التمرين 7

① f خطي $\forall X, Y \in \mathbb{R}^3$; $\forall \alpha, \beta \in \mathbb{R}$; $X = (x_1, y_1, z_1)$, $Y = (x_2, y_2, z_2)$

لنبرهن: $f(\alpha X + \beta Y) = \alpha f(X) + \beta f(Y)$

$*$ $f(\alpha X + \beta Y) = f(\alpha(x_1, y_1, z_1) + \beta(x_2, y_2, z_2))$

$= f(\alpha x_1 + \beta x_2, \alpha y_1 + \beta y_2, \alpha z_1 + \beta z_2)$

$= (\alpha x_1 + \beta x_2 + \alpha y_1 + \beta y_2 - \alpha z_1 - \beta z_2, \alpha x_1 + \beta x_2 - 2\alpha y_1 - 2\beta y_2, 2\alpha x_1 + 2\beta x_2 - \alpha z_1 - \beta z_2)$.

$= (\alpha x_1 + \alpha y_1 - \alpha z_1, \alpha x_1 - 2\alpha y_1, 2\alpha x_1 - \alpha z_1) + (\beta x_2 + \beta y_2 - \beta z_2, \beta x_2 - 2\beta y_2, 2\beta x_2 - \beta z_2)$

$= \alpha(x_1 + y_1 - z_1, x_1 - 2y_1, 2x_1 - z_1) + \beta(x_2 + y_2 - z_2, x_2 - 2y_2, 2x_2 - z_2)$.

$= \alpha f(X) + \beta f(Y)$

② تعيين $M_1(f)$ بالنسبة لـ B_1:

$$\begin{cases} f(e_1) = (1, 1, 2) = e_1 + e_2 + 2e_3 \\ f(e_2) = (1, -2, 0) = e_1 - 2e_2 \\ f(e_3) = (-1, 0, -1) = -e_1 - e_3 \end{cases}$$

$$M_1(f) = \begin{pmatrix} 1 & 1 & -1 \\ 1 & -2 & 0 \\ \end{pmatrix}$$

③ B_2 تشكل أساسا لأن : $\dim B_2 = 3 = \dim \mathbb{R}^3$

والأشعة w_1 و w_2 و w_3 مستقلة خطيا أي : $\forall \alpha, \beta, \gamma \in \mathbb{R}$:

$$\alpha w_1 + \beta w_2 + \gamma w_3 = 0 \Rightarrow \alpha = \beta = \gamma = 0.$$

④ حساب $P = P_{B_1 B_2}$ أي نكتب الأشعة w_1, w_2, w_3 بدلالة الأشعة e_1, e_2, e_3 :

$$w_1 = (1, 1, 0) = \alpha e_1 + \beta e_2 + \gamma e_3 \Rightarrow \alpha = \beta = 1, \quad \gamma = 0.$$

$$w_2 = (0, 1, 1) = \alpha e_1 + \beta e_2 + \gamma e_3 \Rightarrow \alpha = 0, \quad \beta = \gamma = 1.$$

$$w_3 = \alpha e_1 + \beta e_2 + \gamma e_3 \Rightarrow \alpha = \gamma = 1, \quad \beta = 0.$$

$$\Rightarrow P = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}.$$

حساب $Q = P_{B_2 B_1}$ أي نكتب e_1 و e_2 و e_3 بدلالة w_1, w_2, w_3

$$e_1 = (1, 0, 0) = \alpha w_1 + \beta w_2 + \gamma w_3 \Rightarrow \alpha = \gamma = \frac{1}{2}, \quad \beta = -\frac{1}{2}.$$

$$e_2 = \alpha w_1 + \beta w_2 + \gamma w_3 \Rightarrow \alpha = \beta = \frac{1}{2}, \quad \gamma = -\frac{1}{2}.$$

$$e_3 = \alpha w_1 + \beta w_2 + \gamma w_3 \Rightarrow \beta = \gamma = +\frac{1}{2}, \quad \alpha = -\frac{1}{2}.$$

$$Q = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ +\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

لدينا : $PQ = QP = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

ومنه $P^{-1} = Q$.

⑤ تعيين $M_2(f)$ بالنسبة إلى B_2 :

$f(w_1) = (2,-1,2) = \alpha w_1 + \beta w_2 + \gamma w_3 \Rightarrow \alpha = -\frac{1}{2} = \beta \,;\, \gamma = \frac{5}{2}$

$f(w_2) = (0,-2,-1) = \alpha w_1 + \beta w_2 + \gamma w_3 \Rightarrow \alpha = -\frac{1}{2}, \beta = -\frac{3}{2}, \gamma = \frac{1}{2}$

$f(w_3) = (0,1,1) = \alpha w_1 + \beta w_2 + \gamma w_3 \Rightarrow \alpha = \gamma = 0, \beta = 1.$

⑥ ايجاد مركبات v_{B_2} بحيث احداثياته موجودة في الأساس B_1

لدينا القانون : $v_{B_2} = P_{B_2,B_1} v_{B_1}$.

$$v_{B_1} = P_{B_1,B_2} v_{B_2}.$$

$$v_{B_1} = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix}$$

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