force vector:

$$\mathbf{F} = F \cdot \mathbf{u}_{AB} = 120\ \text{lb} \cdot \{0.717\mathbf{i} - 0.666\mathbf{j} - 0.205\mathbf{k}\}$$

$$= \{86.1\mathbf{i} - 79.9\mathbf{j} - 24.6\mathbf{k}\}\ \text{lb}$$

Ans.

Question 2: If force **F** has magnitude of 450 N and is directed from point A to B as shown, determine the force in Cartesian vector form.



(a) $\{-150\mathbf{i}+129\mathbf{j}+171\mathbf{k}\}$cm (b) $\{-258\mathbf{i}+221\mathbf{j}+295\mathbf{k}\}$N

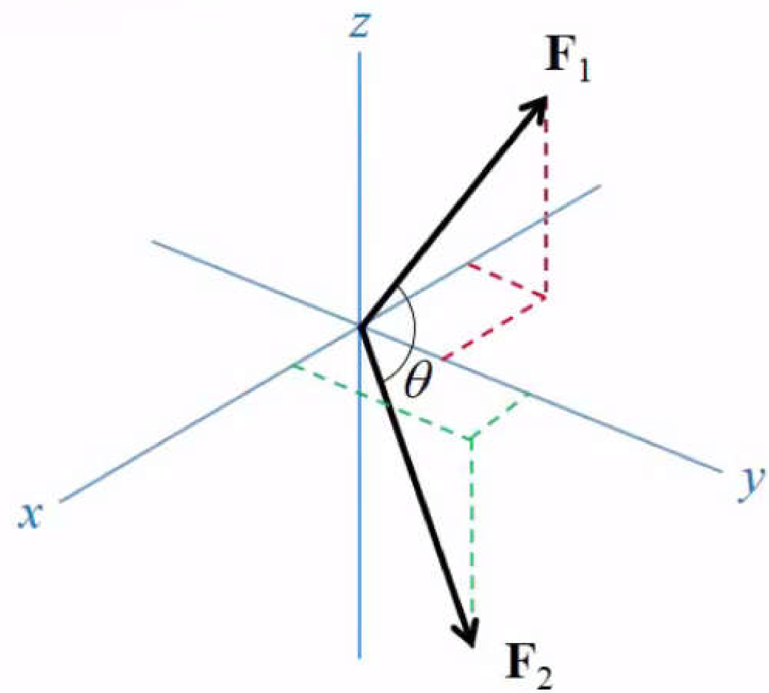
(c) $\{-150\mathbf{i}+129\mathbf{j}+171\mathbf{k}\}$N (d) $\{-21\mathbf{i}+18\mathbf{j}+24\mathbf{k}\}$ cm

Dot Product
of Cartesian Vectors

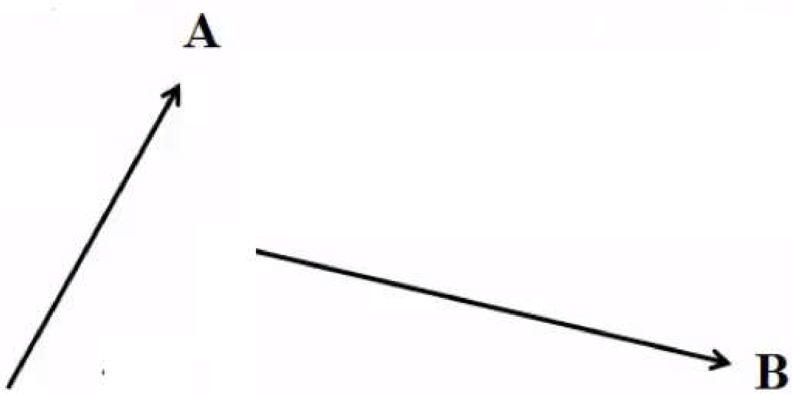
Objectives :

- To revisit the concept of dot product.
- To determine the angle between two vectors using their dot product.
- To calculate the projection of a force along a specified axis using dot product.

Question 1: If you are to use trigonometry to determine the angle between the two forces $\mathbf{F}_1 = \{-4.2i + 2.8j + 5.4k\}$ kip and $\mathbf{F}_2 = \{2.5i + 5.8j - 7.1k\}$ kip, how do you plan to do it? Do you think there's a better way?



Dot product



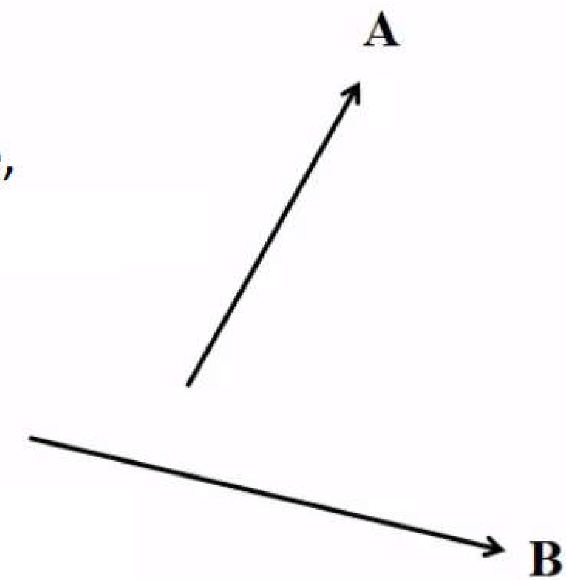
$$\mathbf{A} = A_x\mathbf{i} + A_y\mathbf{j} + A_z\mathbf{k}$$

$$\mathbf{B} = B_x\mathbf{i} + B_y\mathbf{j} + B_z\mathbf{k}$$

For two arbitrary vectors **A** and **B** expressed as Cartesian vectors, their **dot product** is a **scalar** and is defined **algebraically** as:

$$\mathbf{A} \cdot \mathbf{B} = A_x B_x + A_y B_y + A_z B_z$$

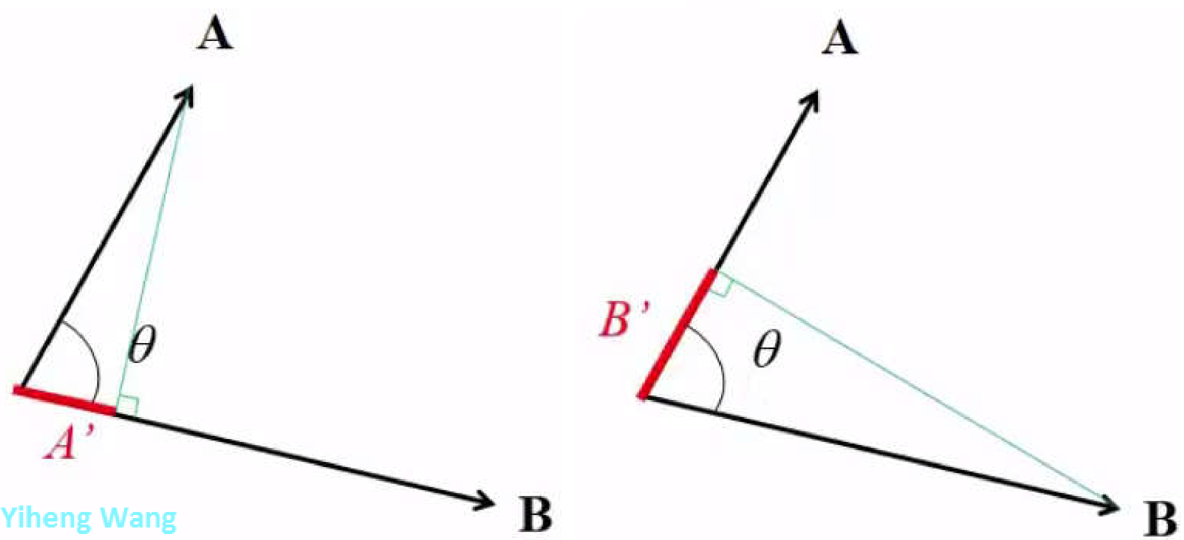
Dot product



Geometrically, even if the two vectors are not on the same plane, they can be parallel transported to be **concurrent**.

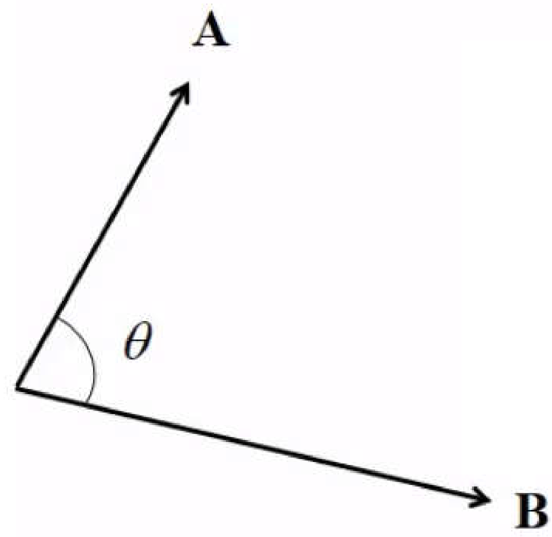
They form an angle θ

Their **dot product** equals to: $\mathbf{A} \cdot \mathbf{B} = AB\cos\theta$



Application of dot product

Because of the **algebraic** and **geometric** definitions of dot product, dot product can now be used **to find the angle** between the two vectors **A** and **B**.



$$\theta = \cos^{-1}\left(\frac{\mathbf{A}\cdot\mathbf{B}}{AB}\right)$$

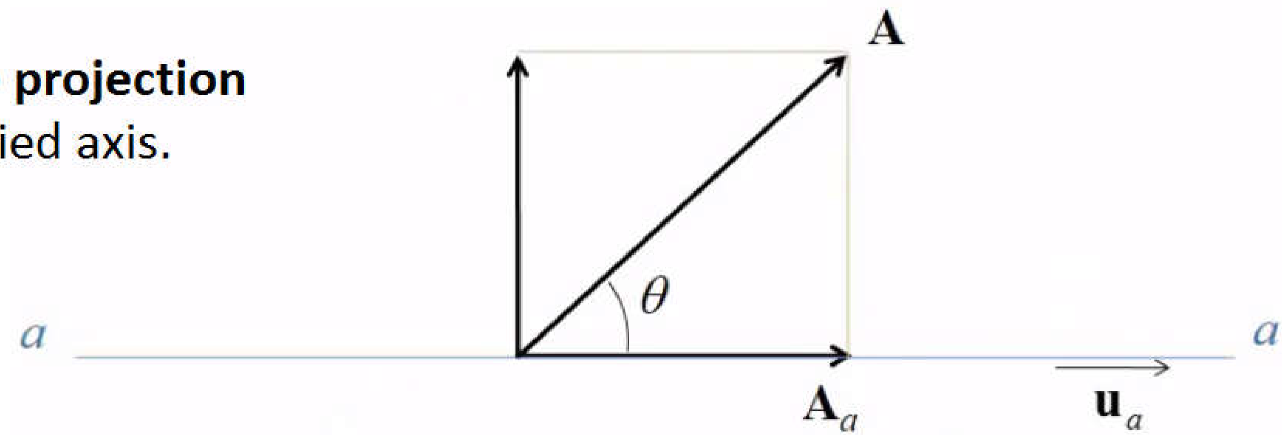
$$= \cos^{-1}\left(\frac{A_x B_x + A_y B_y + A_z B_z}{AB}\right)$$

$$(0^\circ \leq \theta \leq 180^\circ)$$

Application of dot product

We can use dot product to find the **projection vector** of any vector along a specified axis.

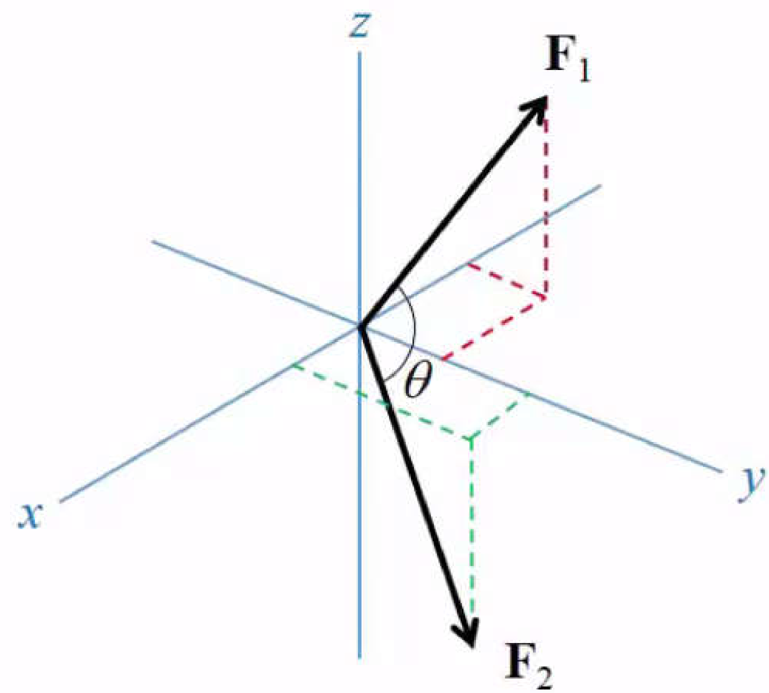
$\mathbf{u}_a$ is the unit vector along the axis.



$$A_a = \mathbf{A} \cdot \mathbf{u}_a$$

$$\mathbf{A}_a = A_a \mathbf{u}_a = A \cos\theta \, \mathbf{u}_a$$

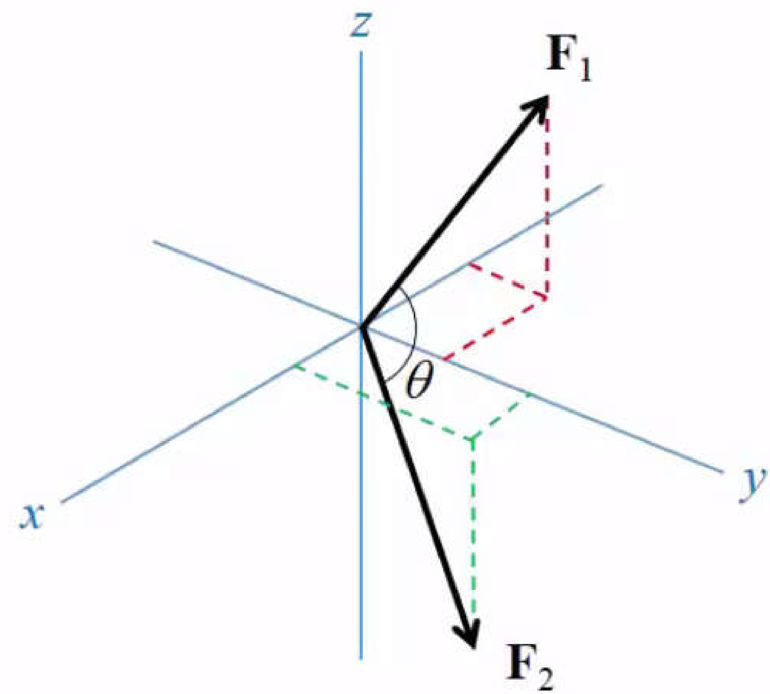
Example: For two forces $\mathbf{F}_1 = \{-4.2\mathbf{i} + 2.8\mathbf{j} + 5.4\mathbf{k}\}$ kip and $\mathbf{F}_2 = \{2.5\mathbf{i} + 5.8\mathbf{j} - 7.1\mathbf{k}\}$ kip, determine the angle between them and the magnitude of the projection of $\mathbf{F}_1$ along the line of action of $\mathbf{F}_2$.



Magnitude:

$$F_1 = \sqrt{(-4.2)^2 + 2.8^2 + 5.4^2} = 7.4\,(\text{kip})$$

$$F_2 = \sqrt{2.5^2 + 5.8^2 + (-7.1)^2} = 9.5\,(\text{kip})$$



Dot product:

$$\mathbf{F}_1 \cdot \mathbf{F}_2 = (-4.2)\cdot 2.5 + 2.8\cdot 5.8 + 5.4\cdot(-7.1)$$

$$= -32.6\,(\text{kip}^2)$$

Angle θ:

$$\theta = \cos^{-1}\left(\frac{\mathbf{F}_1 \cdot \mathbf{F}_2}{F_1 F_2}\right) = \cos^{-1}\left(\frac{-32.6}{7.4\cdot 9.5}\right) = 118°$$

Projection:

$$\left|F_{1\,\text{on}\,2}\right| = \left|F_1 \cdot \cos\theta\right| = \left|7.4\cdot\cos 118°\right| = 3.4\,(\text{kip})$$

Magnitude: $F_1 = 7.4\,(\text{kip}) \quad F_2 = 9.5\,(\text{kip})$

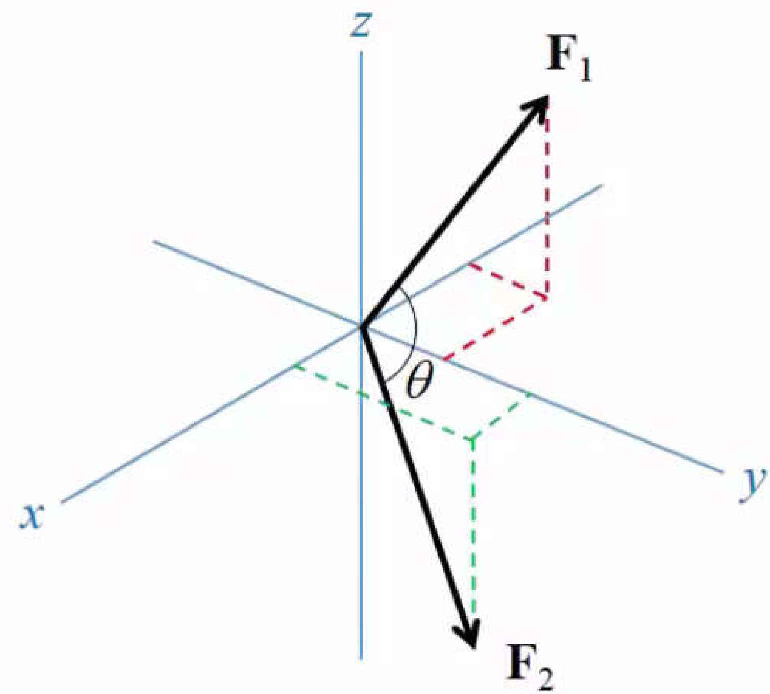
Alternatively:

Unit vectors:

$$\mathbf{u}_{F_1} = \frac{\mathbf{F}_1}{F_1} = -0.568\mathbf{i} + 0.378\mathbf{j} + 0.730\mathbf{k}$$

$$\mathbf{u}_{F_2} = \frac{\mathbf{F}_2}{F_2} = 0.263\mathbf{i} + 0.611\mathbf{j} - 0.747\mathbf{k}$$

Angle θ: $\theta = \cos^{-1}\left(\mathbf{u}_{F_1} \cdot \mathbf{u}_{F_2}\right) = 118°$



Projection:

$$\left|F_{1\,\text{on}\,2}\right| = \left|\mathbf{F}_1 \cdot \mathbf{u}_{F_2}\right| = 3.4\ \text{kip}$$ Ans.