Chapter 2

IMPROPER INTEGRAL

In first year, we studied the integral of defined and continuous functions on a compact interval (closed bounded) $[a, b]$ with $-\infty < a < b < +\infty$, there existed a so-called primitive function F of f such that:

$$\acute{F}(x) = f(x) \forall x \in [a, b] \text{ and } \int_a^b f(t)dt = F(b) - F(a).$$

Which represents the area delimited by the graph of the function f on $[a, b]$. In this chapter, we will learn how to calculate the integrals of unbounded domains, either because the integration interval is infinite (going up to $+\infty$ or $-\infty$), or because the function to be integrated tends towards infinite at the limits of the interval. These integrals are called improper integrals or generalized integrals.

We end our introduction by explaining the plan of this chapter. When we do not know how to calculate an antiderivative, we resort to two types of method: either the function has a constant sign in the vicinity of the uncertain point, or it changes sign an infinite number of times in this vicinity (we then say that it "oscillates"). We will also distinguish the case where the uncertain point is $\pm\infty$ or a finite value. There are therefore four distinct cases, depending on the type of the uncertain point, and the sign, constant or not, of the function to be integrated. These four types are schematized in the following figures:

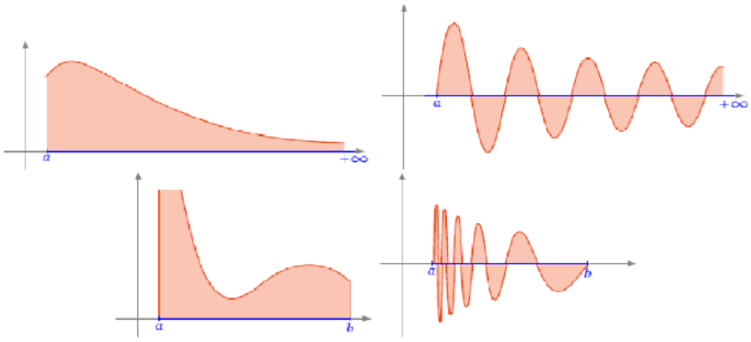

Figure 2.1: Different types of integrals.

2.1 Definitions and properties

2.1.1 Uncertain points

- We first start by identifying the uncertain points, either $+\infty$ or $-\infty$ on the one hand, and on the other hand the point(s) in the vicinity of which the function is not bounded.
- We then divide each integration interval into as many intervals as are necessary so that each of them contains only one uncertain point, placed at one of the two limits.
- the integral over the complete interval is the sum of the integrals over the intervals of the division.
- The only goal is to isolate the difficulties: the choices of cutting points are arbitrary.

2.1.2 Convergence/divergence

Definition 2.1.1 *Let f be a continuous function on $[a, +\infty[$. We say that the integral $\int\limits_{a}^{+\infty} f(t)dt$ converges if the imitates, when $x \longrightarrow +\infty$, of the primitive $\int\limits_{a}^{x} f(t)dt$ exists and is finished, i.e.*

$$\int\limits_{a}^{+\infty} f(t)dt = \lim_{x\longrightarrow+\infty} \int\limits_{a}^{x} f(t)dt.$$

Otherwise, we say that the integral diverges.

Let f be a continuous function on $]a, b]$. We say that the integral $\int\limits_{a}^{b} f(t)dt$ converges if the right *limit, when $x \longrightarrow a$, of the primitive $\int\limits_x^b f(t)dt$ exists and is finished, i.e.*

$$\int\limits_a^b f(t)dt = \lim_{x \longrightarrow a^+} \int\limits_a^b f(t)dt.$$

Otherwise, we say that the integral diverges.

Example 2.1.1 *The integral* $\int\limits_0^{+\infty} e^{-t}dt = \lim\limits_{x \longrightarrow +\infty} \int\limits_0^x e^{-t}dt = \lim\limits_{x \longrightarrow +\infty} \left[-e^{-t}\right]_0^x = 1$. *So the integral converges.*

The integral $\int\limits_0^{+\infty} \sin(t)dt = \lim\limits_{x \longrightarrow +\infty} \int\limits_0^x \sin(t)dt = \lim\limits_{x \longrightarrow +\infty} [-\cos(t)]_0^x = \nexists$ *because* $\lim\limits_{x \longrightarrow +\infty} \cos(x) = \nexists$. *So the integral diverges.*

The integral $\int\limits_0^1 \ln(t)dt = \lim\limits_{x \longrightarrow +0} \int\limits_x^1 \ln(t)dt = \lim\limits_{x \longrightarrow +0} [t\ln(t) - t]_x^1 = -1$. *So the integral converges.*

The integral $\int\limits_0^1 \frac{1}{t}dt = \lim\limits_{x \longrightarrow +0} \int\limits_x^1 \frac{1}{t}dt = \lim\limits_{x \longrightarrow +0} [\ln(t)]_x^1 = +\infty$. *So the integral diverges.*

Remark 2.1.1 - *The generalized integral is considered as the limit of a definite integral.*

Remark 2.1.2 - *Convergence is therefore equivalent to finite limit. Divergence means either there is no limit or the limit is infinite.*

2.1.3 Relationship of Chasles

Proposition 2.1.1 (Relation de Chasles) *Let $f : [a, +\infty[\longrightarrow \mathbb{R}$ be a continuous function. For all $c \in [a, +\infty[$ the improper integrals $\int\limits_a^{+\infty} f(t)dt$ and $\int\limits_c^{+\infty} f(t)dt$ are of the same nature, and in the case of convergence we have:*

$$\int\limits_a^{+\infty} f(t)dt = \int\limits_a^c f(t)dt + \int\limits_c^{+\infty} f(t)dt.$$

Preuve. Using the Chasles relation for the usual Riemann integrals, with $a \leq c \leq x$:

$$\int\limits_a^x f(t)dt = \int\limits_a^c f(t)dt + \int\limits_c^x f(t)dt.$$

Then passing to the limit $x \longrightarrow +\infty$. Now, if we are in the case of a continuous function

$f :]a, b] \longrightarrow \mathbb{R}, c \in]a, b]$, then we have a similar result, and in the case of convergence:

$$\int_x^b f(t)dt = \int_a^c f(t)dt + \int_c^b f(t)dt.$$

Then passing to the limit $x \longrightarrow a^+$ ■

Remark 2.1.3 - *"Being of the same nature" means that the two integrals are convergent at the same time or divergent at the same time.*

- The Chasles relation therefore implies that convergence does not depend on the behavior of the function on bounded intervals, but only on its behavior in the neighborhood of $+\infty$.

2.1.4 Linearity

Proposition 2.1.2 (Linearity of the improper integral) *Let f and g be two continuous functions on $[a, +\infty[$, and λ, μ two real numbers. If the integrals $\int_a^{+\infty} f(t)dt$ and $\int_a^{+\infty} g(t)dt$. converge, then $\int_a^{+\infty} [\lambda f(t) + \mu g(t)]\, dt$ converges and we have:*

$$\int_a^{+\infty} [\lambda f(t) + \mu g(t)]\, dt = \lambda \int_a^{+\infty} f(t)dt + \mu \int_a^{+\infty} g(t)dt$$

The linearity relation is valid for the functions of an interval $]a, b]$, not bounded in a.

Remark 2.1.4 *The converse in the linearity relation is false, we can find two functions f, g such that*

$\int_a^{+\infty} (f(t) + g(t))\, dt$ converges, without $\int_a^{+\infty} f(t)dt$ nor $\int_a^{+\infty} g(t)dt$ converges.

2.1.5 Positivity

Proposition 2.1.3 (Positivity of the improper integral) *Let $f, g : [a, +\infty[\longrightarrow \mathbb{R}$ be continuous functions, having a convergent integral.*

$$\text{if } f \leq g \text{ then } \int_a^{+\infty} f(t)dt \leq \int_a^{+\infty} g(t)dt.$$

In particular, we also have:

$$\text{if } f \geq 0 \text{ then } \int_a^{+\infty} f(t)dt \geq 0.$$

The positivity relation is valid for the functions of an interval $]a,b]$, not bounded in a.

Remark 2.1.5 *If we do not wish to distinguish the two types of improper integrals on an interval $[a,+\infty[$ (or $]-\infty,b]$) on the one hand and $]a,b]$ (or $[a,b[$) on the other hand, then it is practical to add the two ends to the number line:*

$$\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}.$$

Thus the interval $[a,b[$ with $a \in \mathbb{R}$ and $b \in \overline{\mathbb{R}}$ designates the infinite interval $[a,+\infty[$ (if $b = +\infty$) or the finite interval $[a,b[$ (if $b < +\infty$). Likewise for the interval $]a,b]$ with $a = +\infty$ or $a \in \mathbb{R}$.

2.1.6 Cauchy criterion

Theorem 2.1.1 (Cauchy criterion) *Let $f : [a,+\infty[\longrightarrow \mathbb{R}$ be a continuous function. The improper integral $\int_a^{+\infty} f(t)dt$ converges iff*

$$\forall \epsilon > 0, \exists M \geq a \qquad u, v \geq M \implies \left| \int_a^{+\infty} f(t)dt \right| < \epsilon.$$

Preuve. It is enough to apply the Cauchy criterion for the limits to the function $F(x) = \int_a^{+\infty} f(t)dt$. Let $F : [a,+\infty[\longrightarrow \mathbb{R}$. Then $\lim\limits_{x \longrightarrow +\infty} F(x) = \lim\limits_{x \longrightarrow +\infty} \int_a^x f(t)dt$ exists and is finite iff:

$$\forall \epsilon > 0, \exists M \geq a \qquad u, v \geq M \implies |F(u) - F(v) = | \left| \int_u^v f(t)dt \right| < \epsilon.$$

■

2.1.7 Case of two uncertain points

When both ends of the definition interval are uncertain points. It is just a matter of reducing ourselves to two integrals each having a single uncertain point.

Definition 2.1.2 *Let $a, b \in \overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$ with $a < b$. Let $f :]a,b[\longrightarrow \mathbb{R}$ be a continuous* *function. We say that the integral*

$\int\limits_a^b f(t)dt$ converges if there exists $c \in]a,b[$ such that the two improper integrals $\int\limits_a^c f(t)dt$ and $\int\limits_c^b f(t)dt$ converge. The value of this doubly improper integral is then

$$\int_a^b f(t)dt = \int_a^c f(t)dt + \int_c^b f(t)dt.$$

Remark 2.1.6 - *Chasles' relations imply that the nature and value of this doubly improper integral do not depend on the choice of c, with $a < c < b$.*

- *If one of the two integrals $\int\limits_a^c f(t)dt$ diverges, or $\int\limits_c^b f(t)dt$ diverge, alors $\int\limits_a^b f(t)dt$ diverge.*

Example 2.1.2

$$\int_{-\infty}^{+\infty} \frac{t}{(1+t^2)^2}dt = \int_{-\infty}^{2} \frac{t}{(1+t^2)^2}dt + \int_{2}^{+\infty} \frac{t}{(1+t^2)^2}dt.$$

We choose $c = 2$ at random. We start with the first integral

$$\begin{aligned}\int\limits_{-\infty}^{2} \frac{t}{(1+t^2)^2}dt &= \lim_{x \longrightarrow -\infty} \int\limits_{x}^{2} \frac{t}{(1+t^2)^2}dt \\ &= -\frac{1}{2} \lim_{x \longrightarrow -\infty} \left[\frac{1}{(1+t^2)^2}\right]_x^2 \\ &= -\frac{1}{2} \lim_{x \longrightarrow -\infty} \left[\frac{1}{5} - \frac{1}{(1+x^2)}\right]. \\ &= -\frac{1}{10}.\end{aligned}$$

Then $\int\limits_{-\infty}^{2} \frac{t}{(1+t^2)^2}dt$ converges. Likewise for $\int\limits_{2}^{+\infty} \frac{t}{(1+t^2)^2}dt$ which converges to $\frac{1}{10}$. Thus the integral $\int\limits_{-\infty}^{+\infty} \frac{t}{(1+t^2)^2}dt$ converges and is worth 0.

2.2 Improper integrals on an unbounded interval

2.2.1 Positive functions

We will assume that the function is positive or zero on the integration interval $[a, +\infty[$. The convergence criteria for positive functions are also valid for negative functions, you just need to replace f by $(-f)$. Recall that, by definition,

$$\int_a^{+\infty} f(t)dt = \lim_{x\longrightarrow+\infty} \int_a^x f(t)dt.$$

As f is positive, then the primitive is increasing, or else $\lim\limits_{x\longrightarrow+\infty} \int_a^x f(t)dt$ is bounded, and therefore the integral $\int_a^{+\infty} f(t)dt$ converges, or $\lim\limits_{x\longrightarrow+\infty} \int_a^x f(t)dt$ tends towards $+\infty$ therefore diverges. If we cannot (or do not want to) calculate an primitive of f, we study the convergence using comparaison criteria which allow us to deduce its nature without explicitly calculating them.

2.2.2 Comparison criterion

Theorem 2.2.1 *Let f and g be two positive and continuous functions on $[a,+\infty[$. As:*

$$\exists A \geq a, \forall t > A \qquad f(t) \leq g(t).$$

1. *If $\int_a^{+\infty} g(t)dt$ converges $\Longrightarrow \int_a^{+\infty} f(t)dt$ converges.*
2. *If $\int_a^{+\infty} f(t)dt$ diverges $\Longrightarrow \int_a^{+\infty} g(t)dt$ diverge.*

Preuve. The convergence of the integrals does not depend on the left bound of the interval, and we can simply study $\int_A^x f(t)dt$ and $\int_A^x g(t)dt$. Now using the positivity of the integral, we obtain that, for all $x \geq A$,

$$\int_A^x f(t)dt \leq \int_A^x g(t)dt.$$

If $\int_A^x g(t)dt$ converges, then $\int_A^x f(t)dt$ is an increasing and bounded function and therefore converges. Conversely, if $\int_A^x f(t)dt$ tends towards $+\infty$, then $\int_A^x g(t)dt$ tends towards $+\infty$ too. ■

Example 2.2.1 *The integral $\int_1^{+\infty} e^{-t^2}dt$ is convergent because: $\forall t \in [1,+\infty[, -t^2 \leq t$, as e^t is an increasing function then $e^{-t^2} \leq e^t$. And since $\int_1^{+\infty} e^{-t}dt = \lim\limits_{x\longrightarrow+\infty} \left[-e^{-t}\right]_1^x$, therefore the integral $\int_1^{+\infty} e^{-t}dt$ converges.*

2.2.3 Equivalence criterion

Theorem 2.2.2 (Equivalence criterion) *Let f and g be two strictly positive and continuous functions on $[a, +\infty[$. Suppose they are equivalent in the neighborhood of $+\infty$, that is:*

$$\lim_{t \longrightarrow +\infty} \frac{f(t)}{g(t)} = 1.$$

Then the integral $\int_a^{+\infty} f(t)dt$ converges if is only if $\int_a^{+\infty} g(t)dt$ converges.

Preuve. To say that two functions are equivalent in the neighborhood of $+\infty$, is to say that their ratio tends towards 1, or again:

$$\forall \epsilon > 0 \qquad \exists A > a \qquad \forall t > A \qquad \left| \frac{f(t)}{g(t)} - 1 \right| < \epsilon,$$

or again:

$$\forall \epsilon > 0 \qquad \exists A > a \qquad \forall t > A \qquad (1-\epsilon)\, g(t) < f(t) < (1+\epsilon)\, g(t).$$

Let us set $\epsilon < 1$, and apply the comparison theorem on the interval $[A, +\infty[$. If the integral $\int_A^{+\infty} f(t)dt$ converges, then the integral $\int_A^{+\infty} (1-\epsilon)\, g(t)dt$ converges, therefore the integral $\int_A^{+\infty} g(t)dt$ also converges by linearity. Conversely, if $\int_A^{+\infty} f(t)dt$ diverges, then $\int_A^{+\infty} (1+\epsilon)\, g(t)dt$ diverges, therefore $\int_A^{+\infty} g(t)dt$ also diverges. ∎

Example 2.2.2 *The integral $\int_1^{+\infty} \frac{1}{1+t^2}dt$ converges because: $\lim_{t \longrightarrow +\infty} \frac{\frac{1}{1+t^2}}{\frac{1}{t^2}} = 1 \Longrightarrow \frac{1}{1+t^2} \sim \frac{1}{t^2}$ and as $\int_1^{+\infty} \frac{1}{t^2}dt = \lim_{x \longrightarrow +\infty} \int_1^{x} \frac{1}{t^2}dt = \lim_{x \longrightarrow +\infty} \left[-\frac{1}{t}\right]_1^x = 1$, then the integral $\int_1^{+\infty} \frac{1}{t^2}dt$ converges by the equivalence criterion the integral $\int_1^{+\infty} \frac{1}{1+t^2}dt$ converges.*

Proposition 2.2.1 *Let f and g be two strictly positive and continuous functions on $[a, +\infty[$ such that:*

$$\lim_{t \longrightarrow +\infty} \frac{f(t)}{g(t)} = l.$$

If $l \neq 0$ and $l \neq +\infty$, $f(t) \underset{+\infty}{\sim} lg(t)$. Then the two integrals $\int_a^{+\infty} f(t)dt$ and $\int_a^{+\infty} g(t)dt$ are of the same nature.

If $l = 0$, $f(t) \leq g(t)$. Then if the integral $\int_a^{+\infty} g(t)dt$ converges $\Longrightarrow$ $\int_a^{+\infty} f(t)dt$ converges.

If $l=+\infty$, $f(t)\geq g(t)$. Then if the integral $\int\limits_a^{+\infty} g(t)dt$ diverges $\Longrightarrow$ $\int\limits_a^{+\infty} f(t)dt$ diverges.

Example 2.2.3 *The integral $\int\limits_1^{+\infty} \frac{\ln(t)}{1+t^2}dt$ converges because: $\lim\limits_{t\longrightarrow+\infty} \frac{\frac{\ln(t)}{1+t^2}}{\frac{1}{t^{\frac{3}{2}}}}=0$. So $\frac{\ln(t)}{1+t^2}\leq \frac{1}{t^{\frac{3}{2}}}$, and as $\int\limits_1^{+\infty} \frac{1}{t^{\frac{3}{2}}}dt = \lim\limits_{x\longrightarrow+\infty}\left(\frac{-2}{\sqrt{t}}\right)_1^x = 2$ converges $\Longrightarrow$ $\int\limits_1^{+\infty} \frac{\ln(t)}{1+t^2}dt$ converges.*

2.2.4 Riemann integrals

Definition 2.2.1 *A Riemann integral is an integral which is written in the form:*

$$\int\limits_1^{+\infty} \frac{1}{t^\alpha}dt, \text{ où } \alpha\in\mathbb{R}_+^*.$$

In this case, the primitive is explicit:

$$\int\limits_1^{+\infty} \frac{1}{t^\alpha}dt = \begin{cases} \lim\limits_{x\longrightarrow+\infty}\left[\frac{1}{-\alpha+1}\frac{1}{t^{\alpha-1}}\right]_1^x & \text{if } \alpha\neq 1 \\ \\ \lim\limits_{x\longrightarrow+\infty}\left[\ln(t)\right]_1^x & \text{if } \alpha=1 \end{cases}.$$

So we deduce the nature of Riemann integrals

$$\text{if } \alpha>1 \text{ then } \int\limits_1^{+\infty}\frac{1}{t^\alpha}dt \text{ converges.}$$

$$\text{if } \alpha\leq 1 \text{ then } \int\limits_1^{+\infty}\frac{1}{t^\alpha}dt \text{ diverges.}$$

Proposition 2.2.2 *Let f be a positive and continuous function on $[a,+\infty[$.*

- If $f(t) \underset{+\infty}{\sim} \frac{1}{t^\alpha}$ where ($l\neq 0$ and $l\neq+\infty$) then $\int\limits_a^{+\infty} f(t)dt$ $\begin{cases} \text{converges if } \alpha>1 \\ \text{diverges if } \alpha\leq 1\end{cases}$.

- If $\lim\limits_{t\longrightarrow+\infty} t^\alpha f(t)dt = 0$ then $\int\limits_a^{+\infty} f(t)dt$ converges if $\alpha>1$.

- If $\lim\limits_{t\longrightarrow+\infty} t^\alpha f(t)dt = +\infty$ then $\int\limits_a^{+\infty} f(t)dt$ diverges if $\alpha\leq 1$.

Example 2.2.4 *Let $\int\limits_1^{+\infty} \frac{|\sin t|}{t^2}dt$. The function $\frac{|\sin t|}{t^2}$ is continuous and positive on $[1,+\infty[$*

$\lim\limits_{t\longrightarrow+\infty} t^{3/2}\frac{|\sin t|}{t^2}=0$, so $\int\limits_1^{+\infty}\frac{|\sin t|}{t^2}dt$ converges because $\alpha=\frac{3}{2}$.

2.2.5 Bertrand integral

Definition 2.2.2 *A Bertrand Integral is an integral of the form:*

$$\int_e^{+\infty} \frac{1}{t^\alpha (\ln t)^\beta} dt, \text{ where } \alpha \in \mathbb{R}_+^*, \beta \in \mathbb{R}.$$

- If $\alpha > 1$, the integral converges.

- If $\alpha < 1$, the integral diverges.

- If $\alpha = 1$ and $\begin{cases} \beta > 1, \text{ the integral converges} \\ \beta > 1, \text{ the integral diverges.} \end{cases}$

Theorem 2.2.3

$$\int_e^{+\infty} \frac{1}{t^\alpha (\ln t)^\beta} dt \text{ converges} \Leftrightarrow (\alpha > 1) \text{ or } (\alpha = 1 \text{ and } \beta > 1).$$

$$\int_e^{+\infty} \frac{1}{t^\alpha |\ln t|^\beta} dt \text{ diverges} \Leftrightarrow (\alpha < 1) \text{ or } (\alpha = 1 \text{ and } \beta > 1).$$

Example 2.2.5 *Let* $\int_1^{+\infty} \frac{1}{t}\sin(\frac{1}{t})dt$. *The function* $\frac{1}{t}\sin(\frac{1}{t})$ *is continuous and positive on* $[1, +\infty[$ $\frac{1}{t}\sin(\frac{1}{t}) \underset{+\infty}{\sim} \frac{1}{t^2}$, *and as* $\int_1^{+\infty} \frac{1}{t^2}dt$ *is Riemann* $\alpha = 2 > 1$ *therefore converges by equivalence the integral* $\int_1^{+\infty} \frac{1}{t}\sin(\frac{1}{t})dt$ *converges.*

Example 2.2.6 *Let* $\int_1^{+\infty} \sqrt{t^2+3t}\ln\left[\cos(\frac{1}{t})\right]\sin^2\left(\frac{1}{\ln t}\right)dt$. *The function* $\sqrt{t^2}+3t\ln\left[\cos(\frac{1}{t})\right]\sin^2\left(\frac{1}{\ln t}\right)$ *is continuous and positive on* $[2, +\infty[$.

$$\begin{array}{ll} \sqrt{t^2+3t} = t\sqrt{1+\frac{3}{t}} & \underset{+\infty}{\sim} t \\ \ln\left[\cos(\frac{1}{t})\right] & \underset{+\infty}{\sim} -\frac{1}{2t^2} \\ \sin^2\left(\frac{1}{\ln t}\right) & \underset{+\infty}{\sim} \left(\frac{1}{\ln t}\right)^2 \end{array}.$$

So

$$\sqrt{t^2+3t}\ln\left[\cos(\frac{1}{t})\right]\sin^2\left(\frac{1}{\ln t}\right) \underset{+\infty}{\sim} -\frac{1}{2t(\ln t)^2}$$

and as $\int_2^{+\infty} -\frac{1}{2t(\ln t)^2}dt$ *is a Bertrand integral* $\alpha = 1, \beta = 2$ *therefore converges by equivalence* *the integral* $\int\limits_1^{+\infty} \sqrt{t^2+3t}\ln\left[\cos(\frac{1}{t})\right]\sin^2\left(\frac{1}{\ln t}\right)dt$ *converges.*

2.2.6 Absolute convergence of an improper integral

Definition 2.2.3 *Let a real function f, locally integrable on an interval* $[a,+\infty[$. *We say that the integral* $\int\limits_a^{+\infty} f(t)dt$ *s absolutely convergent if the integral* $\int\limits_a^{+\infty} |f(t)|\,dt$ *is convergent.*

Theorem 2.2.4 *An absolutely convergent integral is convergent.*

Example 2.2.7 *Study of the integral* $\int\limits_0^{+\infty} \frac{\sin t}{1+\cos t+e^t}dt$. *The function* $\frac{\sin x}{1+\cos x+e^x}$ *is locally integrable on the interval* $[a,+\infty[$. *When* x *tends towards* $+\infty$, *on a:* $\left|\frac{\sin x}{1+\cos x+e^x}\right| \leq \frac{1}{1+\cos x+e^x}$ *and* $\frac{1}{1+\cos x+e^x} \sim e^{-x}$. *We have:* $\int\limits_0^{+\infty} \frac{\sin t}{1+\cos t+e^t}dt$ *is therefore absolutely convergent, therefore convergent.*

Example 2.2.8 *Study of* $\int\limits_1^{+\infty} \frac{\sqrt{t}\sin\left(\frac{1}{t^2}\right)}{\ln(1+t)}dt$. *When* x *tends towards* $+\infty$, *we have:* $0 < f(x) \leq \sqrt{x}\sin\left(\frac{1}{x^2}\right)$ *and* $\sqrt{x}\sin\left(\frac{1}{x^2}\right) \sim \frac{1}{x^{3/2}}$. *The integral* $\int\limits_1^{+\infty} \frac{\sqrt{t}\sin\left(\frac{1}{t^2}\right)}{\ln(1+t)}dt$ *is absolutely convergent.*

2.3 Integration by parts

Theorem 2.3.1 (Integration by parts) *Let u and v two functions of class* C^1 *on the interval* $[a,+\infty[$. *Suppose that* $\lim\limits_{t\longrightarrow+\infty} u(t)v(t)$ *exists and is finite. Then the integrals* $\int\limits_a^{+\infty} u(t)\acute{v}(t)dt$ *and* $\int\limits_a^{+\infty} \acute{u}(t)v(t)dt$ *are of the same nature. In case of convergence we have:*

$$\int\limits_a^{+\infty} u(t)\acute{v}(t)dt = \left[\lim_{t\longrightarrow+\infty} u(t)v(t) - u(a)v(a)\right] - \int\limits_a^{+\infty} \acute{u}(t)v(t)dt.$$

Preuve. This is the usual formula for integration by parts

$$\int\limits_a^{+\infty} u(t)\acute{v}(t)dt = [u(t)v(t)]_a^x - \int\limits_a^{x} \acute{u}(t)v(t)dt.$$

noting that by hypothesis that $\lim\limits_{x\longrightarrow+\infty} uv$ has a finite limit. ■

Example 2.3.1 *Let the integral be* $\int\limits_0^{+\infty} \lambda te^{-\lambda t}dt$ *ou* $\lambda > 0$. *We carry out the integration by parts* with $u = \lambda t, \acute{v} = e^{-\lambda t}$. We therefore have $\acute{u} = \lambda, v = -\frac{1}{\lambda} e^{-\lambda t}$. Also

$$\int_0^x \lambda t e^{-\lambda t} dt = \left[-t e^{-\lambda t}\right]_0^x + \int_0^x e^{-\lambda t} dt$$

$$= -x e^{-\lambda x} - \frac{1}{\lambda}\left(e^{-\lambda x} - 1\right)$$

$$\int_0^{+\infty} \lambda t e^{-\lambda t} dt = \lim_{x \longrightarrow +\infty} \int_0^x \lambda t e^{-\lambda t} dt = \frac{1}{\lambda} \text{ (then the integral converges).}$$

2.4 Change of variable

Theorem 2.4.1 (Change of variable) *Let f be a function defined on an interval $I = [a, +\infty[$. Let $J = [\alpha, \beta[$ be an interval with $\alpha, \beta \in \mathbb{R}$ or $\beta = +\infty$. Let $\varphi : J \longrightarrow I$ be a diffeomorphism*[1] *of class C^1. The integrals $\int_a^{+\infty} f(x)dx$ and $\int_\alpha^\beta f(\varphi(t))\acute{\varphi}(t)\, dt$ sare of the same nature. In case of convergence, we have:*

$$\int_a^{+\infty} f(x)dx = \int_\alpha^\beta f(\varphi(t))\acute{\varphi}(t)\, dt$$

Example 2.4.1 *The following example is very interesting: the function $f(t) = \sin t^2$ has a convergent integral, but does not tend to 0 (when $t \longrightarrow +\infty$). This is to be contrasted with the case of series: for a convergent series the general term always tends towards 0. Let the integral be $\int_1^{+\infty} \sin(t^2)\, dt$. We carry out the change of variable $u = t^2$, which gives $t = \sqrt{u}$, $dt = \frac{1}{2\sqrt{u}} du$ with φ is a diffeomorphism.*

$$\begin{array}{cccc} \varphi: & [1, x^2] & \longrightarrow & [1, x] \\ & u & \longrightarrow & t \end{array}$$

$$\int_1^{+\infty} \sin(t^2)\, dt = \int_1^{x^2} \sin(u) \frac{1}{2\sqrt{u}} du$$

Now by Abel's theorem $\int_1^{+\infty} \sin(u) \frac{1}{2\sqrt{u}} du$ converges, therefore $\int_1^{x^2} \sin(u) \frac{1}{2\sqrt{u}} du$ admits a finite limit, which proves that $\int_1^x \sin(t^2)\, dt$ also admits a finite limit. Then $\int_1^{+\infty} \sin(t^2)\, dt$ converges.

Example 2.4.2 *Let the integral be $\int_1^2 \frac{dt}{\sqrt{t-1}}$. We carry out the change of variable $u = \sqrt{t-1}$,*

[1] We recall that $\varphi : J \longrightarrow I$ a diffeomorphism of class $\varphi : J \longrightarrow I$ if φ is a bijective C^1 map, whose reciprocal bijection is also C^1.

which gives $t = u^2 + 1$, $dt = 2udu$ with φ is a diffeomorphism:

$$\begin{array}{cccc} \varphi: & [\sqrt{x-1}, 1] & \longrightarrow & [x, 2] \\ & u & \longrightarrow & t \end{array}$$

$$\lim_{x\longrightarrow 1}\int_1^2 \frac{dt}{\sqrt{t-1}} = \lim_{x\longrightarrow 1}\int_1^2 2udu = 2\left[u\right]_{\sqrt{x-1}}^1 = \lim_{x\longrightarrow 1} 2\left[1-\sqrt{x-1}\right] = 2$$

$\int_0^1 2du$ *converges, which proves that* $\int_1^2 \frac{dt}{\sqrt{t-1}}$ *also admits a finite limit. Then* $\int_1^2 \frac{dt}{\sqrt{t-1}}$ *converges.*

Example 2.4.3 *We will calculate the value of the following two improper integrals:*

$$I = \int_0^{\pi/2} \ln(\sin(t))dt, \qquad J = \int_0^{\pi/2} \ln\left[\cos(t)\right]dt.$$

- Show that the integral I converges: *As* $\ln(\sin(t)) \underset{0^+}{\sim} \ln(t) \leq \frac{1}{\sqrt{t}}$, *let us carry out an integration by parts of* $\int_0^{\pi/2} \ln(t)dt$ *the integral converges. By equivalence* $\int_0^{\pi/2} \ln(\sin(t))dt$ *converges.*

- Check that $I = J$: *We carry out the change of variable* $t = \frac{\pi}{2} - u$. *We have* $dt = -du$ *and a diffeomorphism between* $t \in \left[x, \frac{\pi}{2}\right]$ *and* $u \in \left[\frac{\pi}{2} - x, 0\right]$. *So*

$$\int_x^{\pi/2} \ln(\sin(t))dt = \int_{\frac{\pi}{2}-x}^{0} \ln(\sin(\frac{\pi}{2} - x))\,(-du) = \int_0^{\frac{\pi}{2}-x} \ln\left[\cos(u)\right]du$$

$$I = \int_0^{\pi/2} \ln\left[\sin(t)\right]dt = \lim_{x\longrightarrow 0^+}\int_x^{\pi/2} \ln\left[\sin(t)\right]dt = \lim_{x\longrightarrow 0^+}\int_0^{\frac{\pi}{2}-x} \ln\left[\cos(u)\right]du$$

Cela prouve $I = J$. Donc J converge.

- Calculer $I + J$: *This proves $I = J$. So J converges.*

$$\begin{aligned} I + J &= \int_0^{\pi/2} \ln\left[\sin(t)\right]dt + \int_0^{\pi/2} \ln\left[\cos(t)\right]dt = \int_0^{\pi/2} \ln\left[\sin(t)\right] + \ln\left[\cos(t)\right]dt \\ &= \int_0^{\pi/2} \ln\left[\sin(t).\cos(t)\right]dt = -\frac{\pi}{2}\ln 2 + \int_0^{\pi/2} \ln\left[\sin(2t)\right]dt \end{aligned}$$

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And since $I = J$, we have $2I = -\frac{\pi}{2}\ln 2 + I$. We still have to evaluate $L = \int\limits_0^{\pi/2} \ln[\sin(2t)]\,dt$:

Let us change the variable $u = 2t$, the integral L becomes:

$$\begin{aligned} L &= \frac{1}{2}\int\limits_0^{\pi} \ln[\sin(u)]\,du \\ &= \frac{1}{2}I + \frac{1}{2}\int\limits_{\pi/2}^{\pi} \ln[\sin(u)]\,du \end{aligned}$$

we carry out the change of variable $v = \pi - u$, we will have

$$\begin{aligned} L &= \frac{1}{2}I + \frac{1}{2}\int\limits_{\pi/2}^{0} \ln[\sin(\pi - u)]\,(-dv) \\ &= \frac{1}{2}I + \frac{1}{2}\int\limits_0^{\pi/2} \ln[\sin(v)]\,dv \\ &= \frac{1}{2}I + \frac{1}{2}I \\ &= I. \end{aligned}$$

So, as $2I = \frac{\pi}{2}\ln 2 + L$ and $L = I$, we find:

$$I = J = \int\limits_0^{\pi/2} \ln[\sin(t)]\,dt = -\frac{\pi}{2}\ln 2.$$

2.5 Application of improper integrals

2.5.1 Gamma function

Theorem 2.5.1 (Gamma function Γ) *We call Euler's Gamma function denoted Γ Euler the application:*

$$\begin{array}{rccc} \Gamma: &]0,+\infty[& \longrightarrow & \mathbb{R} \\ & x & \longrightarrow & \Gamma(x) = \int\limits_0^{+\infty} t^{x-1}e^{-t} \end{array}$$

The integral $\int\limits_0^{+\infty} t^{x-1}e^{-t}$ converges for all strictly positive x.

Preuve. Indeed:

$$\int_0^{+\infty} t^{x-1}e^{-t}dt = \int_0^1 t^{x-1}e^{-t}dt + \int_0^{+\infty} t^{x-1}e^{-t}dt = I_1 + I_2.$$

- If $x \geq 1$, the 0 is not an improper, therefore I_1 converges and $\lim\limits_{t \longrightarrow +\infty} t^2 t^{x-1}e^{-t} = 0$ which ensures the convergence of I_2.

- If $0 < x < 1$, we have $t^{x-1}e^{-t} \underset{0^+}{\sim} t^{x-1} = \dfrac{1}{t^{1-x}}, \int\limits_0^1 \dfrac{1}{t^{1-x}}dt$ is a Riemann integral converges if $1 - x < 1$, for all $x > 0$ by equivalence $I_1 = \int\limits_0^1 t^{x-1}e^{-t}dt$ converges, in addition $\int\limits_1^{+\infty} t^{x-1}e^{-t}dt$ converges because $\lim\limits_{t \longrightarrow +\infty} t^2 t^{x-1}e^{-t} = 0$. ■

Propretie 2.5.1 *1)* $\Gamma(x+1)+x\Gamma(x)$*, for all* $x > 0$*. In particular* $\Gamma(2) = 1\Gamma(1) = \int\limits_0^{+\infty} e^{-t}dt = 1.$

2) For all $x > 0$*,* $\Gamma(x+n+1) = x(x+1)(x+2)...(x+n-1)(x+n)\Gamma(x).$

3) For all $n > 0$*,* $\Gamma(x+1) = n!.$

4) $\Gamma(\frac{1}{2})=\sqrt{\pi}.$

2.5.2 Beta Function

Theorem 2.5.2 (Beta function β) *For all strictly positive real numbers p and q, we define Euler's Beta function denoted* β *by:*

$$\beta(p,q) = \int_0^1 u^{p-1}(1-u)^{q-1}du$$

$\beta(p,q)$ *converges if* $p > 0$ *and* $q > 0$*.*

Preuve. Indeed:

$$\beta(p,q) = \int_0^1 u^{p-1}(1-u)^{q-1}du = \int_0^{1/2} u^{p-1}(1-u)^{q-1}du + \int_{1/2}^1 u^{p-1}(1-u)^{q-1}du$$

Near 0, $u^{p-1}(1-u)^{q-1} \sim \dfrac{1}{u^{p-1}}$, so $\int\limits_0^{1/2} u^{p-1}(1-u)^{q-1}du$ converges for $p > 0$.

Near 1, $u^{p-1}(1-u)^{q-1} \sim \dfrac{1}{(1-u)^{1-q}}$, so $\int\limits_{1/2}^1 u^{p-1}(1-u)^{q-1}du$ converges for $q > 0$. ■

Propretie 2.5.2 *1)* $\forall p,q > 0, \beta(p,q) = \beta(q,p).$

2) $\forall p,q>0, \beta\left(p,q\right)=\dfrac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}$.

3) $\forall p,q>0, \beta\left(p,q\right)=\int\limits_{0}^{+\infty}\dfrac{u^{p-1}}{(1-u)^{p+q}}du$.

4) Si $p\notin\mathbb{Z}$,$\beta\left(p,1-p\right)=\Gamma(p)\Gamma(1-p)=\dfrac{\pi}{\sin\pi x}$.