Solution

Exo 1 :

* $\cos\left(\frac{\pi}{12}\right) = \cos\left(\frac{\pi}{3} - \frac{\pi}{4}\right) = \cos\left(\frac{\pi}{3}\right)\cos\left(\frac{\pi}{4}\right) - \sin\frac{\pi}{3}\sin\frac{\pi}{4} = \frac{\sqrt{6} + \sqrt{2}}{4}$

* $\cos^2\left(\frac{\pi}{8}\right) = \frac{1}{2}\left(1 + \cos\left(2 \times \frac{\pi}{8}\right)\right) = \frac{2+\sqrt{2}}{4}$. car : $\cos\left(\frac{\pi}{8}\right) > 0$.

  donc : $\cos\left(\frac{\pi}{8}\right) = \frac{1}{2}\sqrt{2+\sqrt{2}}$.

* $\arccos\left(\frac{\sqrt{2}}{2}\right) = \arccos\left(\cos\frac{\pi}{4}\right) = \frac{\pi}{4}$.

* $\arctan\frac{1}{\sqrt{3}} = \arctan\left(\tan\frac{\pi}{6}\right) = \frac{\pi}{6}$.

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Exo 2 : d'après l'hôpitale.

① $\lim\limits_{x\to 0} \frac{\arctan x}{1-e^x} = \lim\limits_{x\to 0} \frac{\frac{1}{1+x^2}}{-e^x} = \frac{1}{-1} = -1$

③ $\lim\limits_{x\to 0} \frac{\arctan(ax) - \arctan(bx)}{\ln(1+x)} = \lim\limits_{x\to 0} \frac{\frac{a}{1+(ax)^2} - \frac{b}{1+(bx)^2}}{\frac{1}{1+x}} = a-b$

② $\lim\limits_{x\to 0} \frac{\arcsin(x)}{x} = 1$, ④ $\lim\limits_{x\to 0} \frac{\arcsin(2x) - \arcsin x}{x} = 1$.

Exo 3 :

① $\forall x \in \mathbb{R}$: $(\arctan)' = \frac{1}{1+x^2} > 0$.

② On a : $0 < \frac{1}{3} < 1 \Rightarrow \arctan(0) < \arctan\left(\frac{1}{3}\right) < \arctan(1)$

(car arctan est strictement croissante) donc :

$0 < \arctan\left(\frac{1}{3}\right) < \frac{\pi}{4}$.

$$\Rightarrow \quad 0 < 2\arctan\left(\frac{1}{3}\right) < \frac{\pi}{2}$$

③ $\forall a \in \mathbb{R} - \left\{\frac{\pi}{2} + k\pi, k \in \mathbb{Z}\right\}$

$$\frac{2\tan(a)}{1+\tan^2(a)} = \frac{2\sin(a)}{\cos(a)} \times \cos^2(a) = 2\sin(a)\cos(a) = \sin(2a).$$

④ d'après ③, on a :

$$\sin\left(2\arctan\left(\frac{1}{3}\right)\right) = \frac{2\tan\left(\arctan\left(\frac{1}{3}\right)\right)}{1+\tan^2\left(\arctan\frac{1}{3}\right)} = \frac{\frac{2}{3}}{1+\frac{1}{9}} = \frac{3}{5}$$

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EXo4 :

$$\ln(\cosh(x)+\sinh(x)) + \ln(\cosh(x)-\sinh(x))$$
$$= \ln\left[(\cosh(x)+\sinh(x))(\cosh(x)-\sinh(x))\right]$$
$$= \ln\left[\cosh^2(x) - \sinh^2(x)\right] = \ln(1) = 0.$$

② $$\frac{\cosh(\ln x) + \sinh(\ln x)}{x} = 1$$

③ $$\frac{1+\tanh(x)}{1-\tanh(x)} = e^{2x}.$$

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EXo5 : ① $D_f =]-1,0[\cup]0,+\infty[$

② $$\lim_{x\to 0} \frac{\sin x}{\ln(1+x)} = \lim_{x \to 0} \frac{\cos(x)}{\frac{1}{1+x}} = 1.$$

③ $f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \dots + x^n \frac{f^{(n)}(0)}{n!} + o(x^n)$

$\sin(x) = x + o(x^2)$, $\ln(1+x) = x - \frac{x^2}{2} + o(x^2)$.

④ $f(x) = 1 + \frac{x}{2} + o(x)$

$\Rightarrow f(0{,}02) = 1{,}01$.

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$$
\begin{array}{r|l}
x & x - \frac{x^2}{2} \\
\hline
-\left(x - \frac{x^2}{2}\right) & 1 + \frac{x}{2} \\
\frac{x^2}{2} & \\
-\frac{x^2}{2} & \\
\hline
0 &
\end{array}
$$

EXo6 :

① $f(x) = \sin(2x) + \cos(x^2)$

$= 2x - \frac{(2x)^3}{3!} + \frac{(2x)^5}{5!} - \frac{(2x)^7}{7!} + o(x^7) + 1 - \frac{(x^2)^2}{2!} + o(x^7)$

$= 1 + 2x - \frac{4}{3}x^3 - \frac{1}{2}x^4 + \frac{4}{15}x^5 - \frac{8}{315}x^7 + o(x^7)$.

③ $f(x) = (X+1)^2 \ln(X+1)$ | Posons $X = x-1$. $x = X+1$

$= (1+X+X^2)\ln(X+1)$.

$= (1+X+X^2)\left(X - \frac{X^2}{2} + \frac{X^3}{3} - \frac{X^4}{4} + \frac{X^5}{5} + o(X^5)\right)$

$= X + \frac{3}{2}X^2 + \frac{1}{3}X^3 - \frac{7}{12}X^4 + \frac{1}{30}X^5 + o(X^5)$.

$= (x-1) + \frac{3}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \frac{7}{12}(x-1) + \frac{1}{30}(x-1)^5 + o(x^5)$

⑤ $f(x) = \dfrac{\sqrt{x+2}}{\sqrt{x}}$ in $+\infty$, $n=3$.

on pose : $t = \dfrac{1}{x} \xrightarrow[x \mapsto \infty]{} 0$. donc : $f(x) = \sqrt{1+2u}$

$\Rightarrow f(x) = 1 + u - \dfrac{u^2}{2} + \dfrac{u^3}{3} + o(u^3)$

$= 1 + \dfrac{1}{x} - \dfrac{1}{2x^2} + \dfrac{1}{2x^3} + o\left(\dfrac{1}{x^3}\right)$.

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Exo 7 :

$\lim\limits_{x \mapsto 0} \dfrac{\sinh(x)}{\cosh(x)} = \dfrac{x + o(x)}{x + o(x)} = 1$.
