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First Year (MI+Mathematics)
Module of Algebra 2
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Series No. 01 (Vector Spaces)

Exercise 1: We define on $\mathbb{R}^*$ an internal law $\oplus$ and an external law $\otimes$ as follows:

$$\oplus: \mathbb{R}^* \times \mathbb{R}^* \to \mathbb{R}^* \qquad (x, y) \mapsto x \oplus y = xy$$

$$\otimes: \mathbb{R} \times \mathbb{R}^* \to \mathbb{R}^* \qquad (\lambda, x) \mapsto \lambda \otimes x = x^\lambda$$

Shows that $(\mathbb{R}^*, \oplus, \otimes)$ is a vector space on the field $(\mathbb{R}, +, .)$.

Exercise 2: We define on $E = \mathbb{R}^2$ the two operations:

$$\forall (x, y), (x', y') \in E: \qquad (x, y) + (x', y') = (x + x', y + y')$$

$$\forall \alpha = a + ib \in \mathbb{C}, \qquad (a + ib).(x, y) = (ax - by, ay + bx)$$

Shows that $(E, +, .)$ is a vector space on the field $\mathbb{C}$.

Exercise 3: In each of the following cases, is $(\mathbb{R}^2, +, .)$ a vector space on the field $\mathbb{R}$?

1) $(x, y) + (x', y') = (y + y', x + x');$ $\qquad \alpha.(x, y) = (\alpha x, y)$
2) $(x, y) + (x', y') = (x + x', y + y');$ $\qquad \alpha.(x, y) = (\alpha x, -\alpha y)$
3) $(x, y) + (x', y') = (x + x', y + y');$ $\qquad \alpha.(x, y) = (\alpha^2 x, \alpha^2 y)$

Exercise 4: In each of the following cases, check whether the subsets F_i form a vector subspace of the vector space E

$$E = \mathbb{R}^2$$

$F_1 = \{(x, y) \in E / \ 3x - y = 0\}$
$F_2 = \{(x, y) \in E / \ e^x e^y = 0\}$
$F_3 = \{(x, y) \in E / \ xy = 0\}$

$$E = \mathbb{R}^3$$

$F_4 = \{(x, y, z) \in E / \ x + y + 3z = 0\}$
$F_5 = \{(x, y, z) \in E / \ x + y + 3z = 2\}$
$F_6 = \{(x, y, z) \in E / \ z(x^2 + y^2) = 0\}$

$$E = \mathbb{P}_1[X]$$

$F_7 = \{P \in E, P'(0) = 3\}$
$F_8 = \{P \in E, P(2) = P'(2)\}$
$F_9 = \{P \in E, P(-1) = P(2)\}$

$$E = \mathcal{F}(\mathbb{R}, \mathbb{R})$$

$F_{10} = \{f \in E / \ f \ even\}$
$F_{11} = \{f \in E / \ f \ increasing\}$
$F_{12} = \{f \in E / \forall x \in \mathbb{R}: f(1 - x) = f(x)\}$

Exercise 05: Let $\mathbb{R}^3$ be a vector space on $\mathbb{R}$ and let $U = (1, -3, 2)$,$V = (2, -1, 1)$

1. Write the vector *X= (1,7,-4)* as a linear combination of *U* and *V*.
2. Is the vector $Y = (2, -5, 4)$ a linear combination of *U* and *V*?
3. Find m such that $Z = (3,1, m) \in [\{U, V\}]$

Exercise 06:

1. Which of the following families generates the vector space *E*?

a) $E = \mathbb{R}^2$:

$$\mathcal{F}_1 = \{(3, -1), (1,1)\}, \mathcal{F}_2 = \{(-1,1), (3, -3)\}, \mathcal{F}_3 = \{(3, -1), (1,1), (1, -2)\}$$

b) $E = \mathbb{P}_2[\mathrm{X}]$:

$$\mathcal{F}_1 = \{X^2, 3X, -1\}, \mathcal{F}_2 = \{X^2 + X, X - 1\}, \mathcal{F}_3 = \{2X^2 - 5{,}7X, 3X^2 + 4\}$$

c) $E = \mathbb{R}^3$:

$$\mathcal{F}_1 = \{(1,0,1), (-1,1,0)\}, \mathcal{F}_2 = \{(1,0,-1), (2,0,3), (3,1,-1)\}$$

2. Which of the following families are linearly independent in *E*?

a) $E = \mathbb{R}^2$:

$$\mathcal{F}_1 = \{(-1,3), (0,1)\}, \mathcal{F}_2 = \{(1,2), (-1,1), (-1,2)\}$$

b) $E = \mathbb{P}_2[\mathrm{X}]$:

$$\mathcal{F}_1 = \{X^2 + 1, X - 2\}, \mathcal{F}_2 = \{X, X + 1, X - 1\}, \mathcal{F}_3 = \{X^2 - 1, X^2 + 1{,}2X\}$$

c) $E = \mathbb{R}^3$:

$$\mathcal{F}_1 = \{(1,0,1), (0,2,2), (3,7,1)\}, \mathcal{F}_2 = \{(1,0,0), (0,1,1), (1,1,1)\}$$

d) $E = \mathcal{F}(\mathbb{R}, \mathbb{R})$:

$$\mathcal{F}_1 = \{e^x, xe^x\}, \mathcal{F}_2 = \{\cos x, \sin x\}, \mathcal{F}_3 = \{x, \sin x\}$$

Exercise 07: Let $\mathbb{P}_2[\mathrm{X}]$ the vector space of polynomials with a degree less than or equal to 2. Let the set $\mathcal{F} = \{P_1, P_2, P_3\}$ such that:

$$P_1(X) = \frac{1}{2}(X - 1)(X - 2), P_2(X) = -X(X - 2), P_3(X) = \frac{1}{2}X(X - 1)$$

1) Show that $\mathcal{F}$ is a basis of $\mathbb{P}_2[\mathrm{X}]$.
2) Let $Q(X) = aX^2 + bX + c \in \mathbb{P}_2[\mathrm{X}]$, write $Q(X)$ in base $\mathcal{F}$.

Exercise 08: $\mathbb{R}^3$ is a vector space on field $\mathbb{R}$, and $G = [\{(1,1,0), (0,0,1), (1,1,1)\}]$ a vector subspace and the set F defined as follows: $F = \{(x, y, z) \in \mathbb{R}^3 / 2x + y - z = 0\}$

1. Show that F is a vector subspace of $\mathbb{R}^3$.
2. Find a basis for: $F \cap G, F + G, G, F$ (if it exists), specifying their dimensions.
3. Is $\mathbb{R}^3 = F \oplus G$?