Particle Equilibrium

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Objectives :

- To apply Newton's first law to solve 2D and 3D particle equilibrium problems.

Recall:

Newton's second law: $\mathbf{F} = m\mathbf{a}$

Newton's first law:

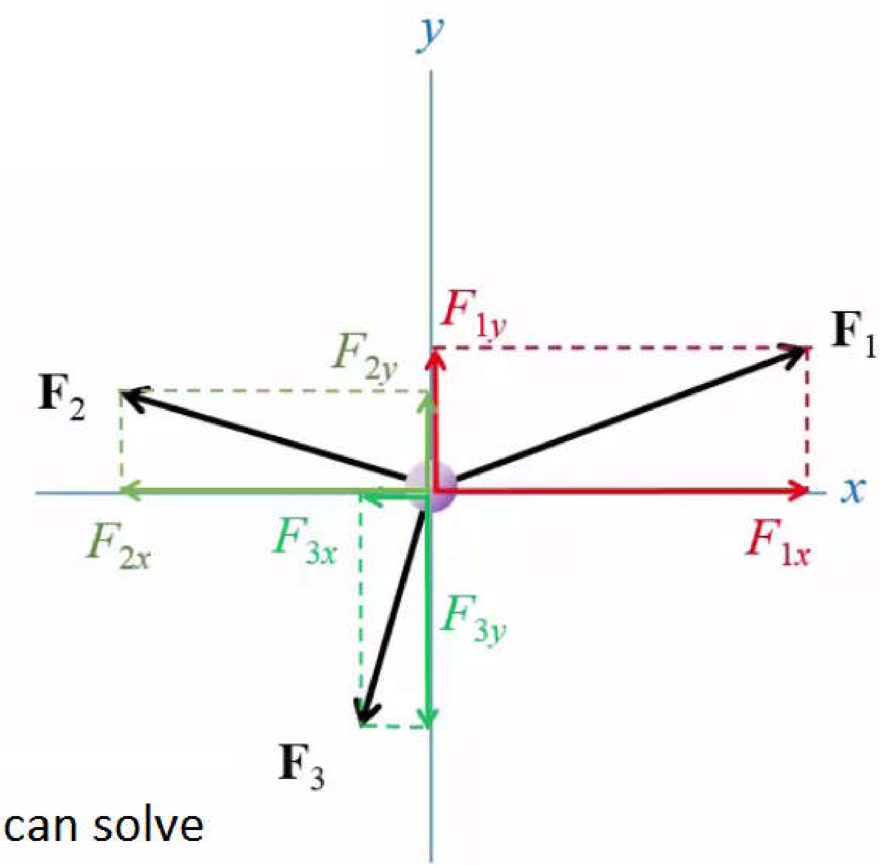
$$\mathbf{F}_R = \mathbf{0} \longrightarrow \mathbf{a} = \frac{\mathbf{F}_R}{m} = \mathbf{0}$$

The condition for **particle equilibrium** is simply described by Newton's first law, that is the resultant force must be **Zero**.

$$\mathbf{F}_R = \sum \mathbf{F} = \mathbf{0}$$

2-D Particle equilibrium

$$\begin{cases} \sum F_x = 0 \\ \sum F_y = 0 \end{cases}$$



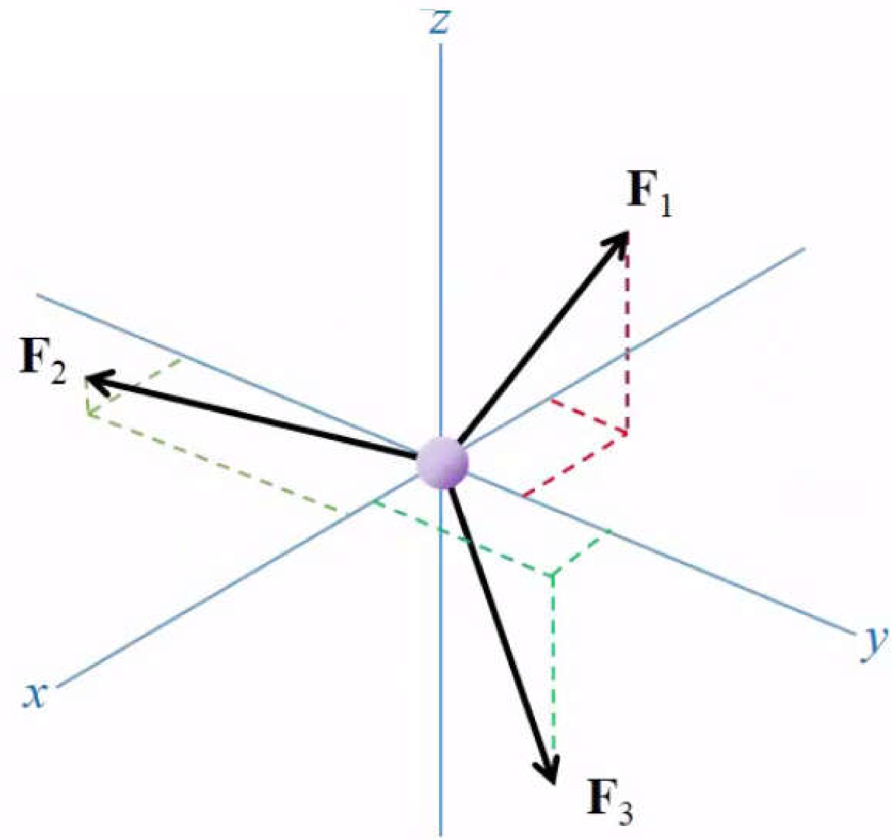
We know that with **two independent equations** we can solve for **a maximum of two unknowns.**

3-D Particle equilibrium

$$\mathbf{F}_R = \sum \mathbf{F} = \mathbf{0}$$

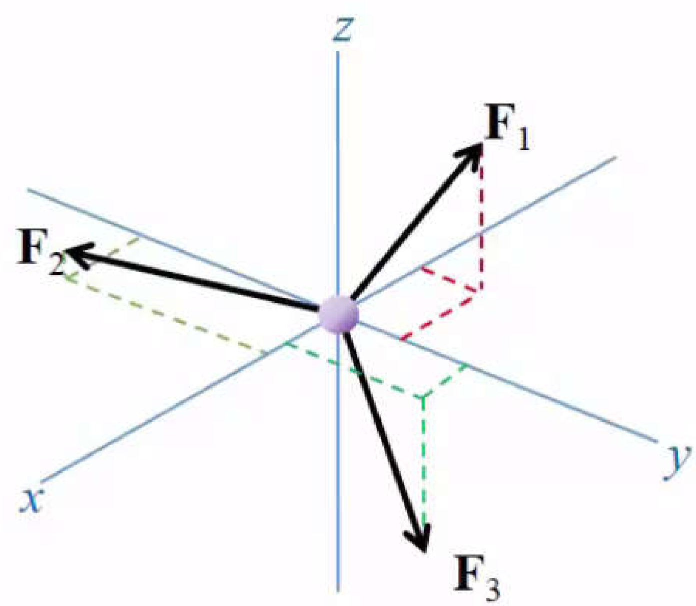
For 3D problem, since each force that acts on a particle can now be resolved into three components along x, y and z directions respectively, the same vector equation can now be rewritten as:

$$\begin{cases} \sum F_x = 0 \\ \sum F_y = 0 \\ \sum F_z = 0 \end{cases}$$

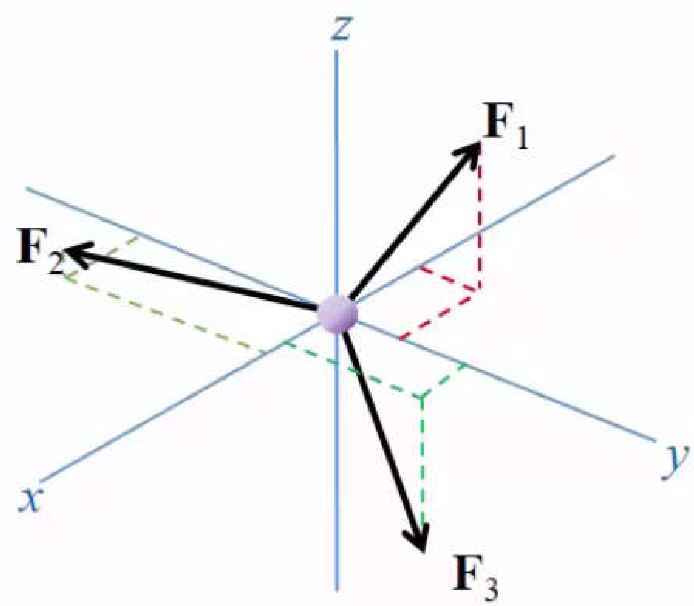


Enabling us to solve for a maximum of 3 unknowns

Example: If the particle is subjected to the three forces and is in equilibrium, $\mathbf{F}_1 = \{-40\mathbf{i} + 30\mathbf{j} + 45\mathbf{k}\}\ N$ and $\mathbf{F}_2 = \{35\mathbf{i} - 65\mathbf{j} + 10\mathbf{k}\}\ N$, determine $\mathbf{F}_3$.



$$\begin{cases} \sum F_x = 0 \\ \sum F_y = 0 \\ \sum F_z = 0 \end{cases}$$



$$\begin{cases} \sum F_x = -40 + 35 + F_{3x} = 0 \\ \sum F_y = 30 - 65 + F_{3y} = 0 \\ \sum F_z = 45 + 10 + F_{3z} = 0 \end{cases} \quad \therefore \quad \begin{cases} F_{3x} = 5\ N \\ F_{3y} = 35\ N \\ F_{3z} = -55\ N \end{cases} \quad \therefore \quad \mathbf{F}_3 = \{5\mathbf{i} + 35\mathbf{j} - 55\mathbf{k}\}\ N$$

Moment of a Force

Objectives :

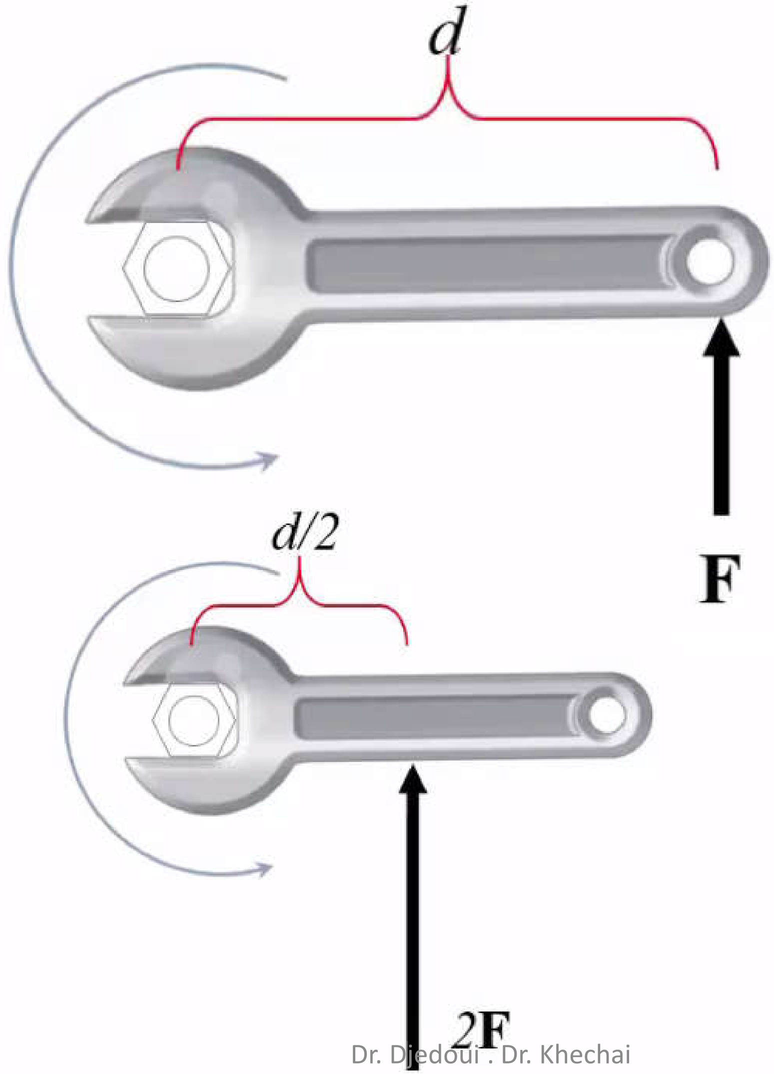
- To define the moment of a force.
- To visually represent the moment of a force in 2D and 3D views.

We know that forces can cause not only **translational motion** but also Rotational Motion.

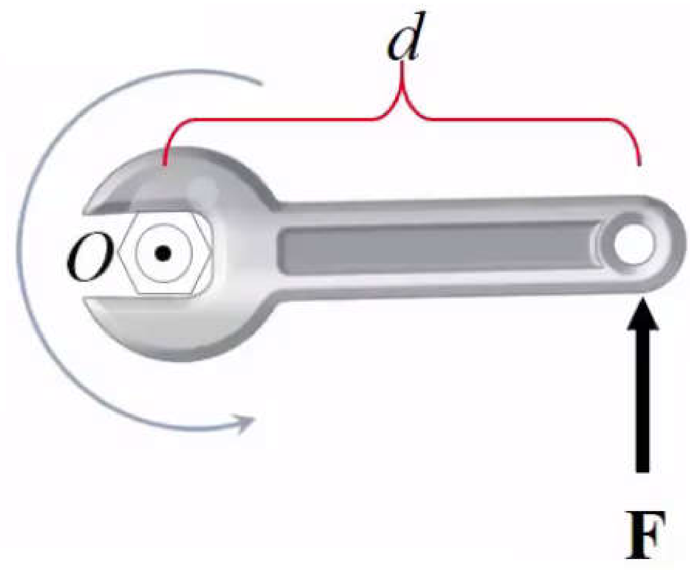
When we apply a force on a handle, we can cause the screw **to rotate**.

We also know almost intuitively to apply the force at **the edge**, creating **a maximum distance** from the screw. Why is that ?

If we shorten the distance by **half**, here we need to **double** that force to get the **same rotational effect** on the screw.



Moment is a physical quantity that describes the *rotational* effect (or rotational tendency) about an *axis* produced by a *force*.

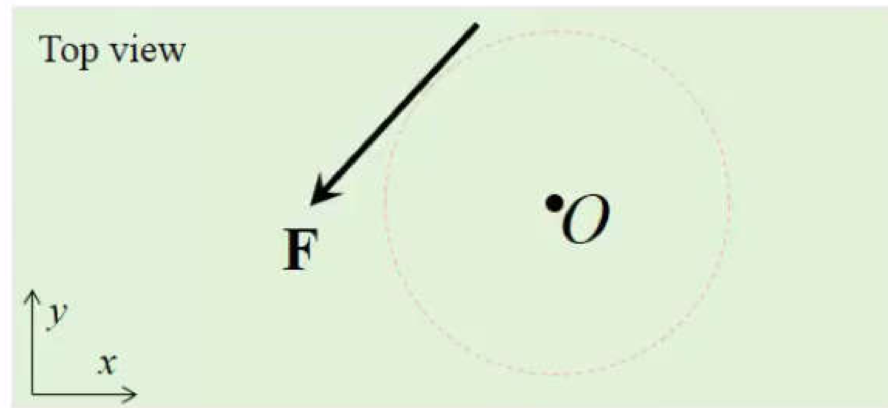


In this example, the axis is **perpendicular** to the plane.

Sometimes a moment is also called a **Torque**.

Just like force, moment is a ***vector***. It follows all vector calculation rules.

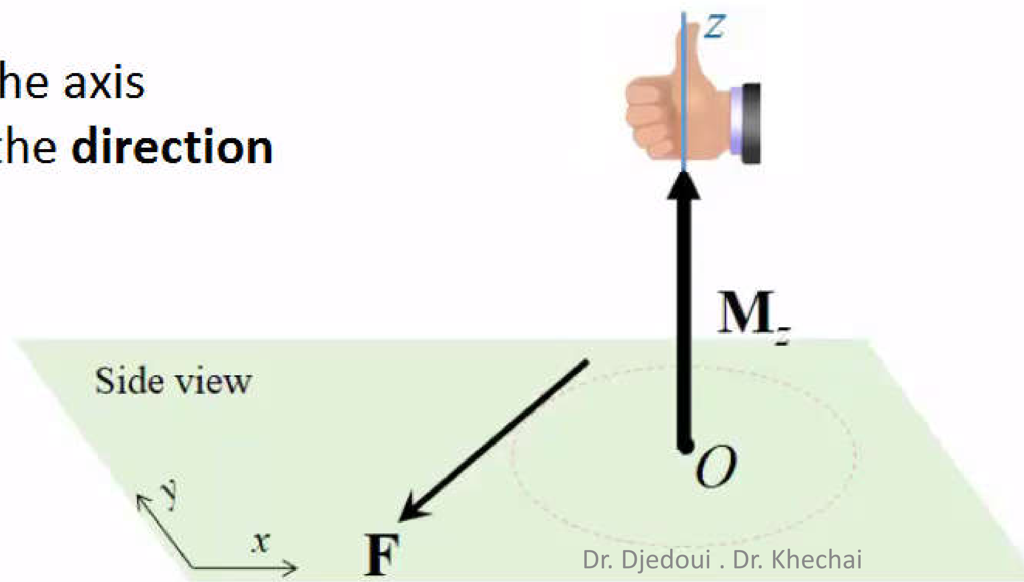
We want to find the **moment** caused by the forece **F** about an axis *z* which is perpendicular to the *xy* plane.



This axis intercepts the plane at point *O*.

The **rotational effect** caused by the force can be determined by the **Right-hand** rule.

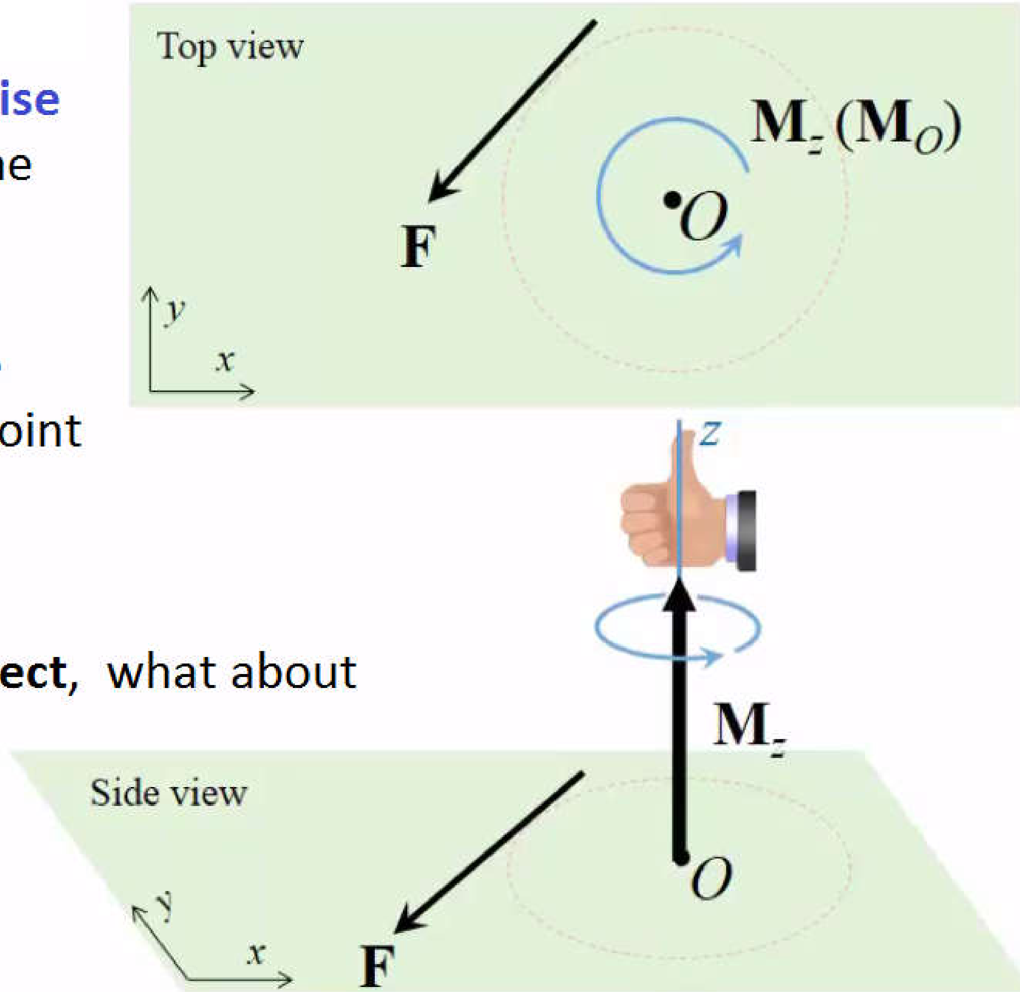
If you extend the four right-hand fingers from the axis towards the force, and then **roll** the fingers to the **direction of the force**, your **thumb** will point towards the direction of the **moment vector** noted by **Mz**.



The rotational effect is always **counterclockwise** about the **moment vector,** also agrees with the rolling of your four right-hand fingers.

For a 2D problem, the rotational effect can be considered to be within the plane about the point *O*. Therefore, the moment **Mz** can also be expressed as **Mo**.

We know the **direction** and the **rotational effect**, what about the magnitude?



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The **magnitude** is determined by the magnitude of the force as well as the **perpendicular distance** between the axis and the force *d* known as the **moment arm**.

In the scalar form:

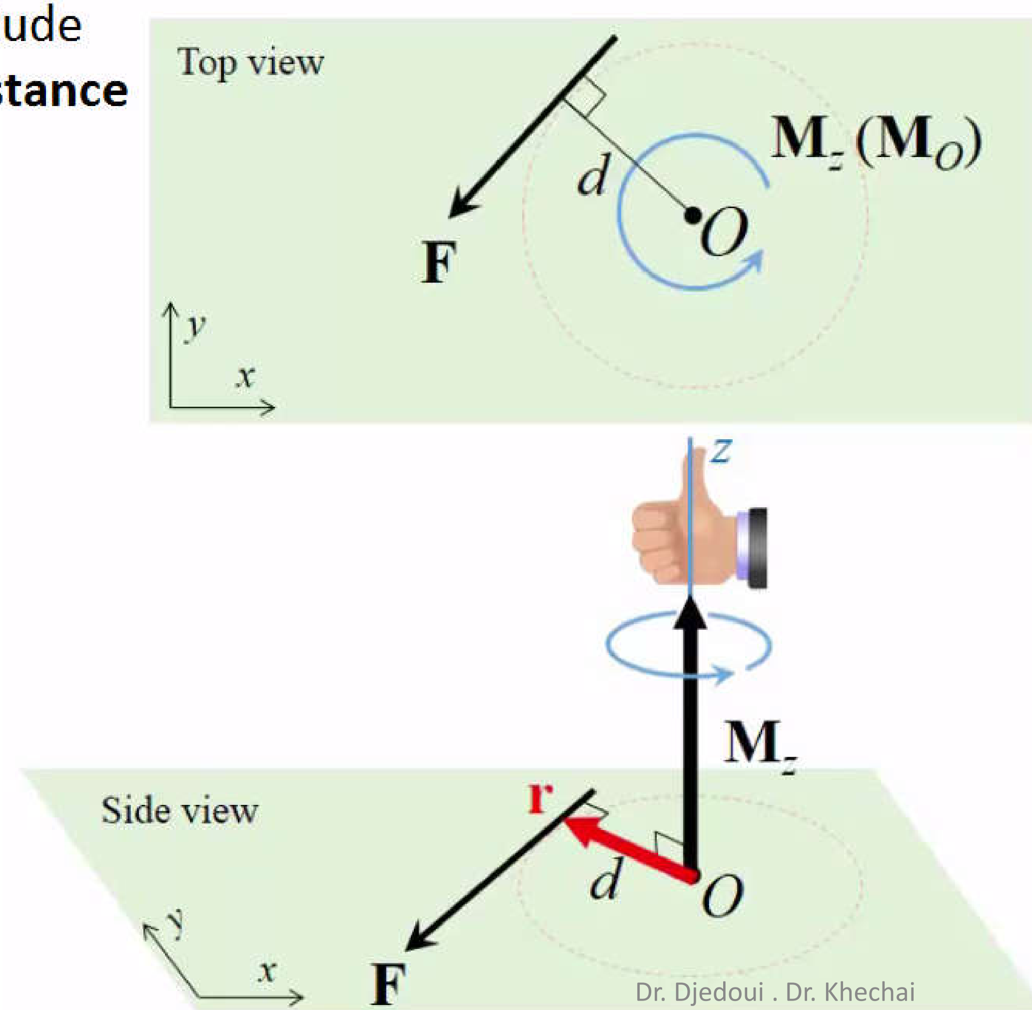
$$M_O = Fd$$

In the vector form:

$$\mathbf{M} = \mathbf{r} \times \mathbf{F}$$

(Cross vector product)

r is the position vector from point *O* to **F**.



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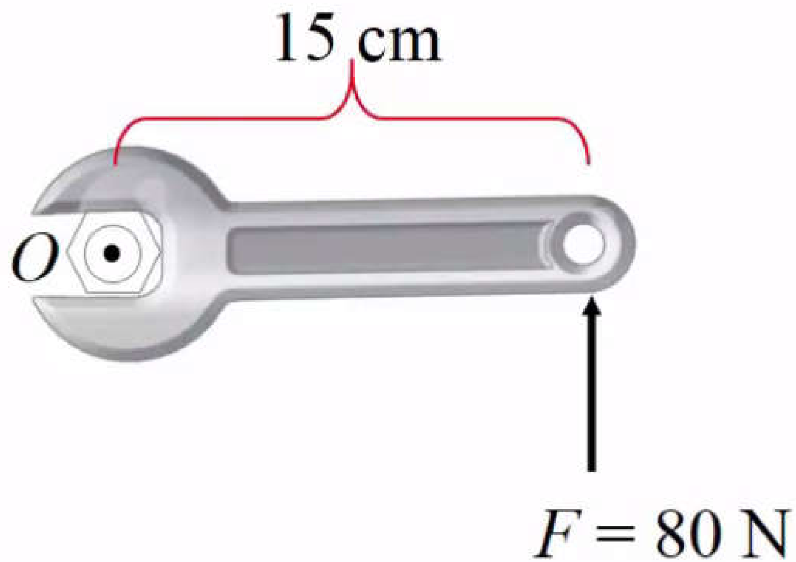
Question 3: Determine the moment about point O caused by force F.



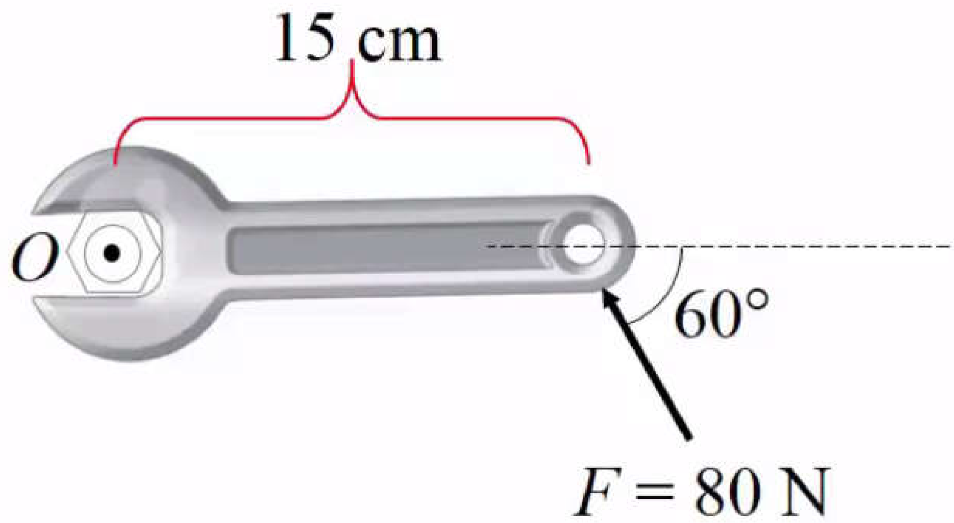
(a) $12\ \mathrm{Pa}$

(b) $12\ \mathrm{N \cdot cm}$

(c) $12\ \mathrm{N \cdot m}$

(d) $1200\ \mathrm{N \cdot m}$

Question 4: Determine the moment about point O caused by force F.



(a) $20.8\ \text{N}\cdot\text{m}$

(b) $10.4\ \text{N}\cdot\text{m}$

(c) $12\ \text{N}\cdot\text{m}$

(d) $6\ \text{N}\cdot\text{m}$

Moment Calculation
Scalar Formulation

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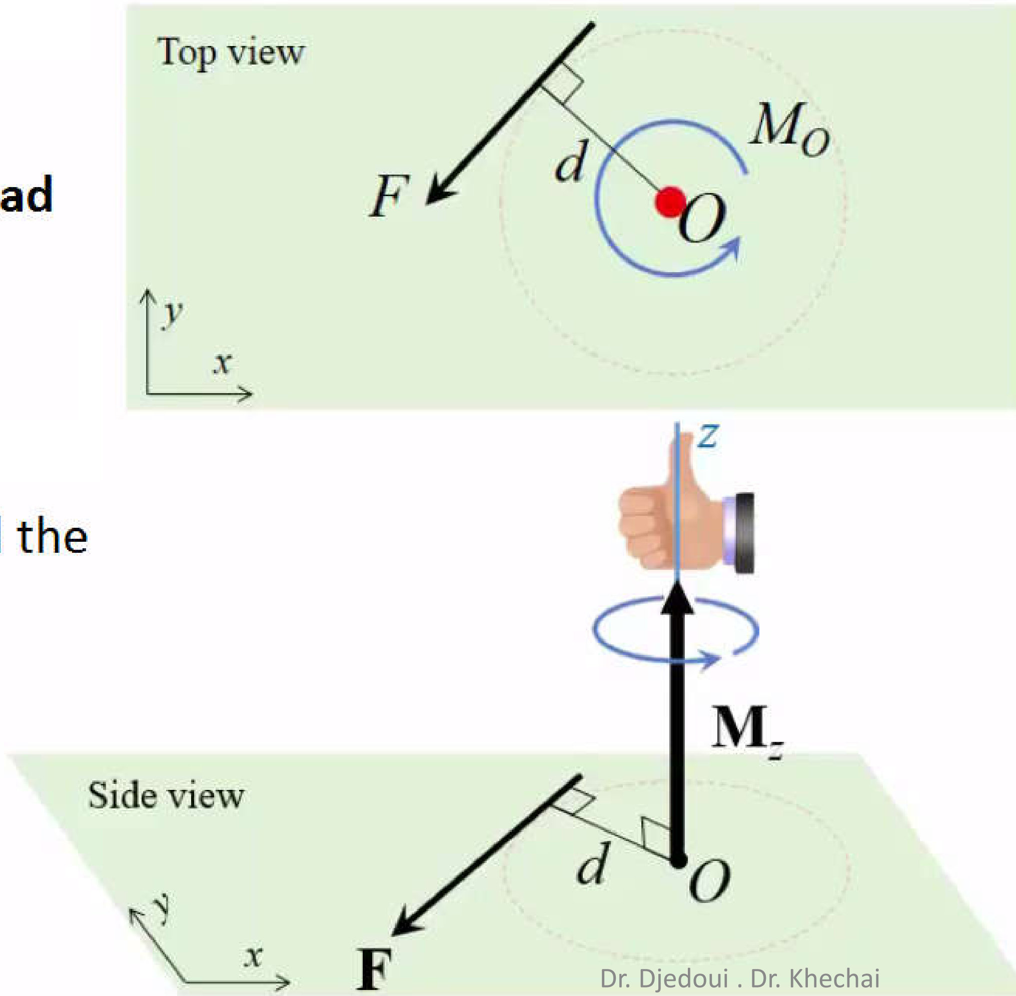
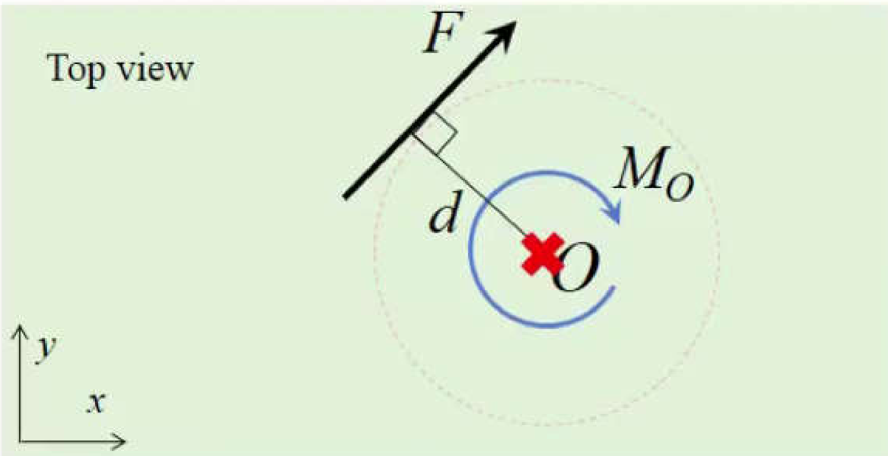
Objectives :

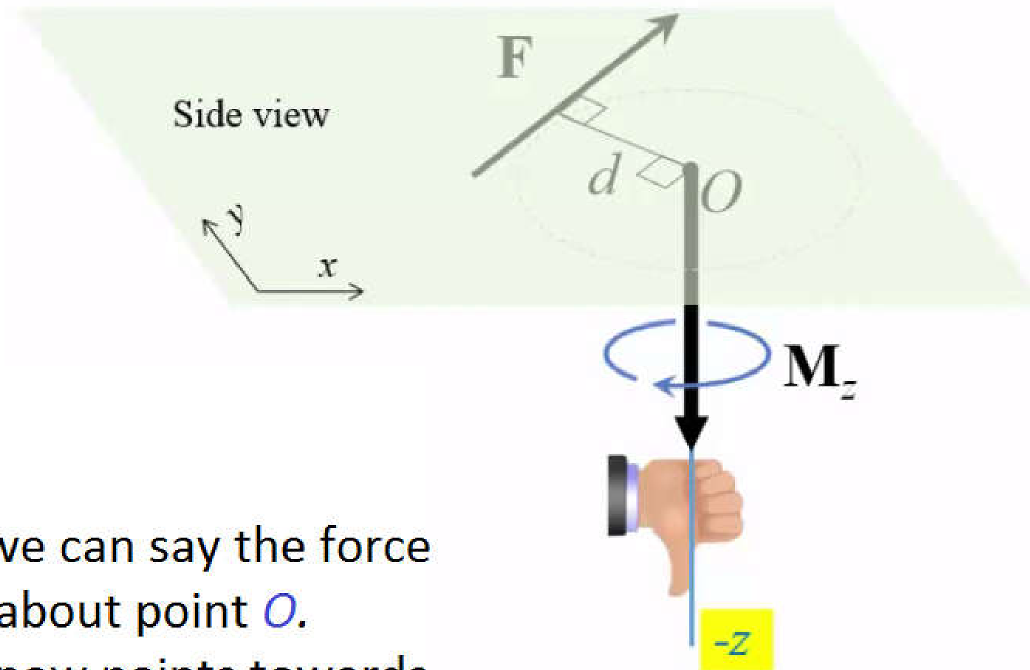
- To illustrate the moment of a force as viewed on a 2D plane.
- To calculate the moment of a force about a point in scalar formulation.

In a **2D** plane the moment vector cannot be visualized but you can imagine it to be **the head** of an arrow **shooting out** of the plane represented by a **dot**.

The rotational effect is **counterclockwise** and the magnitude of the moment is **positive**.



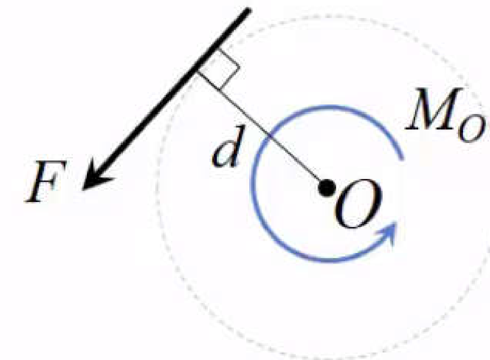




If we reverse the direction of the force, then we can say the force is now creating a **clockwise rotational effect** about point *O*. However, the moment that the force creates now points towards the -z direction still following the **right-hand rule**.

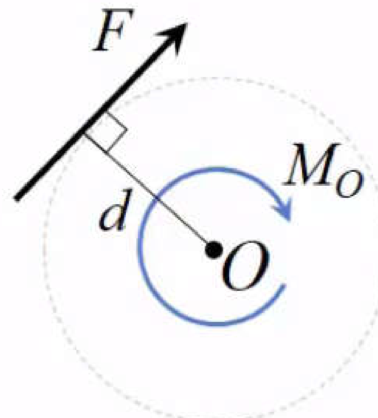
In a 2D plane, you should imagine the moment vector as an arrow **shooting into** the plane and you can only see **the tail** of the arrow.

When we calculate **the moment** caused by a force F about a point *O* in a 2D plane, if the force creates **a counterclockwise rotational effect** about *O* the moment Mo:



$$M_O = Fd$$

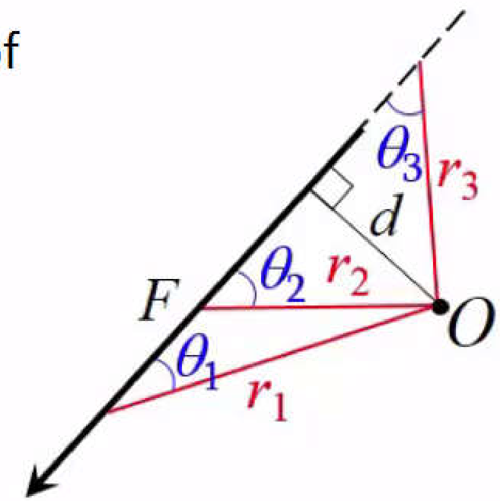
If the force creates **a clockwise effect** about point *O*, the moment Mo equals to:



$$M_O = -Fd$$

d is the moment arm.

We can just draw a line from point O to anywhere on the line of action of the force F, r_1 or r_2 or r_3 and determine the angle between each of these three lines and the force.

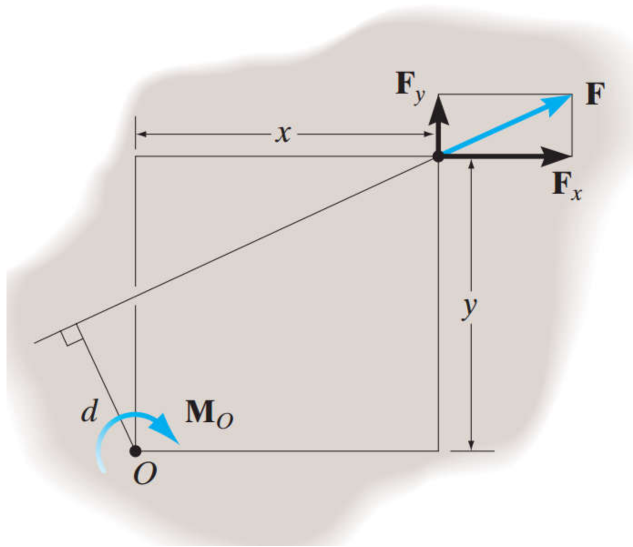


The moment can be determined to be:

$$M_O = Fd = F \cdot r_1 \cdot \sin\theta_1 = F \cdot r_2 \cdot \sin\theta_2 = F \cdot r_3 \cdot \sin\theta_3$$

Varignon's Theorem

- One of the most useful principles of mechanics is *Varignon's theorem*, which states that the moment of a force about any point is equal to the sum of the moments of the components of the force about the same point.

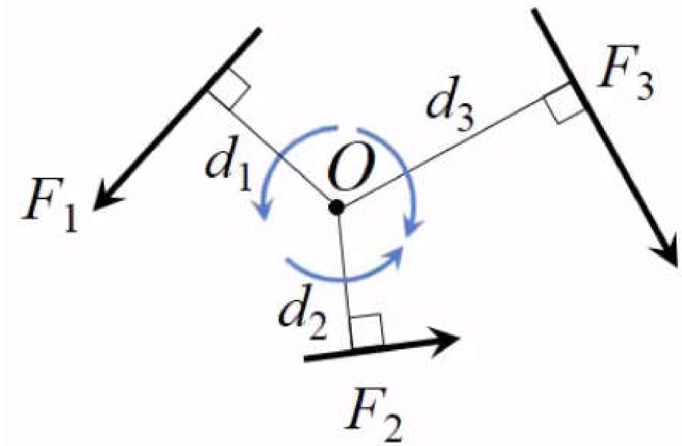


$$M_0 = Fd$$

Or

$$M_0 = F_x y + F_y x$$

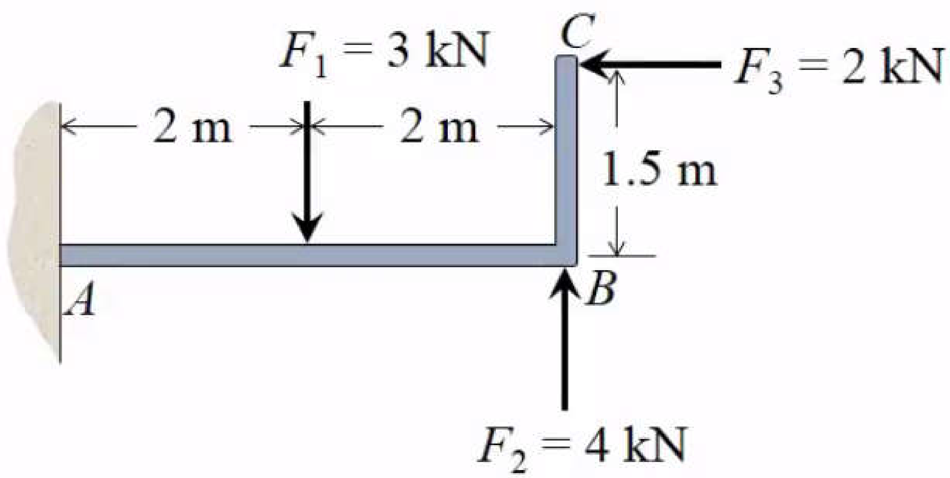
The **resultant moment** caused by **multiple forces** can be determined by simply **adding up** the individual moment caused by each force about the same point.



$$(M_R)_O = \sum Fd = F_1 d_1 + F_2 d_2 - F_3 d_3$$

F_3 : is creating **a clockwise rotational effect** about point *O*.

Question 1: Determine the total moment about **point** A caused by the three forces F_1, F_2 and F_3. Take counterclockwise as positive.



(a) $25\,\text{kN}\cdot\text{m}$ (b) $13\,\text{kN}\cdot\text{m}$

(c) $7\,\text{kN}\cdot\text{m}$ (d) $11\,\text{kN}\cdot\text{m}$

Moment Calculation

Vector Formulation

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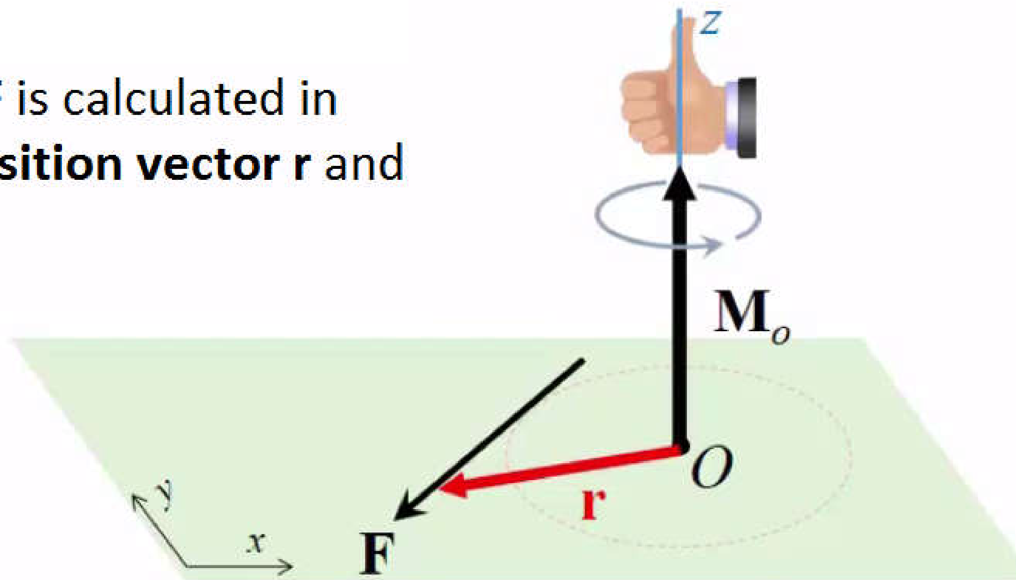
Objectives :

- To review the cross product of two vectors.
- To calculate the moment of a force about a point in vector formulation.

The moment about a point O caused by force **F** is calculated in **vector form** simply as the **Cross-Product** of **position vector r** and the force vector **F.**



$$\mathbf{M}_O = \mathbf{r} \times \mathbf{F}$$

Note that **r** could be any vector as long as it starts from point *O* and ends anywhere on **the line of action** of the force.

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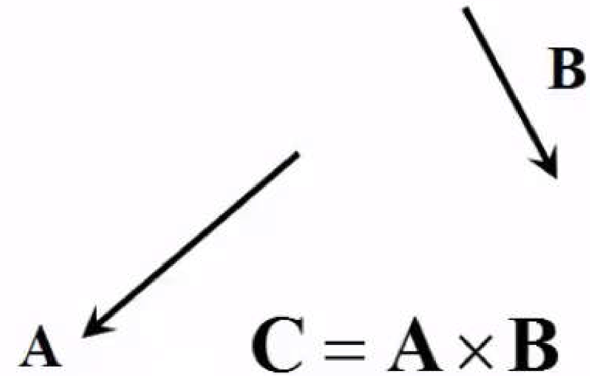
Cross product



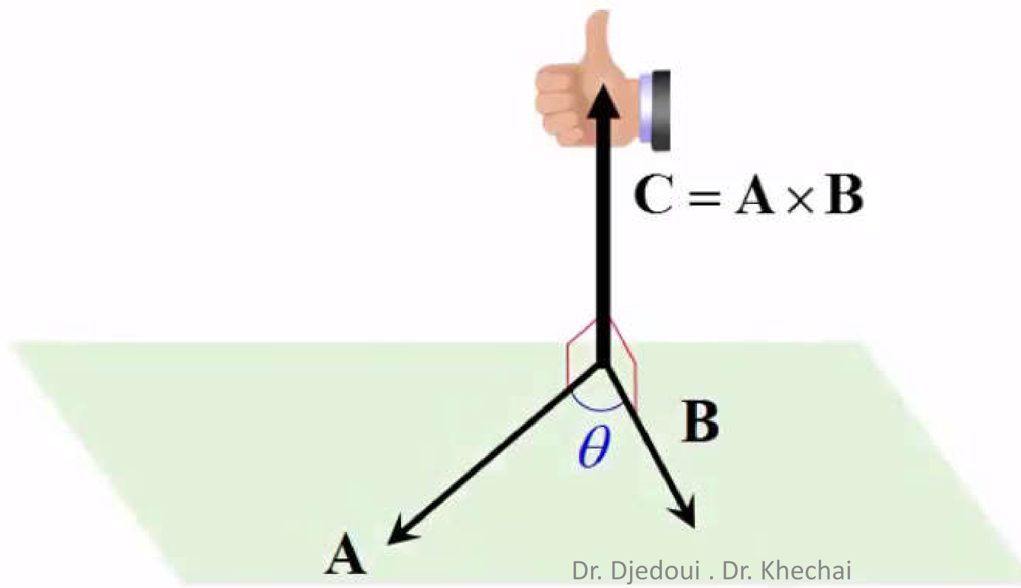
We **join the tails** of the two vectors together and then determine the **angle** between them.

The magnitude of vector **C** is determined as:

$$C = AB \sin \theta$$

The **direction** is determined by the **right-hand rule**. When you roll your four right-hand fingers from vector **A** towards vector **B**, your thumb points to vector **C**'s direction.

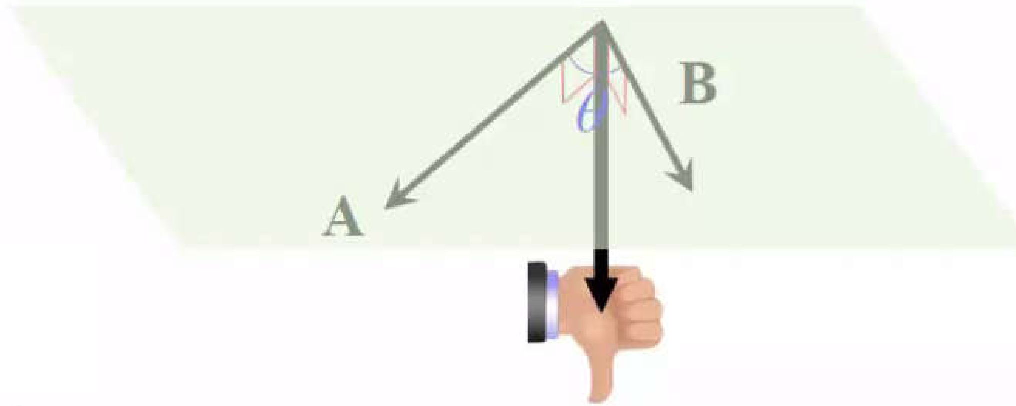
Vector **C** is perpendicular to the plane made by **A** and **B**.



Cross product

A cross **B** is not the same as **B** cross **A**.

$$\mathbf{A} \times \mathbf{B} \neq \mathbf{B} \times \mathbf{A}$$



B cross **A** represents another vector **C'** that is in the opposite direction as vector **C**.

$$\mathbf{C}' = \mathbf{B} \times \mathbf{A} = -\mathbf{C}$$

Cross product

If the vectors **A** and **B** are given in the **Cartesian forms,** then we can use **a matrix** to determine the **Cartesian form** of the cross product of **A** and **B**.

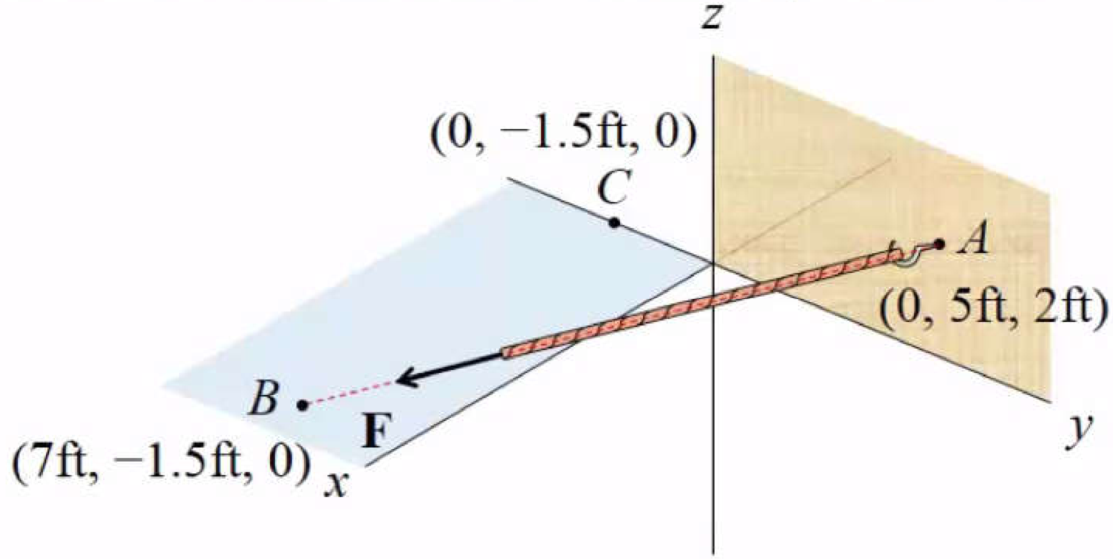
$$\mathbf{A} = A_x\mathbf{i} + A_y\mathbf{j} + A_z\mathbf{k}$$

$$\mathbf{B} = B_x\mathbf{i} + B_y\mathbf{j} + B_z\mathbf{k}$$

$$\mathbf{A} \times \mathbf{B} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

$$= \left(A_yB_z - A_zB_y\right)\mathbf{i} - \left(A_xB_z - A_zB_x\right)\mathbf{j} + \left(A_xB_y - A_yB_x\right)\mathbf{k}$$

Example: The line of action of force **F** directs from point A to point B. If the magnitude of the force is 120 lb, determine the moment of **F** about point C in Cartesian vector form.



Force vector:

$$\mathbf{F} = \{86.1\mathbf{i} - 79.9\mathbf{j} - 24.6\mathbf{k}\}\ \text{lb}$$

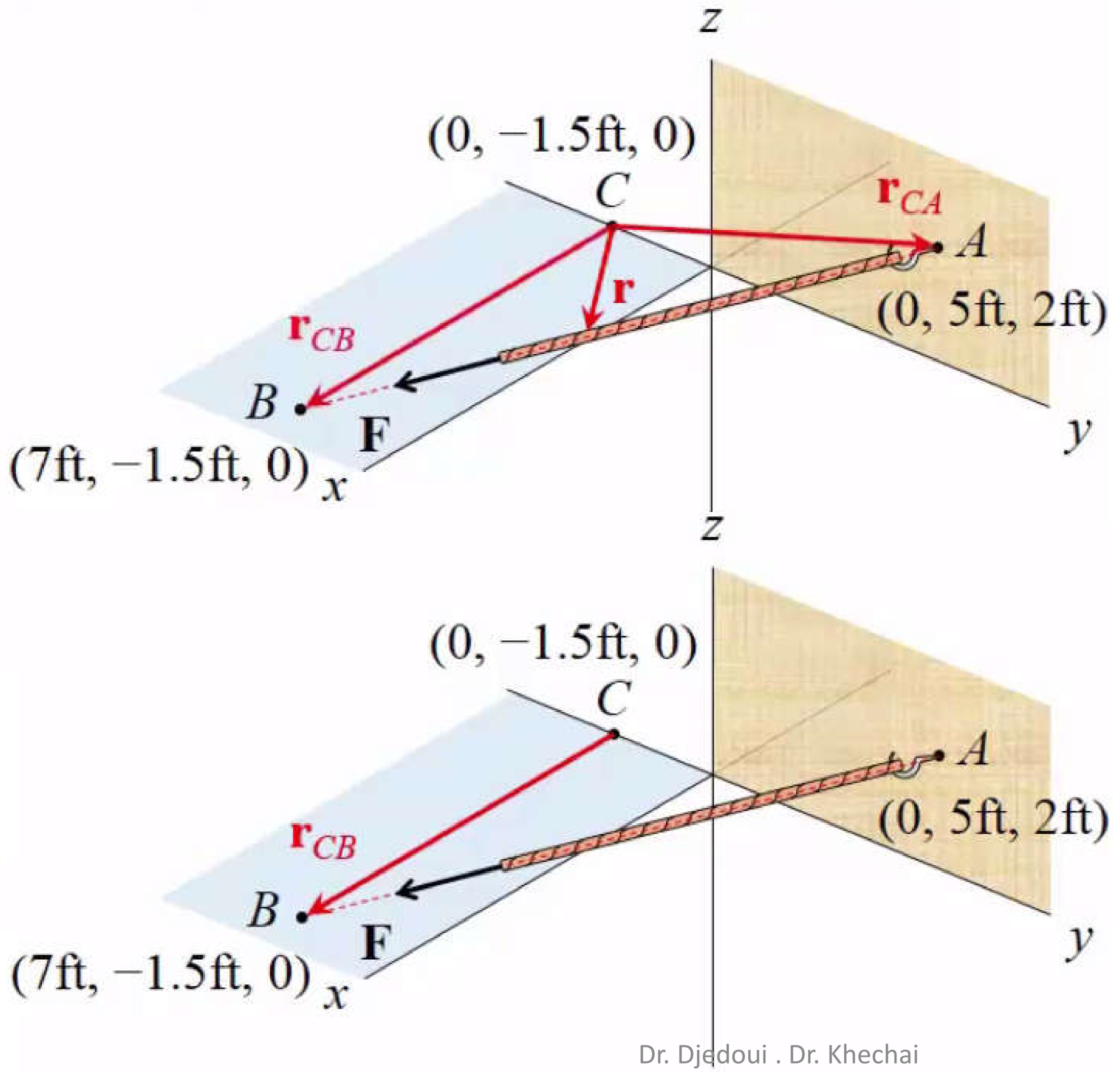
Position vector: $\mathbf{r}_{CB} = 7\mathbf{i}$ ft

Moment vector:

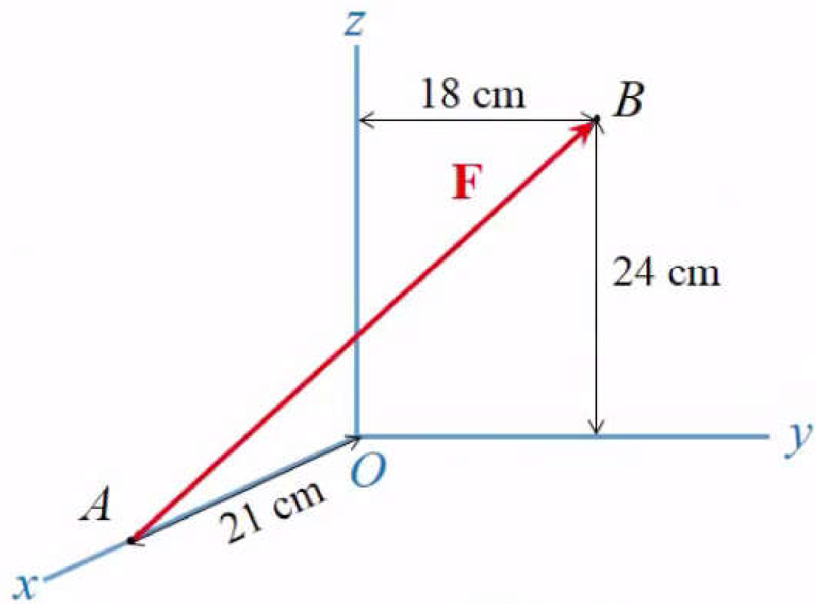
$$\mathbf{M} = \mathbf{r}_{CB} \times \mathbf{F}$$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 7 & 0 & 0 \\ 86.1 & -79.9 & -24.6 \end{vmatrix}$$

$$= \{172\mathbf{j} - 559\mathbf{k}\}\ \text{lb}\cdot\text{ft}$$ Ans.



Question 2: If force **F** has magnitude of 450 N and is directed from point *A* to *B* as shown, determine the moment (Cartesian vector form) caused by **F** about point *O*.



(a) $\{62.0\mathbf{j} - 46.4\mathbf{k}\}\,\text{N}\cdot\text{m}$ (b) $\{-258\mathbf{i} + 295\mathbf{k}\}\,\text{N}\cdot\text{m}$

(c) $\{-15.0\mathbf{i} + 12.9\mathbf{j} + 17.1\mathbf{k}\}\,\text{N}\cdot\text{m}$ (d) $\{-62.0\mathbf{j} + 46.4\mathbf{k}\}\,\text{N}\cdot\text{m}$

Principle of Moments

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Objectives :

To explain the application of the principle of moments to simplify moment calculation.

Used **vector formulation**, the moment caused by **F** about *O* is equal to:

$$\mathbf{M}_O = \mathbf{r} \times \mathbf{F}$$

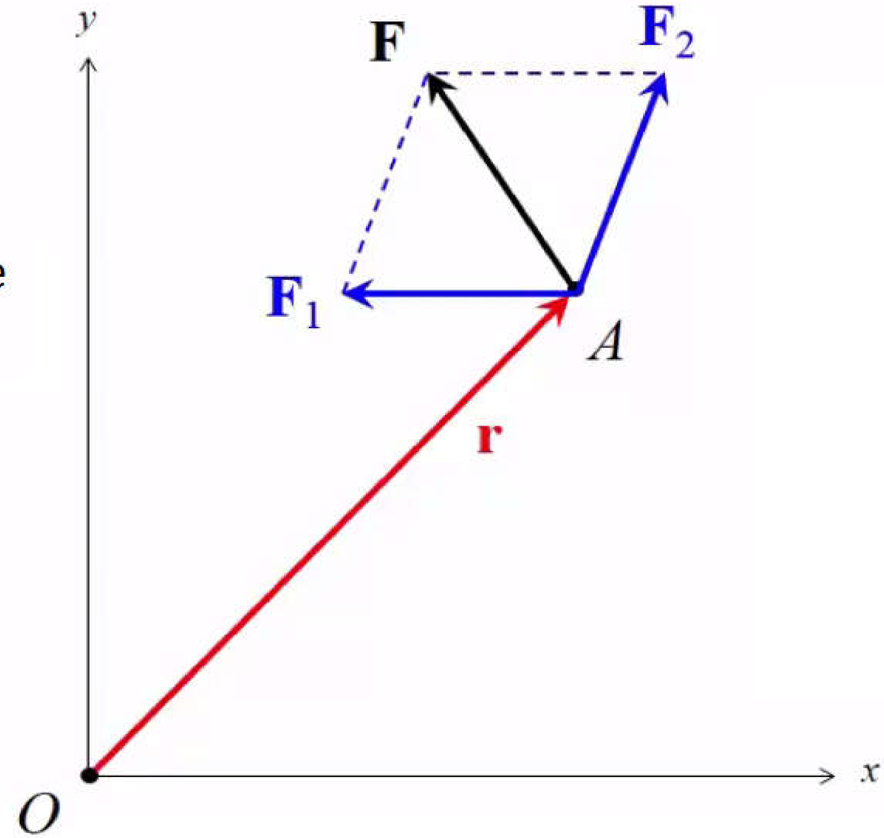
Since **F** is a vector, it can **be resolved** into two or more components following the **parallellogram law**.

$$\mathbf{M}_O = \mathbf{r} \times (\mathbf{F}_1 + \mathbf{F}_2)$$

Following the distributive law:

$$= \mathbf{r} \times \mathbf{F}_1 + \mathbf{r} \times \mathbf{F}_2$$

$$= \mathbf{M}_{O,1} + \mathbf{M}_{O,2}$$



The moment caused by a force can be calculated by summarizing the moments caused by its component forces about the same point and this is the Principle of Moments.

Why we care about the **principle of moments?**

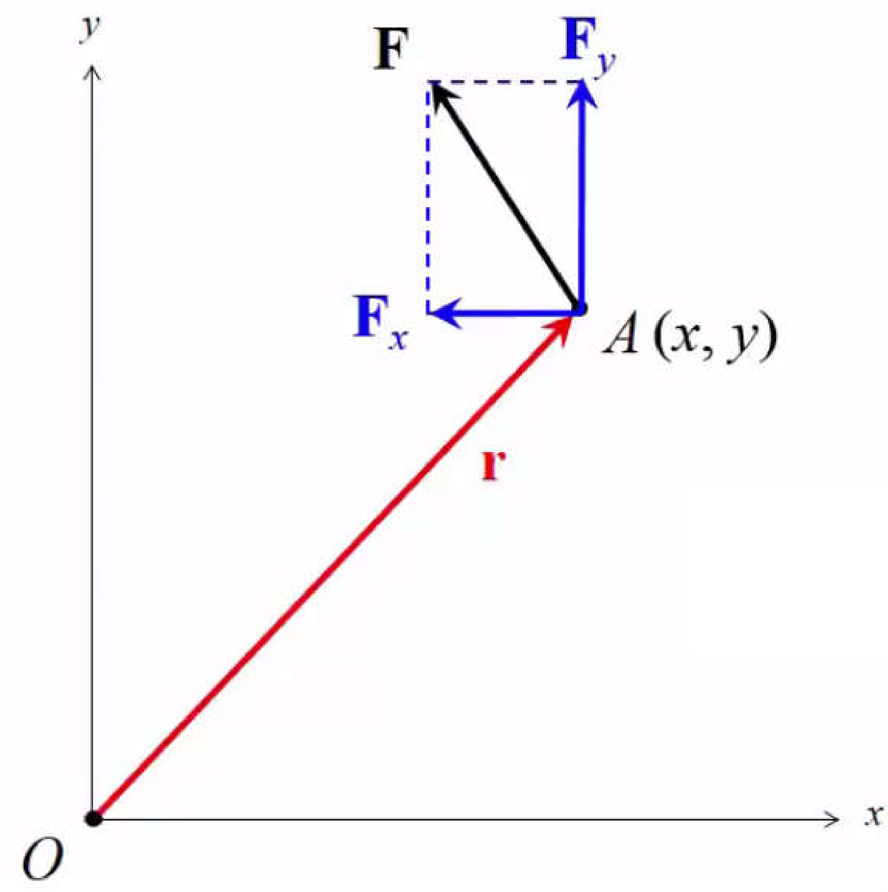
We want to use it to help us simplify the calculation of moment.

If we know the point of action of the force *A*(x,y), we would resolve the force vector into **Fx** and **Fy.**

The moment arms of these two component forces are *x* and *y*, respectively.

The calculation of **the magnitude** of the moment can be easily achieved to be:

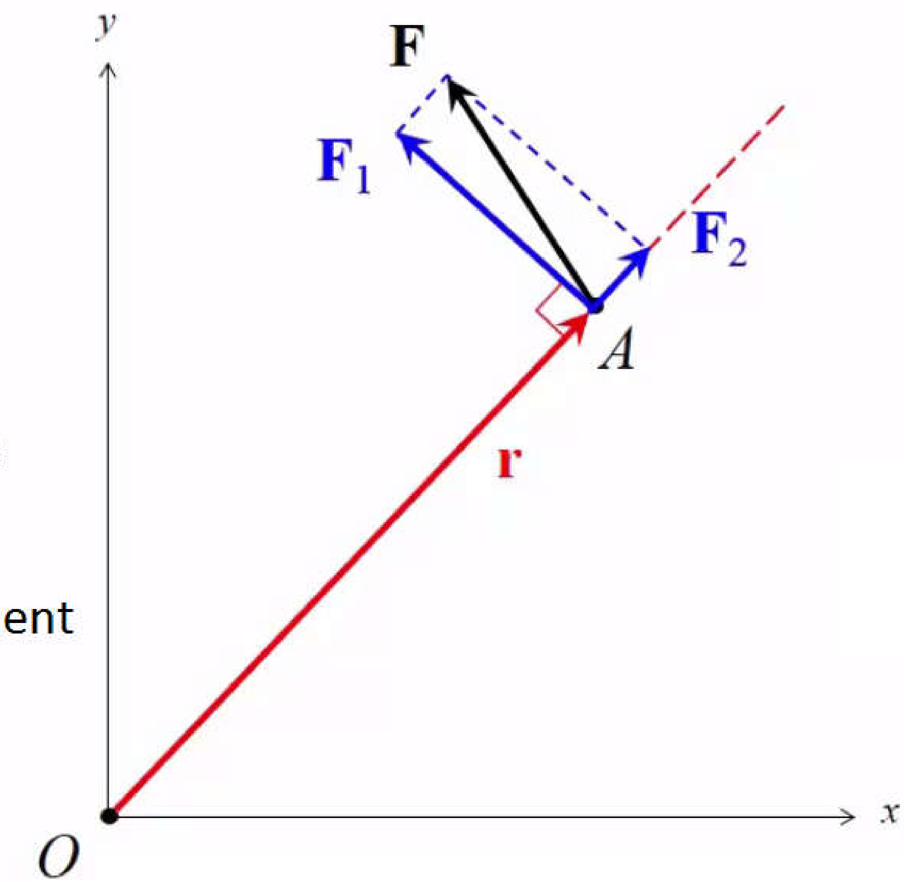
$$M_O = F_x \cdot y + F_y \cdot x$$



Sometimes, we can conveniently resolve the force into one component that is **along** the direction of **r** pointing from *O* towards *A*, and another component that is **perpendicular** to **r**.

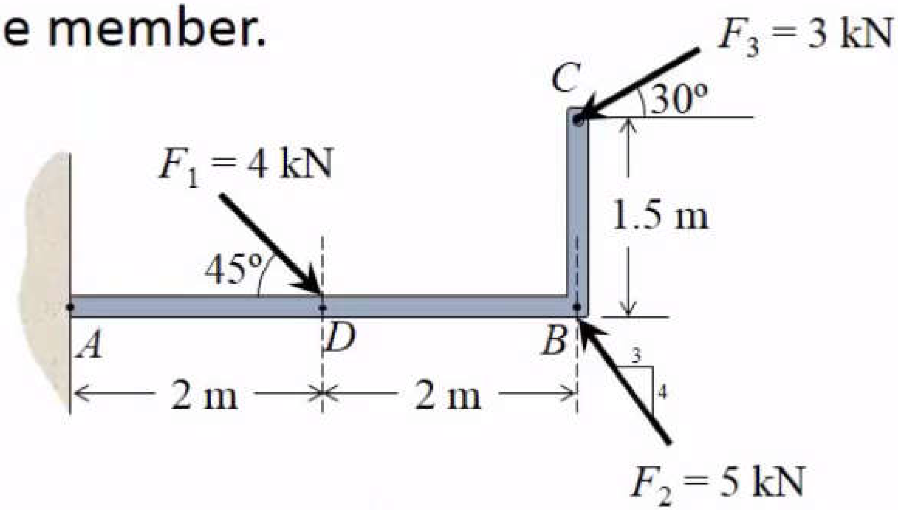
The advantage of this way is the **moment arm** of **F1** is the magnitude of **r** or in other words, the distance from *O* to *A*.

The moment arm of **F2** is simply **Zero**. (The component **F2 does not create** any moment about *O***.**)

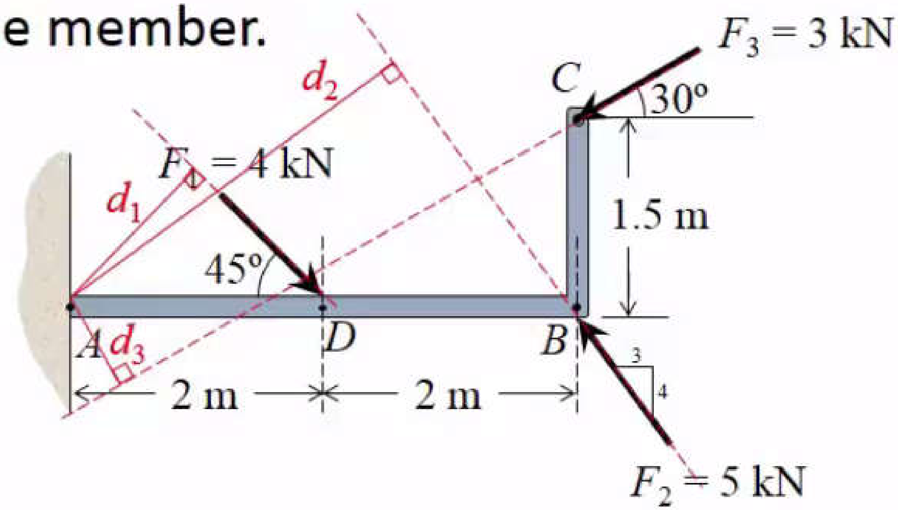


$$M_O = F_1 \cdot r$$

Example: Determine the total moment about **point *A*** caused by the three forces F_1, F_2 and F_3. Take counterclockwise as positive. Neglect the thickness of the member.

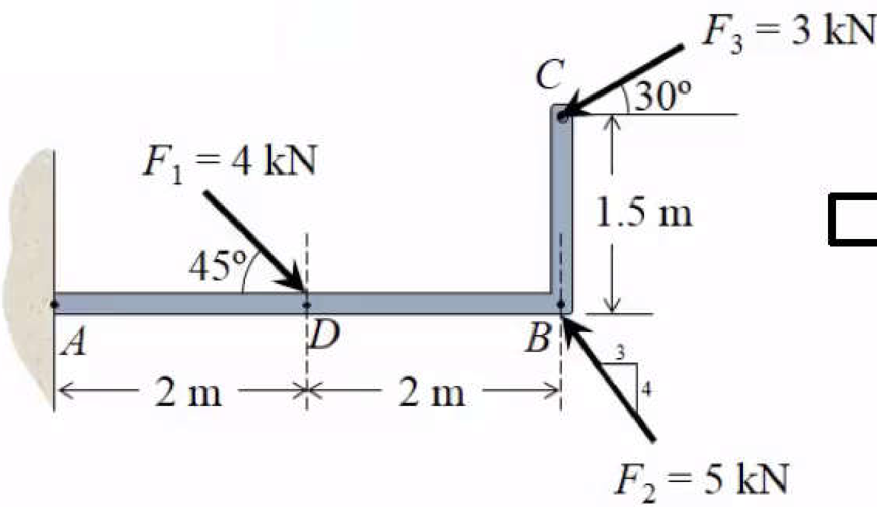


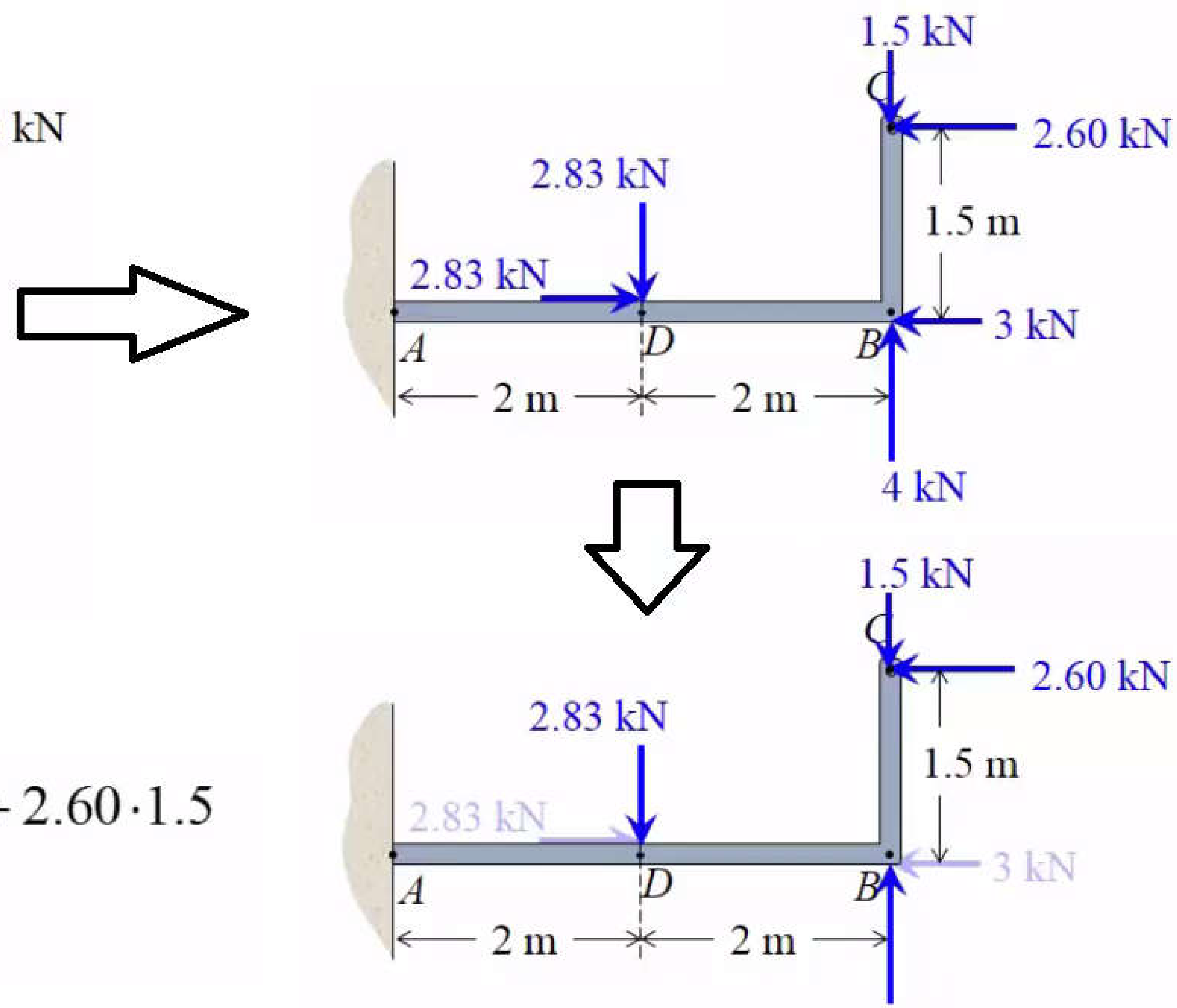
Example: Determine the total moment about **point *A*** caused by the three forces F_1, F_2 and F_3. Take counterclockwise as positive. Neglect the thickness of the member.



$$M_A = -F_1 \cdot d_1 + F_2 \cdot d_2 - F_3 \cdot d_3$$

Principle of Moments

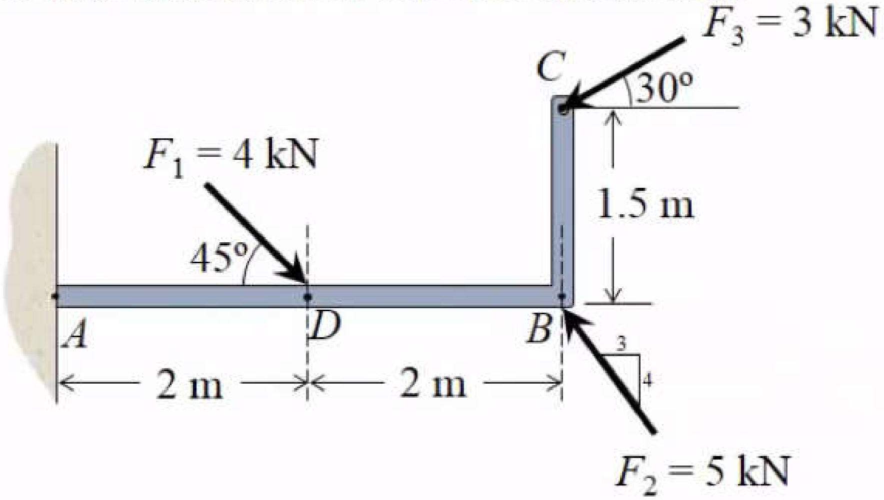




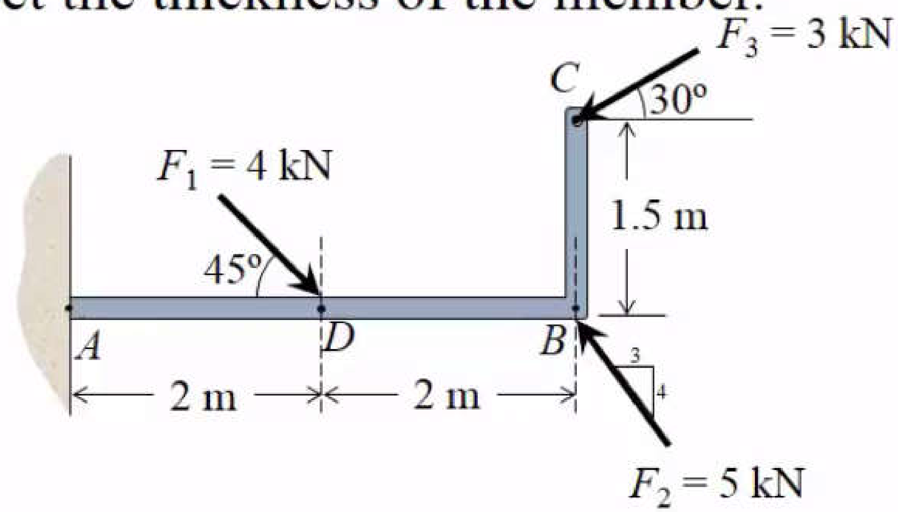
$$M_A = -2.83 \cdot 2 + 4 \cdot 4 - 1.5 \cdot 4 + 2.60 \cdot 1.5$$

$$= 8.24 \, (\text{kN} \cdot \text{m})$$ Ans.

Question 1: Determine the total moment about **point *B*** caused by the three forces F_1, F_2 and F_3. Take counterclockwise as positive. Neglect the thickness of the member.



Question 2: Determine the total moment about **point *C*** caused by the three forces F_1, F_2 and F_3. Take counterclockwise as positive. Neglect the thickness of the member.

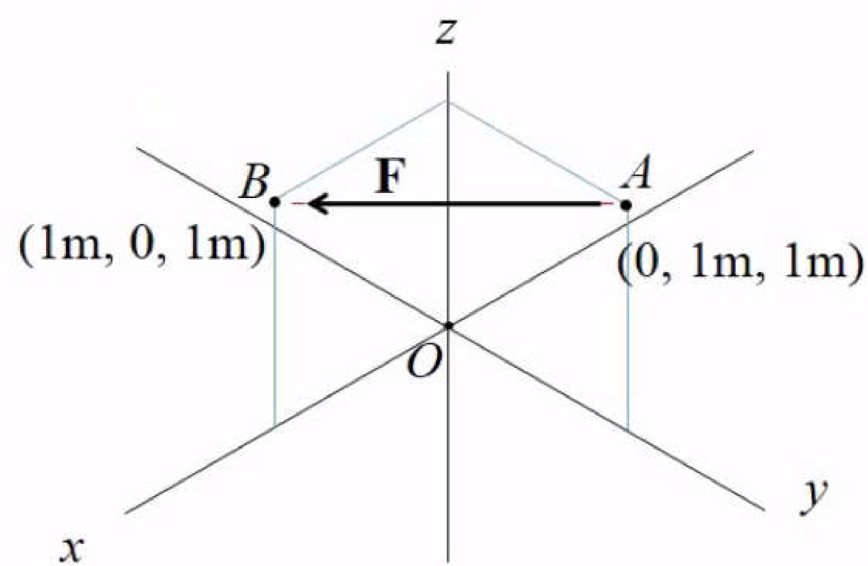


Moment Calculation about a Specified Axis

Objectives :

- To determine the moment caused by a force about a specified axis.
- To compare the moment about a specified axis to the projection of a force.

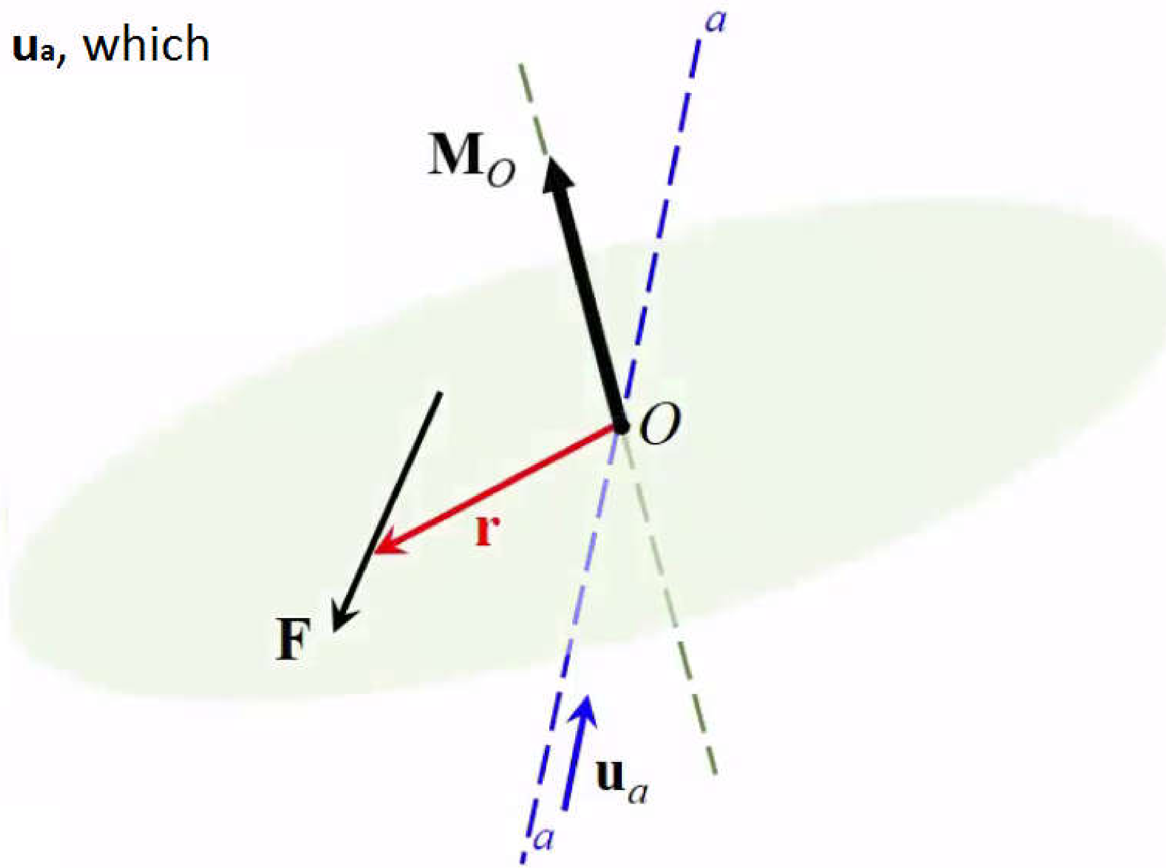
Question 1: If the 100-N force **F** is directed from point *A* to point *B* as shown, what are the moments it causes about the *x*, *y* and *z* axes respectively? Try work it out yourself first. But if you need a hint: what's the Cartesian vector moment of **F** about the origin point *O* and what do the three components (including +/− sign) physically mean?



In a specified axis (aa), the unit vector is **u**a, which specifies the direction of the axis.

To determine **a moment** caused by force **F** about this particular axis, we can draw **an arbitrary position vector r**, as long as **it starts** from **an arbitrary point** *O* an the axis and ends **anywhere** on the **line of action** of force **F**.

$$\mathbf{M}_O = \mathbf{r} \times \mathbf{F}$$



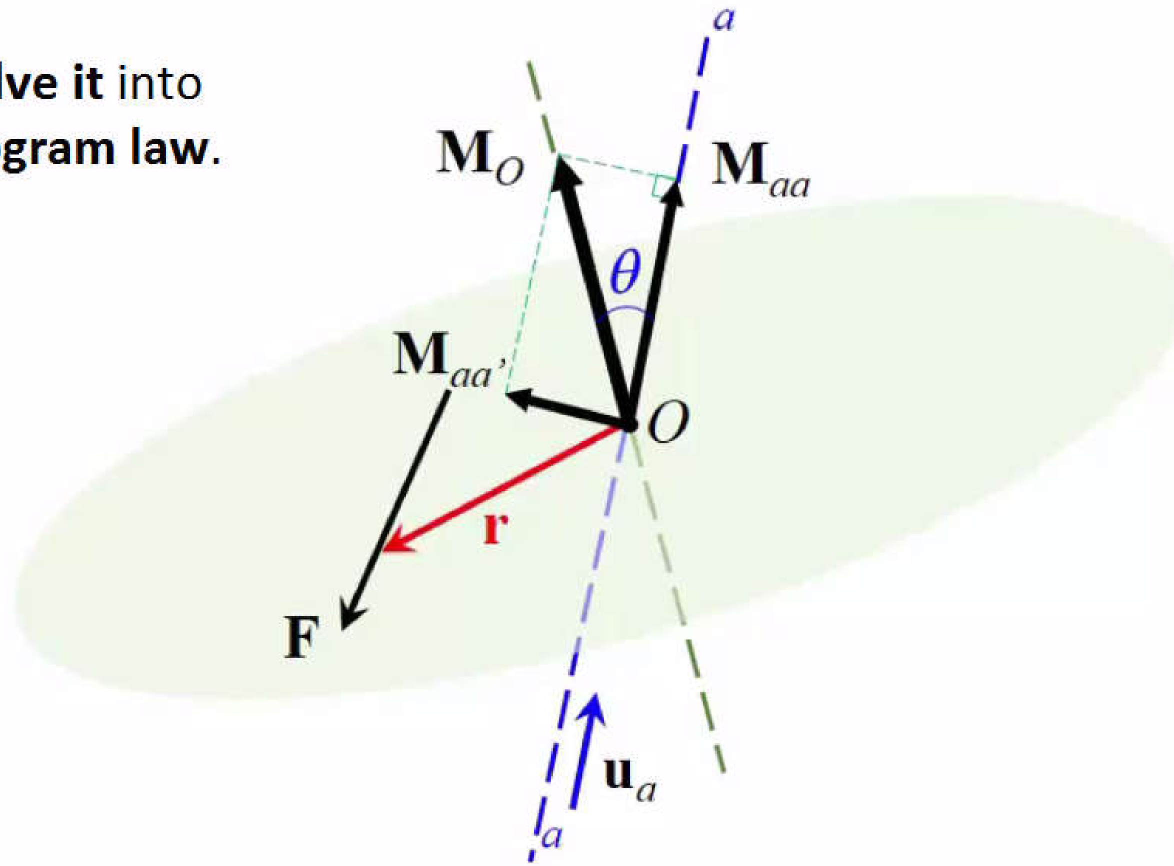
The direction of this moment is not necessarily along the (aa) axis as we want it

Since moment is a vector, we can **resolve it** into components according to the **parallelogram law**.

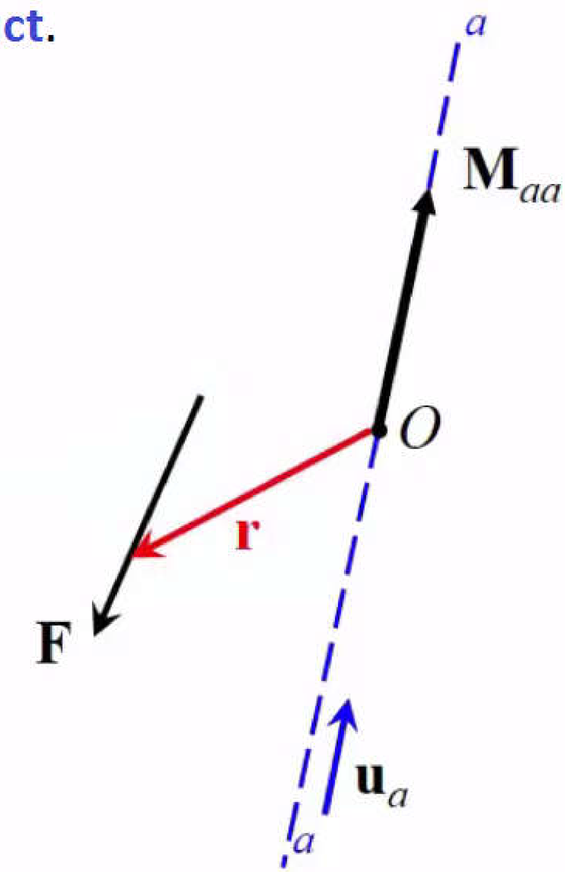
$\mathbf{M}_{aa}$: is along the axis.

$\mathbf{M}_{aa}$: is the projection of the moment along the axis.

$$M_{aa} = M_o \cdot \cos\theta$$



Just like finding **projection** of a force, we can also use **dot product**.



$$M_{aa} = \mathbf{u}_a \cdot \mathbf{M}_O$$

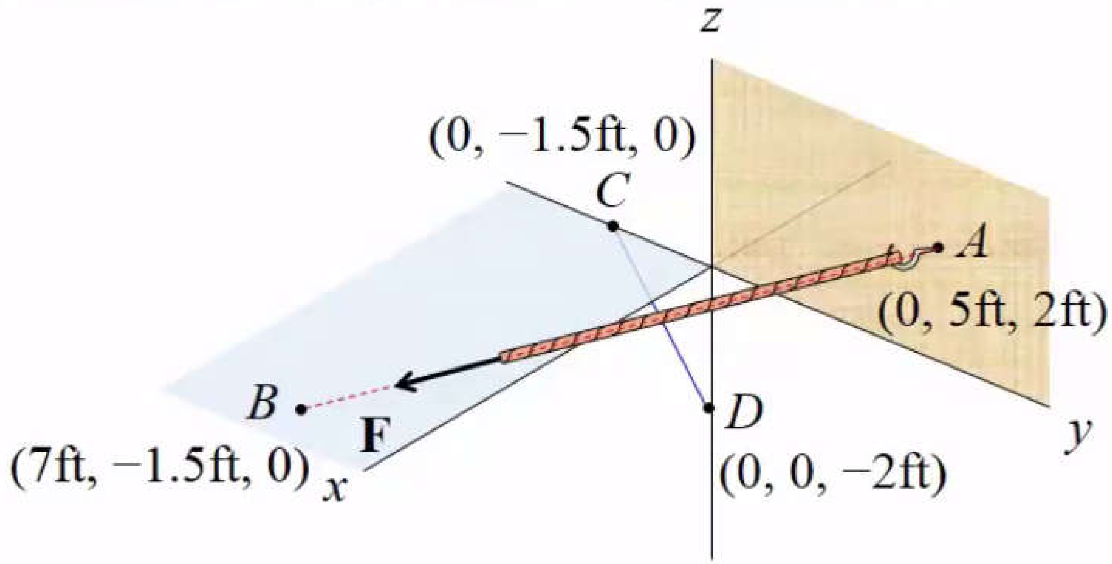
Or more directly:

$$M_{aa} = \mathbf{u}_a \cdot (\mathbf{r} \times \mathbf{F}) = \begin{vmatrix} u_{a_x} & u_{a_y} & u_{a_z} \\ r_x & r_y & r_z \\ F_x & F_y & F_z \end{vmatrix}$$

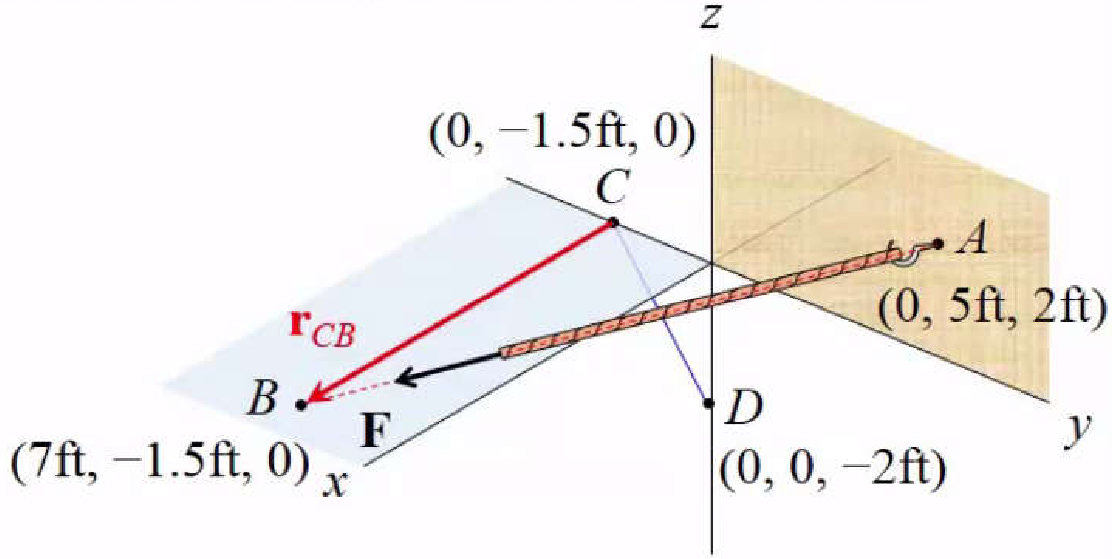
$$= (r_y F_z - r_z F_y) u_{a_x}$$
$$- (r_x F_z - r_z F_x) u_{a_y}$$
$$+ (r_x F_y - r_y F_x) u_{a_z}$$

If you want to find the moment ***Maa*** as a vector: $\mathbf{M}_{aa} = M_{aa} \mathbf{u}_a$

Example: The line of action of force **F** directs from point A to point B. If the magnitude of the force is 120 lb, determine the magnitude of the moment caused by **F** about the CD axis.



Example: The line of action of force $\mathbf{F}$ directs from point A to point B. If the magnitude of the force is 120 lb, determine the magnitude of the moment caused by $\mathbf{F}$ about the CD axis.



Force vector:

$$\mathbf{F} = \{86.1\mathbf{i} - 79.9\mathbf{j} - 24.6\mathbf{k}\}\ \text{lb}$$

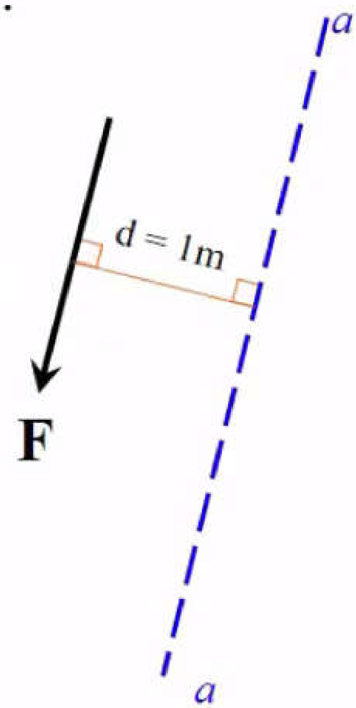
Position vector: $\mathbf{r}_{CB} = 7\mathbf{i}\ \text{ft}$

Unit vector: $\mathbf{u}_{CD} = 0.6\mathbf{j} - 0.8\mathbf{k}$

$$M_{CD} = \mathbf{u}_{CD} \cdot (\mathbf{r}_{CB} \times \mathbf{F}) = \begin{vmatrix} 0 & 0.6 & -0.8 \\ 7 & 0 & 0 \\ 86.1 & -79.9 & -24.6 \end{vmatrix} = 551\,(\text{lb}\cdot\text{ft})$$

Ans.

Question 2: What is the moment of the 100-N force F about the *aa* axis when the force is parallel to the axis and the perpendicular distance between them is 1 meter.



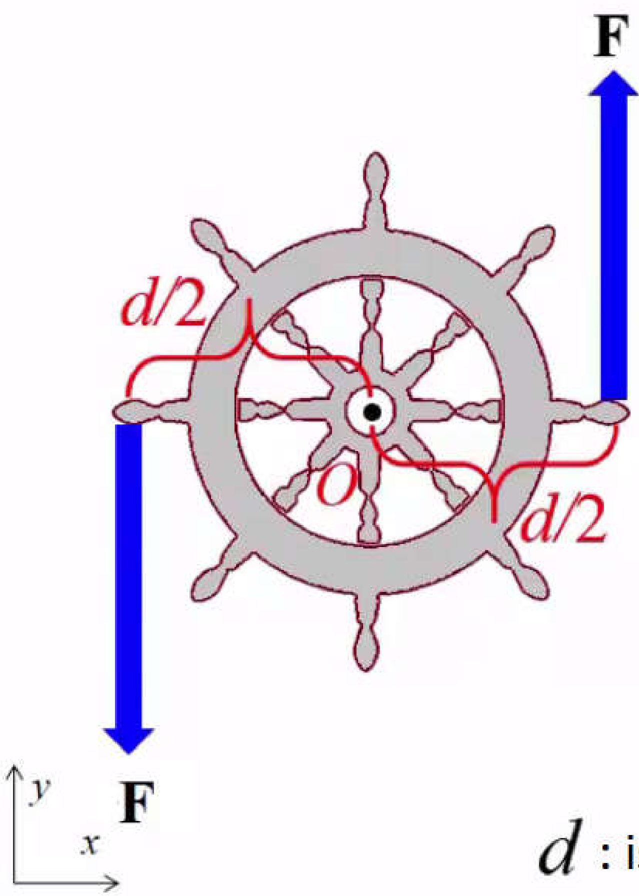
Moment of a Couple

Dr. Djedoui . Dr. Khechai

Objectives :

- To define and explain the moment of a couple.
- To demonstrate the different calculation methods of couple moment through an example.

Question 1: When you are driving, how do you position your hands on the driving wheel? (Please be specific.) In your opinion why is that an optimal positioning?



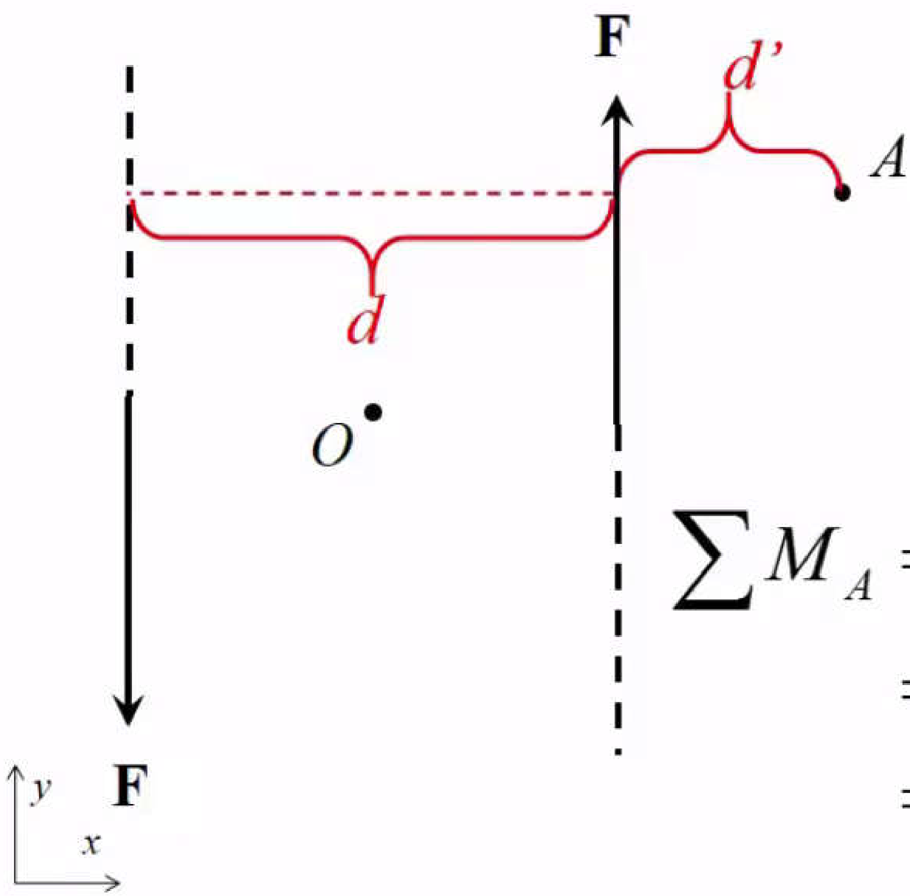
$$\sum F_y = F + (-F) = 0$$

The forces cancel each other out, therefore, have **no translational effect** on the wheel.

These two forces are known as a couple.

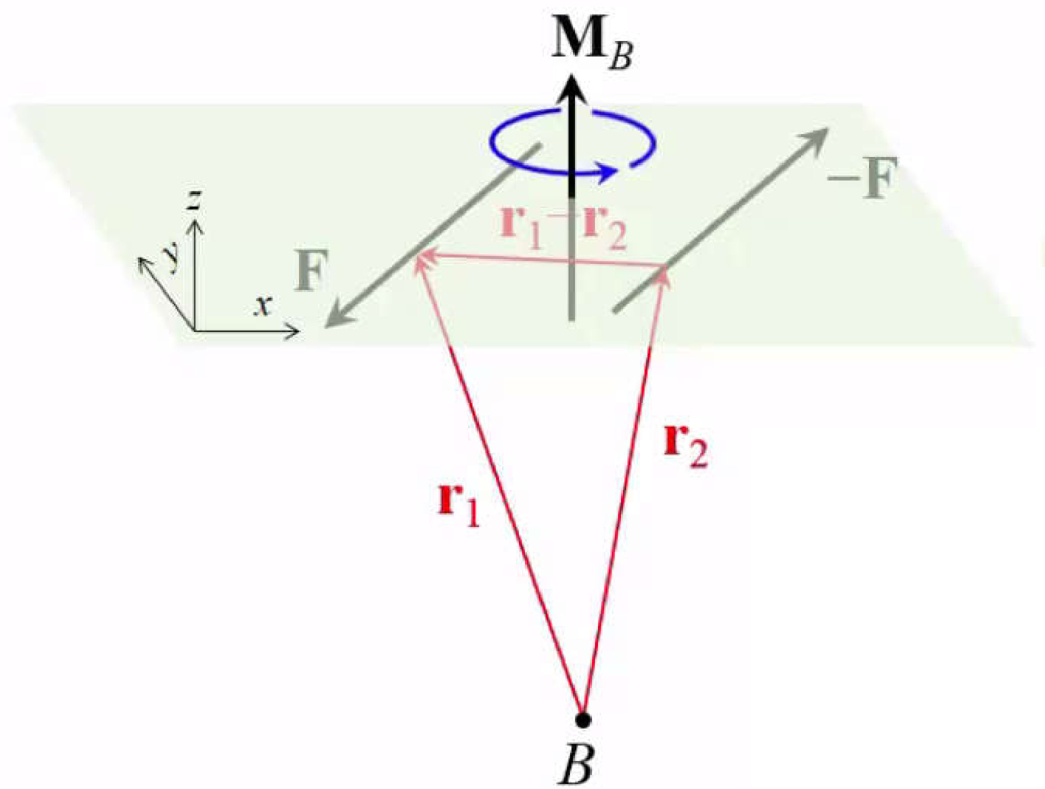
$$\sum M_O = F \cdot \frac{d}{2} + F \cdot \frac{d}{2}$$
$$= F \cdot d$$

d : is the **perpendicular distance** between the lines of action of these two forces.



$$\sum M_A = F \cdot (d + d') - F \cdot d'$$
$$= F \cdot d$$
$$= \sum M_O$$

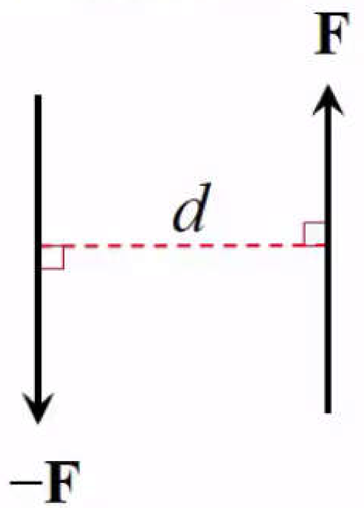
If we want to calculate the **total moment** caused by these **two forces** about point *B*, that is **not** even in the current xy plane,
We use **Vector Formulation**:



$$\sum \mathbf{M}_B = \mathbf{r}_1 \times \mathbf{F} + \mathbf{r}_2 \times (-\mathbf{F})$$
$$= (\mathbf{r}_1 - \mathbf{r}_2) \times \mathbf{F}$$
$$= (F \cdot d)\mathbf{k}$$

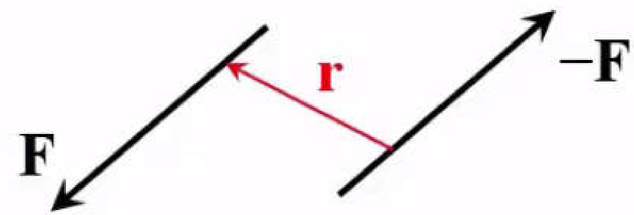
- The moment of a couple is a free vector because it does not depend on the reference point.
- The net external effect of a couple is that the net force equals zero and the magnitude of the net moment equals $F \cdot d$.
- Moments of couples are also vectors.

Scalar formulation



$$M = F \cdot d$$

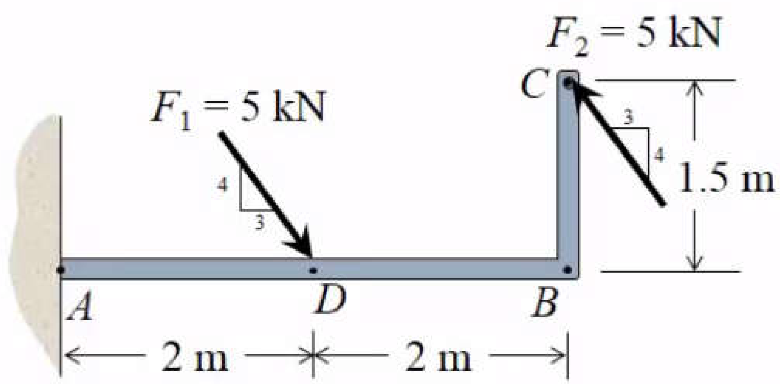
Vector formulation



$$\mathbf{M} = \mathbf{r} \times \mathbf{F}$$

We need to determine if the moment is positive or negative based on if the rotational effect is **counterclockwise** or **clockwise**.

Example: Determine the magnitude of the applied couple moment. Neglect the thickness of the member.



Example: Determine the magnitude of the applied couple moment. Neglect the thickness of the member.

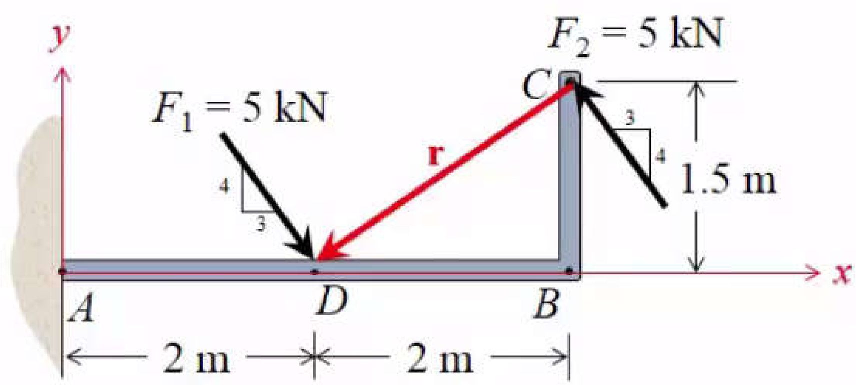
Vector formulation:

$$\mathbf{F}_1 = \{3\mathbf{i} - 4\mathbf{j}\}\ \text{kN}$$

$$\mathbf{r} = \{-2\mathbf{i} - 1.5\mathbf{j}\}\ \text{m}$$

$$\mathbf{M} = \mathbf{r} \times \mathbf{F}_1 = 12.5\mathbf{k}\ \text{kN}\cdot\text{m}$$

$$M = 12.5\ \text{kN}\cdot\text{m}$$ Ans.



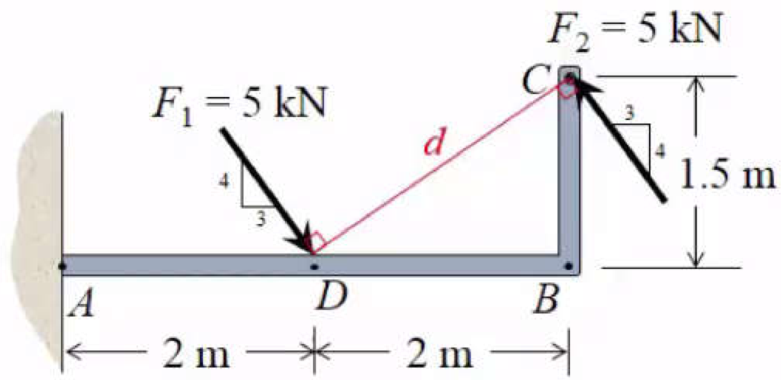
Example: Determine the magnitude of the applied couple moment. Neglect the thickness of the member.

Scalar formulation:

$F = 5\ \text{kN}$

$d = 2.5\ \text{m}$

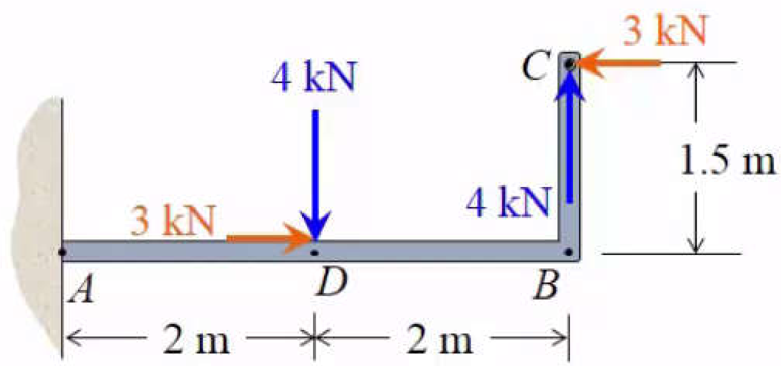
$M = F \cdot d = 12.5\ \text{kN} \cdot \text{m}$ Ans.



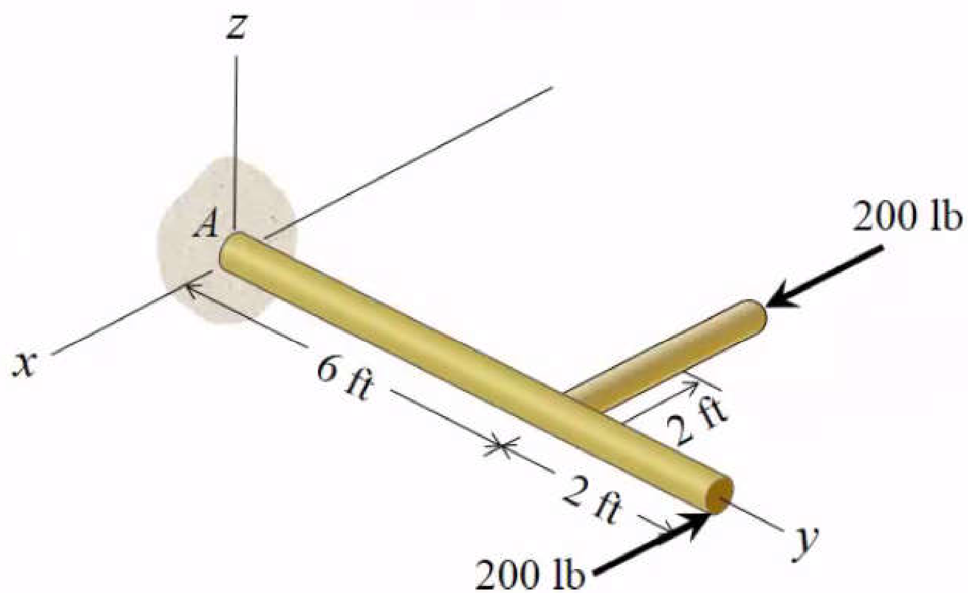
Example: Determine the magnitude of the applied couple moment.
Neglect the thickness of the member.

Principle of moments:

$$M = 4\ \text{kN} \cdot 2\ \text{m} + 3\ \text{kN} \cdot 1.5\ \text{m}$$
$$= 12.5\ \text{kN} \cdot \text{m} \quad \text{Ans.}$$



Question 2: What is the shown couple moment in Cartesian vector form?



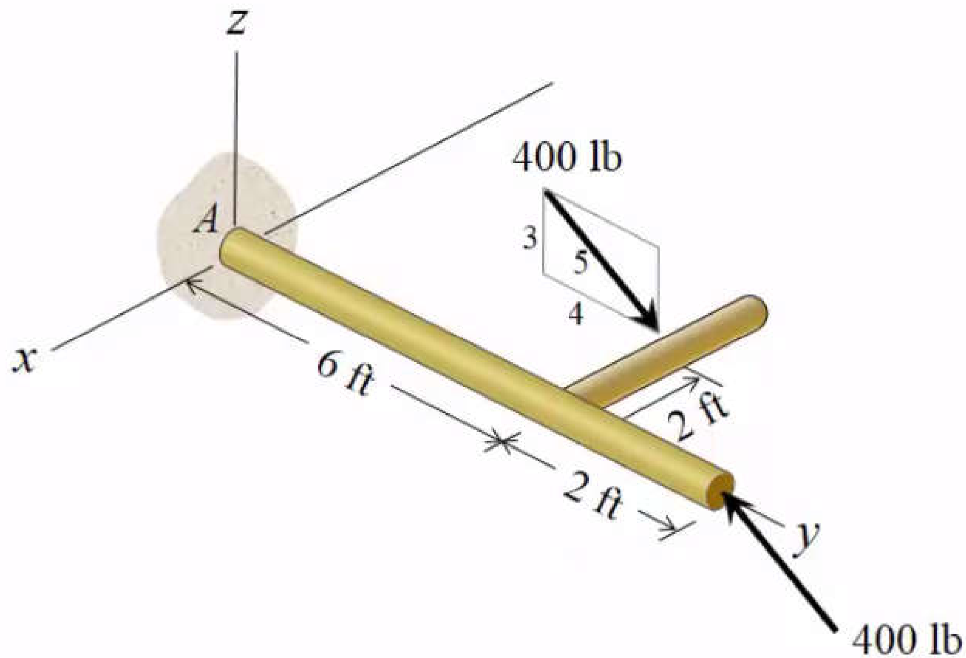
(a) $\{400\mathbf{k}\}\text{lb}\cdot\text{ft}$

(b) $\{-400\mathbf{k}\}\text{lb}\cdot\text{ft}$

(c) $\{800\mathbf{k}\}\text{lb}\cdot\text{ft}$

(d) $\{400\mathbf{j}+400\mathbf{k}\}\text{lb}\cdot\text{ft}$

Question 3: What is the shown couple moment in Cartesian vector form?



(a) $\{960\mathbf{i} - 960\mathbf{j} - 1280\mathbf{k}\}\text{lb}\cdot\text{ft}$ (b) $\{-480\mathbf{j} - 640\mathbf{k}\}\text{lb}\cdot\text{ft}$

(c) $\{-960\mathbf{j} - 1280\mathbf{k}\}\text{lb}\cdot\text{ft}$ (d) $\{480\mathbf{i} - 480\mathbf{j} - 640\mathbf{k}\}\text{lb}\cdot\text{ft}$

Principle of transmissibility

- When dealing with the mechanics of a rigid body; force may be applied at any point on its given **line of action** without altering the resultant effects of the force *external* to the *rigid* body on which it acts