Chapitre 3

DIFFERENTIAL EQUATIONS

Differential equation is an equation : whose unknown is a function (generally denoted $y(x)$ or simply y) and in which appear some of the derivatives of the function (first derivative $\acute{y}$, or derivatives of higher orders $\H{y}$, $y^{(3)}$,...).

3.1 General information on 1st order differential equations

Let's move on to the complete definition of a differential equation and especially a solution of a differential equation.

3.1.1 Definition and Examples

Definition 3.1.1 *Given a function of three variables F, we call a 1^{st} order differential equation any relation of the form :*

$$F(x, y, \acute{y}) = 0, \tag{3.1}$$

between the variable x, the function $y(x)$ and its derivative $\acute{y}(x)$. The function φ, differentiable, is called the solution or integral of the differential equation (3.1) on a set I of $\mathbb{R}$ if

$$\forall x \in I, F\left(x, \varphi(x), \acute{\varphi}(x)\right) = 0.$$

Example 3.1.1 *$\acute{y} + y = x$ admits on $\mathbb{R}$ the solution $\varphi(x) = x - 1$. $x\acute{y} - 1 = 0$ admits on $\mathbb{R}^*$ the solution $\varphi(x)(x) = \ln |x|$.*

Integrating a differential equation means determining all the solutions, specifying, if necessary, the definition set of each.

3.1.2 Cauchy's theorem

If f is continuous and has a continuous derivative with respect to y on an open set Ω de $\mathbb{R}^2$, whatever the point (x_0, y_0) of Ω, there exists a unique solution $\varphi(x)$ of the equation $\acute{y} = f(x, y)$ defined in the neighborhood of x_0 and such that $y(x_0) = y_0$. For given x_0 the solution depends on y_0. The set of solutions of a 1^{st} order differential equation depends on an arbitrary constant λ, $y_\lambda = \varphi(x, \lambda)$. This set of solutions will be called **General Integral**. By giving particular values for λ we obtain particular solutions. The condition $y(x_0) = y_0$ is called **initial condition.**

Example 3.1.2 *Integrate the differential equation $y - \acute{y} = 0$, such that $y(1) = 1$. $y - \acute{y} = 0 \iff \frac{dy}{y} = dx$ hence $y = \lambda e^x$, or $y(1) = 1 \iff 1 = \lambda e$ which gives $\lambda = \frac{1}{e}$. So the solution to the general equation is given $y = e^{x-1}$.*

3.1.3 Geometric interpretation of the solution of a 1st order equation

1. **Contact elements :** Let $y = f(x, y)$. If f verifies Cauchy's hypotheses it defines an application of Ω in $\mathbb{R}$ which associates the $\acute{y}$ derivative with each pair (x, y). $M(x, y) \in \Omega \longrightarrow (MT)$ such that (MT) is the leading coefficient $\acute{y}$. (M, MT) is a contact element. The data of a 1^{st} order equation thus defines a "field" of contact elements in the plane, a curve (C) being an integral curve if all of its contact elements belong to the previous field.

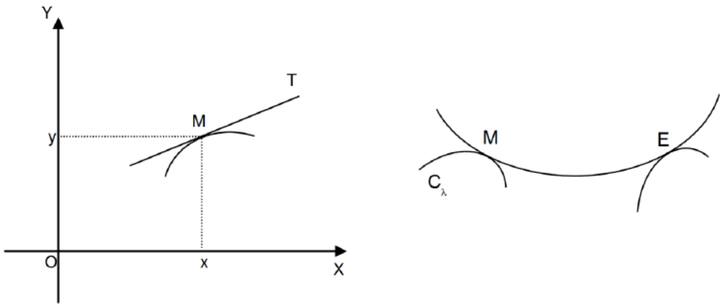

FIG. 3.1 – Contact elements.

2. **Graphical integration :** The construction of neighboring contact elements reveals a polygonal contour which allows an approximate drawing of the integral curves, this is the principle of graphic integration. This plot is also facilitated by the preliminary construction

of isoclines, the set of points of integral curves at which the tangent has a given direction coefficient m. The Cartesian equation of the curves is therefore written $f(x,y)=m$.

Example 3.1.3 *Consider the differential equation $x\acute{y}=2y$. The isoclines with the equation $y=\frac{m}{2}x$ are lines passing through the origin.*

$\frac{dy}{y}=2\frac{dx}{x} \iff y=\lambda x^2$. The integral curves are parabolas with vertex O and axis (oy).

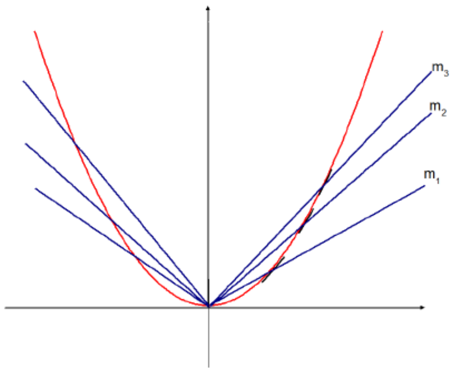

FIG. 3.2 – Graphical integration

3.1.4 Differential equation attached to a family of curves

We have seen that the integral curves of a 1st order differential equation depend on a parameter. Reciprocally a family of curves C_λ depending on a parameter λ and defined by the equation

$$f(x,y,)=0 \tag{3.2}$$

At any point $M(x,y)$ of a certain open $\Omega\subset\mathbb{R}$ passes at least one curve C_λ. The leading coefficient $\acute{y}$ of the tangent at M given by

$$\acute{f}_x(x,y,\lambda)+\acute{f}_y(x,y,\lambda)\acute{y}=0 \tag{3.3}$$

The improvement of λ between equations (3.2) and (3.3) gives a relation which defines the contact elements of the curves C_λ. $F(x,y,\acute{y})=0$, this relation represents the differential equation of the family of curves considered.

1. The discussion of the number of solutions $\acute{y}=0$, of $\acute{y}$ in $F(x,y,\acute{y})=0$ gives the number of integral curves passing through a point and allows a regioning of the plane.

2. The equation $F(x,y,0)$ corresponds to the isocline with zero direction coefficient $\acute{y}=0$, represents the locus of points at tangent parallel to (ox) of integral curves.

3. **The orthogonal trajectories** of the curves C_λ are solution of the differential equation $F(x,y,\frac{-1}{\acute{y}})=0$ obtained by changing $\acute{y}$ to $\frac{-1}{\acute{y}}$ in the equation differential of C_λ, the tangent in $M(x,y)$ of C_λ has slope $m=\acute{y}=f(x,y)$, if there exists a curve Γ orthogonal in M to C_λ, the tangent has slope $\acute{m}=\frac{-1}{m}=\frac{-1}{\acute{y}}$. The differential equation of the Γ curves is therefore written $\frac{-1}{\acute{y}}=f(x,y)$ or $F(x,y,\frac{-1}{\acute{y}})=0$.

Example 3.1.4 *Let the sheaf of circles with base points A and B of equation*

$$x^2+y^2-2\lambda y=1.$$

Differentiating with respect to x, we obtain $2x+2y\acute{y}-2\lambda\acute{y}=0$ eliminating λ gives $(x^2+\acute{y}-2y-1)\acute{y}-2x=0$ the beam differential equation.

Example 3.1.5 *Orthogonal trajectories of the family of hyperbolas H_λ of equations $xy=\lambda$. The hyperbola equation is written $x\acute{y}+y=0$. The equation of the orthogonal trajectories is therefore C such that : $-x\frac{1}{\acute{y}}+y=0$ ou $x-y\acute{y}=0$. This equation fits directly into the form :*

$$x-y\acute{y}=0 \implies x=y\acute{y} \iff x=y\frac{dy}{dx}$$
$$xdx=ydy \implies \frac{x^2}{2}=\frac{y^2}{2}+k$$
$$\implies x^2-y^2=k$$

The orthogonal trajectories are therefore a family of hyperbolas with center 0 and axes (ox) and (oy).

3.2 Integration of 1st order differential equations

3.2.1 Equation with separable variables

A differential equation with separable variables is a 1^{st} order equation that can be written in the form

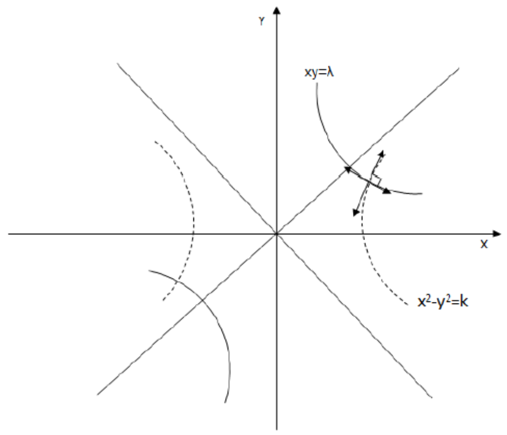

FIG. 3.3 – Orthogonal trajectories of the family of hyperbolas.

$$\acute{y} = \frac{f(x)}{g(x)} \qquad \text{or} \qquad g(y)dy = f(x)dx,$$

the functions f and g are assumed to be continuous, from where $\int g(y)dy = \int f(x)dx + \lambda$ we obtain $G(y) = F(x) + \lambda$

Example 3.2.1

$$\begin{aligned}
\acute{y} + y &= a, \qquad a \in \mathbb{R} \\
\frac{dy}{dx} &= a - y \implies \frac{dy}{a-y} = -dx \\
\ln|y-a| &= x + \lambda \qquad we\ obtain \qquad y = a + Ce^{-x}
\end{aligned}$$

Example 3.2.2

$$\begin{aligned}
y - 2x\acute{y} &= 1, \\
\acute{y} &= \frac{y-1}{2x} \implies \frac{dy}{y+1} = \frac{1}{2}dx \\
we\ obtain\ y - 1 &= \lambda\sqrt{|x|}
\end{aligned}$$

3.2.2 Homogeneous equation

We call a homogeneous differential equation of the 1^{st} order an equation of the form $F(x, y, \acute{y}) = 0$, in which the change of x into λx and y into λy leaves $\acute{y}$ invariant.

Geometric interpretation : Let (M, MT) be a contact element. The equation being homogeneous, the tangent in $\acute{M}(\lambda x, \lambda y)$ is parallel to (MT), that is to say that the set of integral curves is globally invariant in all homothety of center o, $\acute{y}$ therefore depends on $\dfrac{y}{x}$. We assume that $\acute{y} = f(\dfrac{y}{x})$.

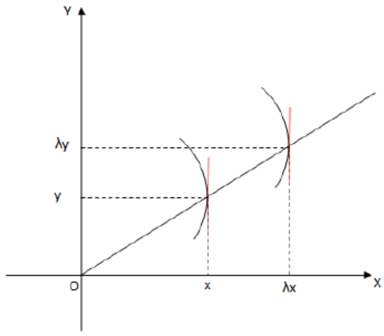

FIG. 3.4 – Geometric interpretation of homogeneous equation.

Integration method : We set $\dfrac{y}{x} = t$ or $y = tx$, we obtain $dy = tdx + xdt$, from where $\dfrac{dy}{dx} = t + x\dfrac{dt}{dx} = f(t)$. By separating the variables, we obtain $\dfrac{dx}{x} = \dfrac{dt}{f(t) - t}$ for $f(t) - t \neq 0$. Then $\ln x = \int \dfrac{dt}{f(t) - t} = \varphi(t)$. We find a parametric representation of the integral curves in the form :

$$\begin{cases} x = \lambda e^{\varphi(t)} \\ y = \lambda t e^{\varphi(t)} \end{cases}$$

Example 3.2.3 *Let the differential equation be* $\acute{y} = \dfrac{y^2 - x^2}{2xy}$, *We set* $\dfrac{y}{x} = t$, *then* $\acute{y} = \dfrac{y^2 - x^2}{2xy}$ *therefore* $y = xt \implies t + x\dfrac{dt}{dx} = \frac{t^2-1}{2t}$ *hence* $x\dfrac{dt}{dx} = \dfrac{-t^2 + 1}{2t}$ *and* $\dfrac{-2tdt}{t^2 + 1} = \dfrac{dx}{x}$ *or* $\dfrac{-d(t^2 + 1)}{t^2 + 1} = \dfrac{dx}{x}$. *So*

$$\begin{cases} x = \dfrac{\lambda}{t^2 + 1} \\ y = \dfrac{\lambda t}{t^2 + 1} \end{cases}$$

3.2.3 Linear equations

We call a linear differential equation of the 1^{st} order an equation in the form

$$a(x)\acute{y} + b(x) = f(x) \tag{3.4}$$

in which the functions a, b and f are assumed to be continuous on the same subset $I \subset \mathbb{R}$. a and b are the coefficients of the equation. We call an equation without an associated second member the equation

$$a(x)\acute{y}+b(x)=0 \tag{3.5}$$

Theorem 3.2.1 (Fundamental theorem) *The general solution of the linear differential equation of the* 1^{st} *order (3.4) is obtained by adding to a particular solution of the complete equation (3.4) the general solution of the equation without an associated second member(3.5).*

Integration of the equation *(3.5)* : The equation $a(x)\acute{y}+b(x)y=0$ has separable variables, we can write : $\frac{dy}{y}=\frac{-b(x)}{a(x)}dx$ from where $\ln y=-\int\frac{b(x)}{a(x)}dx$, i.e. $y(x)=\lambda y_1(x)$ with $y_1(x)=e^{-\int\frac{b(x)}{a(x)}dx}$.

Example 3.2.4 *Let the linear differential equation be* $(1+x^2)\acute{y}-xy=1$. *It is easy to verify that* $y=x$ *is a particular solution of the given equation. Solving the equation without a second member,* $(1+x^2)\acute{y}-xy=0$. *We have*

$$\frac{dy}{dx}=\frac{xdx}{1+x^2}=\frac{1}{2}\left(\frac{d(1+x2)}{1+x^2}\right),$$

hence $y=\sqrt{1+x^2}$. *The general solution is* $x+\lambda\sqrt{1+x^2}$.

3.2.4 Method of variation of the constant

In the case where we do not know the particular solution of the linear differential equation we use the method of variation of the constant. Let $y=\lambda y_1(x)$ be the general solution of the equation without a second member, we propose to seek if there exist solutions of the equation of the form $y=\lambda(x)y_1(x)$, $\lambda(x)$ now represents a function differentiable from the variable x, and we have

$$\begin{aligned}
&\acute{y}=\acute{\lambda}(x)y_1(x)+(x)\acute{y}_1(x)\\
&a(x)\acute{y}+b(x)y=a(x)[\acute{\lambda}(x)y_1(x)+\lambda(x)\acute{y}_1(x)]+b(x)\lambda(x)y_1(x)=f(x)\\
&\lambda(x)a(x)\acute{y}_1(x)+\lambda(x)[a(x)\acute{y}_1(x)+b(x)y_1(x)]=f(x)
\end{aligned}$$

Or $a(x)\acute{y}_1(x)+b(x)y_1(x)=0$ puis donc $\acute{\lambda}(x)=\frac{f(x)}{a(x)y_1(x)}$ d'où $\lambda(x)=\varphi(x)+C$ avec $\varphi(x)=\int\frac{f(x)}{a(x)y_1(x)}dx$. C'est-à-dire $y=\varphi(x)y_1(x)+Cy_1(x)$.

Example 3.2.5 *Integrate the equation :* $\acute{y}\cos x+y\sin x=x$. *The equation without a second member :* $\acute{y}\cos x+y\sin x=0$ *admits as a general solution* $y=\lambda\cos x$. *We put* $y(x)=\lambda(x)\cos x$, *then* $\acute{y} = \acute{\lambda}(x)\cos x - \lambda(x)\sin x = x$ *therefore* $[\acute{\lambda}(x)\cos x - \lambda(x)\sin x]\cos x + \lambda(x)\cos x \sin x = x$, *therefore* $\acute{\lambda}(x) = \dfrac{x}{\cos^2 x}$ *from where* $\lambda(x) = \int \dfrac{x}{\cos^2 x}dx$ *after integration by parts we obtain* $\lambda(x) = x\tan x - \int \tan x dx = x\tan x + \ln|\cos x| + C$. *Eventually*

$$y = C\cos x + x\sin x + \cos x \ln|\cos x|.$$

3.2.5 Equation reducing to a linear equation

1. **Bernoulli equation :** Bernoulli equation is a 1^{st} order differential equation of the form

$$a(x)\acute{y} + b(x)y = f(x)y^{\alpha}$$

- If $\alpha = 1$ the equation is linear.

- If $\alpha \neq 1$, by dividing both sides of the equation by y^{α}, we obtain $a(x)\acute{y}y^{-\alpha} + b(x)y^{1-\alpha} = f(x)$. We set $z = y^{\alpha}$ then $\acute{z} = (1-\alpha)y^{-\alpha}\acute{y}$, hence the linear equation $\dfrac{a(x)}{1-\alpha}\acute{z} + b(x)z = f(x)$.

Example 3.2.6 *Either*

$$y - x\acute{y} = 2xy^2 \tag{3.6}$$

By dividing by y^2 *we obtain,* $\dfrac{1}{y} - x\dfrac{\acute{y}}{y^2} = 2x$, *we set* $z = \dfrac{1}{y}$ *soit* $\acute{z} = -\dfrac{\acute{y}}{y^2}$, *the equation becomes :*

$$z + x\acute{z} = 2x \tag{3.7}$$

we notice that $z = x$ *is a particular solution of (3.6), the equation without a second member gives* $\dfrac{dz}{z} = -\dfrac{dx}{x}$, *we obtain* $z = \dfrac{\lambda}{x}$. *Then* $z = x + \dfrac{\lambda}{x}$ *is the general solution of (3.7), and consequently the general solution of (3.6) is* $\dfrac{1}{x + \frac{\lambda}{x}}$.

2. **Riccati equation :** We call Riccati equation a differential equation of the 1^{st} order of the form

$$\acute{y} = a(x)y^2 + b(x)y + c(x)$$

We can only integrate this equation when we know a particular solution. Suppose y_1 is a particular solution, then

$$\acute{y}_1 = a(x)y_1^2 + b(x)y_1 + c(x)$$
$$\acute{y} - \acute{y}_1 = a(x)\left[y^2 - y_1^2\right] + b(x)(y - y_1).$$

By setting $y - y_1 = z$ we obtain

$$\acute{z} = a(x)z(2y_1 + z) + b(x)z$$
$$\acute{z} = a(x)z^2 + [2a(x) + y_1 + b(x)]z,$$

we have led to a Bernoulli equation.

Example 3.2.7 *Integrate $\acute{y} = y^2 - 2xy + x^2 + 1$. We notice that $y = x$ is a particular solution, we put $y = x + z$, then $1 + \acute{z} = (x + z)^2 - 2x(x + z) + x^2 + 1$, we obtain $\acute{z} = z^2$. We integrate this equation, We write $\frac{\acute{z}}{z^2} = 1$ i.e. $\left(\frac{1}{z}\right)' = -1$ hence $\frac{1}{z} = -x + \lambda$, finally* $y = x + \frac{1}{\lambda - x}$.

3.3 Second order differential equations

3.3.1 Definition and Examples

Definition 3.3.1 *We call a 2^{nd} order differential equation any relation of the form : $F(x, y, \acute{y}, \H{y}) = 0$ between the variable x, the function $y(x)$ and its first and second derivatives. The function φ, twice differentiable, is then called solution or integral over I subset of $\mathbb{R}$ if $\forall x \in I, F(x, \varphi(x), \acute{\varphi}(x), \varphi(x)) = 0$*

Example 3.3.1 *The equation $\H{y} + \omega^2 y = 0$, admits for solution on $\mathbb{R}$, $\varphi_1(x) = \cos x$ and $\varphi_2(x) = \sin x$*

Example 3.3.2 *The equation $\H{y} = 0$, admits for solution on $\mathbb{R}$, any polynomial of the 1^{er} degree $\varphi(x) = ax + b$ with (a, b) arbitrary.*

We will admit without demonstration that, under certain hypotheses, a differential equation of the 2^{nd} order admits an infinity of solutions depending on two arbitrary constants λ_1 and λ_2 : $y = \varphi(x, \lambda_1, \lambda_2)$, all of these solutions constitutes the general integral and represents the equation of a family of curves of two parameters C_{λ_1,λ_2} called integral curves.

3.3.2 Equation reducing to 1st order

1. **Equation not containing y :** Consider a differential equation of 2^{nd} order $F(x, y, \acute{y}, \ddot{y}) = 0$. By setting $z = \acute{y}$ the equation becomes $F(x, z, \acute{z}) = 0$.

Example 3.3.3 *Let the equation* $\acute{\acute{y}}+\acute{y}^2=0$*, by setting* $z=\acute{y}$ *we obtain* $\acute{z}+z^2=0$*, then* $-\dfrac{dz}{z^2}=dx \implies \dfrac{1}{z}=x-x_0$ *(*x_0 *constant), therefore* $z=\dfrac{dy}{dx}=\dfrac{1}{x-x_0}$*, hence* $dy=\dfrac{dx}{x-x_0} \implies y-y_0=\ln|x-x_0|$*, the solution depends on two constants* x_0, y_0*.*

2. **Equation not containing x :** Consider a differential equation of 2^{nd} order $F(x,y,\acute{y},\ddot{y})=0$. If we consider that $\acute{y}$ as a function of y, by setting $\acute{y}=z(y)$ we obtain

$$\ddot{y}=\frac{d\acute{y}}{dx}=\frac{dz}{dx}=\frac{dz}{dy}.\frac{dy}{dx}=\frac{dz}{dy}.z$$

y therefore plays the role of variable, and the equation becomes $F(y,z,z\dfrac{dz}{dy})=0$, it is a 1^{er} order equation for z. Let $z=\varphi(y,\lambda_1)$ be the integral of this equation, then $z=\dfrac{dy}{dx}=\varphi(y,\lambda_1)$ or $\dfrac{dy}{\varphi(y,\lambda_1)}=dx$, then by integrating $x=f(y,\lambda_1)+\lambda_2$ with $f(y,\lambda_1)=\int\dfrac{dx}{\varphi(y,\lambda_1)}$.

Example 3.3.4 *Consider the equation* $y^2\acute{\acute{y}}+\acute{y}=0$*. By setting* $\acute{y}=z(y)$ *or* $\acute{\acute{y}}=z\acute{z}$*, the equation becomes* $y^2z\acute{z}+z=0$*, discarding the banal solution* $z=0$ *(corresponding to* $y=k$*), we have* $y^2\acute{z}+1=0 \implies \acute{z}=\dfrac{1}{y}+\lambda_1$*, we have brought to the first order equation* $\dfrac{dy}{dx}=\dfrac{1}{y}+\lambda_1$*, then*

$$dx=\frac{dy}{\frac{1}{y}+\lambda_1}=\frac{ydy}{\lambda_1 y+1}=\frac{1}{\lambda_1}(1-\frac{1}{\lambda_1 y+1})dy.$$

From where $x=\dfrac{1}{\lambda_1}y-\dfrac{1}{\lambda_1^2}\ln|\lambda_1 y+1|+\lambda_2$.

3.3.3 Second order linear differential equation

Definition 3.3.2 *We call a linear differential equation of the* 2^{nd} *order an equation of the form*

$$a(x)\acute{\acute{y}}+b(x)\acute{y}+c(x)y=f(x). \tag{3.8}$$

a,b,c,f *are functions on* $I\subset\mathbb{R}$*. (*a,b,c *are called coefficients of the equation). We associate with this equation the so-called equation without a second member*

$$a(x)\acute{\acute{y}}+b(x)\acute{y}+c(x)y=0 \tag{3.9}$$

Theorem 3.3.1 (Fundamental theorem) *is obtained by adding to a particular integral of* *the complete equation the integral of the equation without a second member. If y is the general solution of (3.8) and y_0 is a particular solution of (3.8), and Y is the general solution of (3.9), then $y = y_0 + Y$.*

Integration of the equation without a second member

1. **Case where we know two particular solutions** : If y_1, y_2 are two solutions of (3.8), then $y_1 + y_2$ and λy_1 with ($\lambda \in \mathbb{R}$ are solutions of (3.9). y_1 and y_2 are said to be linearly independent if there do not exist two non-zero constants λ_1, λ_2 such that : $\forall x \in I, \lambda_1 y_1(x) + \lambda_2 y_2(x) = 0$ this results in $\lambda_1 \acute{y}_1(x) + \lambda_2 \acute{y}_2(x) = 0$. this results in

$$\begin{cases} \lambda_1 y_1(x) + \lambda_2 y_2(x) = 0 \\ \lambda_1 \acute{y}_1(x) + \lambda_2 \acute{y}_2(x) = 0 \end{cases}$$

admits only $(\lambda_1, \lambda_2) = (0, 0)$ as a solution. So the determinant

$$w(x) = \begin{vmatrix} y_1(x) & y_2(x) \\ \acute{y}_1(x) & \acute{y}_2(x) \end{vmatrix}$$

called Wronskian of y_1, y_2 is not zero; On the contrary if $w(x) = 0$ then y_1 and y_2 are linearly dependent.

Theorem 3.3.2 *The dimension of the vector space of the solutions of the equation $a(x)\acute{\acute{y}} + b(x)\acute{y} + c(x)y = 0$ is equal to 2.*

Consequence : If y_1 and y_2 are two linearly independent solutions of the equation without a second member, the general solution is written $Y(x) = \lambda_1 y_1(x) + \lambda_2 y_2(x)$, λ_1, λ_2 are two arbitrary constants.

Example 3.3.5 *$\acute{\acute{y}} + wy = 0, y_1 = \cos wx$ and $\sin wx$ are two independent solutions because*

$$w(x) = \begin{vmatrix} \cos wx & \sin wx \\ -w \sin wx & w \cos wx \end{vmatrix} = w \neq 0$$

$Y = \lambda_1 \cos wx + \lambda_2 \sin wx$.

Example 3.3.6 *$\acute{\acute{y}} - wy = 0$, $y_1 = \cosh wx$ and $\sinh wx$ are two independent solutions* $Y = \lambda_1 \cosh wx + \lambda_2 \sinh wx$.

2. **Case where we only know a particular solution :** Let y_1 be a solution of (3.9), we set $y = y_1 z$ therefore $\acute{y} = \acute{y}_1 z + y_1 \acute{z}$ and $\H{y} = \H{y}_1 z + 2\acute{y}_1 \acute{z} + y_1 \H{z}$. Let $a(x)[\H{y}_1 z + 2\acute{y}_1 \acute{z} + y_1 \H{z}] + b(x)[\acute{y}_1 z + y_1 \acute{z}] + c(x)yz = 0$ taking into account $a\H{y}_1 + b\acute{y}_1 + cy_1 = 0$, then we obtain $ay_1 \H{z} + (2a\acute{y}_1 + by_1)\acute{z} = 0$, is an equation that can be integrated easily.

Example 3.3.7 *Let the equation* $x^2 \H{y} + x\acute{y} - y = 0, y_1 = x.y = xz \implies \acute{y} = z + x\acute{z} \implies \H{y} = 2\acute{z} + xz$*, then* $x^2(2\acute{z} + x\H{z}) + x(z + x\acute{z}) - xz = 0$ *where* $x\H{z} + 3\acute{z} = 0$ *or* $\frac{\H{z}}{\acute{z}} = -3x$*, which gives* $\acute{z} = \frac{\lambda_1}{x^3}$*, so* $z = -\frac{\lambda_1}{x^2} + \lambda_2$*, we obtain* $y = \frac{\lambda_1}{2}\frac{1}{x} + \lambda_2 x$ *or even* $y = C_1 \frac{1}{x} + C_2 x$*.*

3.3.4 Integration of the full equation

1. **In the case where we know a particular solution y_0** : Simply apply the fundamental theorem, $y = y_0 + Y$.

Example 3.3.8 *Given the equation* $x^2 \H{y} + x\acute{y} - y = x^3$*, the solution of the equation without a second member is* $Y = C_1 \frac{1}{x} + C_2 x$*, we are looking for a particular solution in the form of a polynomial of* $3^{\text{ème}}$ *degree* $y_0 = ax^3$ *therefore* $\acute{y}_0 = 3ax^2$*, and* $\H{y}_0 = 6ax$ *which gives* $a = \frac{1}{8}$*, therefore the general solution of the given equation is* $y = \frac{x^3}{8} + C_1 \frac{1}{x} + C_2 x$*.*

2. **If we do not know a particular solution :** We apply the method of variation of constants. Let y_1, y_2 be two independent solutions of (3.9) $y = \lambda_1 y_1 + \lambda_2 y_2$ the general solution of the equation without a second member. We set $y = \lambda_1(x) y_1 + \lambda_2(x) y_2$ where λ_1, λ_2 are functions, then $\acute{y} = \acute{\lambda}_1(x) y_1 + \acute{\lambda}_2(x) y_2 + \lambda_1(x) \acute{y}_1 + \lambda_2(x) \acute{y}_2$, by imposing the condition $\acute{\lambda}_1(x) y_1 + \acute{\lambda}_2(x) y_2 = 0$ on obtient $\H{y} = \acute{\lambda}_1(x)\acute{y}_1 + \acute{\lambda}_2(x)\acute{y}_2 + \lambda_1(x)\H{y}_1 + \lambda_2(x)\H{y}_2$, or by reporting in (3.8)

$$a\left[\acute{\lambda}_1(x)y_1 + \acute{\lambda}_2(x)y_2 + \lambda_1(x)\acute{y}_1(x) + \lambda_2(x)\acute{y}_2(x)\right] + b\left[\lambda_1(x)\acute{y}_1 + \lambda_2(x)\acute{y}_2\right] + c\left[\lambda_1(x)y_1 + \lambda_2(x)y_2\right] = f(x),$$

but we have

$$\begin{cases} a\H{y}_1 + b\acute{y}_1 + cy_1 = 0 \\ a\H{y}_2 + b\acute{y}_2 + cy_2 = 0 \end{cases}$$

we obtain

$$\begin{cases} \acute{\lambda}_1(x)y_1(x) + \acute{\lambda}_2(x)y_2(x) = 0 \\ \acute{\lambda}_1(x)\acute{y}_1(x) + \acute{\lambda}_2(x)\acute{y}_2(x) = \frac{1}{a(x)} f(x) \end{cases}.$$

$$w(x) = \begin{vmatrix} y_1 & y_2 \\ y'_1 & y'_2 \end{vmatrix} \neq 0 \text{ because } y_1,\, y_2 \text{ are linearly independent.}$$

Example 3.3.9 *Consider the equation* $y''+y = \tan x$. *We have* $y''+y = 0 \implies y = \lambda_1 \cos x + \lambda_2 \sin x$. *Then* $y' = -\lambda_1 \sin x + \lambda_2 \cos x$ *if*

$\lambda'_1 \cos x + \lambda'_2 \sin x = 0$. *And* $y''= -\lambda'_1 \sin x + \lambda'_2 \cos x - \lambda_1 \cos x + \lambda_2 \sin x$. *By reporting in the equation we obtain :* $y''+y' = -\lambda'_1 \sin x + \lambda'_2 \cos x = \tan x$, *we have the system :*

$$\begin{cases} \lambda'_1 \cos x + \lambda'_2 \sin x = 0 \\ -\lambda'_1 \sin x + \lambda'_2 \cos x = \tan x = 0 \end{cases}$$

hence $\lambda'_1 = -\dfrac{\sin x}{\cos x} = \cos x - \dfrac{1}{\cos x}$, et $\lambda'_2 = \sin x$, Then

$$\lambda_1 = \sin x - \ln|\tan(\frac{x}{2} + \frac{\pi}{4})| + C_1$$
$$\lambda_2 = \cos x + C_2.$$

The general solution is written : $y = C_1 \cos x + C_2 \sin x - \cos x \ln|\tan(\frac{x}{2} + \frac{\pi}{4})|$.

3.4 Linear equation with constant coefficients

Definition 3.4.1 *We call a linear differential equation of the second order with constant coefficients a differential equation of the form*

$$ay'' + by' + cy = f(x) \tag{3.10}$$

in which a, b, c *are constants.*

We associate with (3.10) the equation without a second member

$$ay'' + by' + cy = 0 \tag{3.11}$$

If y_0 is a solution of (3.11) and Y the general solution of (3.11), then $y = y_0 + Y$.

3.4.1 Integration

We set $y = e^{rx}$, then $\acute{y} = re^{rx}$ et $\ddot{y} = r^2 e^{rx}$, by reporting in (3.11) we obtain $ar^2 e^{rx} + bre^{rx} + ce^{rx} = 0$ or $e^{rx} \neq 0$ so we obtain

$$ar^2 + br + c = 0. \tag{3.12}$$

Then $y = e^{rx}$ is solution of the differential equation if and only if r is root of (3.12).

<u>*Discussion*</u> :

1. If (3.12) admits two different roots $r_1 \neq r_2$, then $y1 = e^{r_1 x}$ and $y2 = e^{r_2 x}$ are two particular integrals of (3.12) linearly independent because $r_1 \neq r_2$, the general integral is written $y = \lambda_1 e^{r_1 x} + \lambda_2 e^{r_2 x}$.

2. If $\Delta < 0$, r_1 and r_2 are complex conjugates, the general solution is written $y = \lambda_1 e^{r_1 x} + \lambda_2 e^{r_2 x}$, we choose $\lambda_2 = \overline{\lambda_1}$ then $y = \lambda_1 e^{r_1 x} + \overline{\lambda_1} e^{\overline{r_1 x}} = 2\mathcal{R}e(\lambda_1 e^{r_1 x})$.

   Example 3.4.1 *a)* $\ddot{y} - 2\acute{y} - 3y = 0$, *we have* $r^2 - 2r - 3 = 0 \implies r_1 = -1, r_2 = 3$, *then* $y = \lambda_1 e^{-x} + \lambda_2 e^{3x}$.

   b) $\ddot{y} - 2\acute{y} + 5y = 0$, so $r^2 - 2r + 5 = 0 \implies r_1 = 1 - 2i, r_2 = 1 + 2i$, hence $y = e^x(\mu_1 \cos 2x + \mu_2 \sin 2x)$.

3. If (3.12) admits a double root $r_1 = r_2 = \frac{-b}{2a}$, $y = e^{rx}$ is a particular solution. We look for the general solution in the form $y = e^{rx} z$, where z is an unknown function of x, we have $\acute{y} = e^{rx}(rz + \acute{z})$ and $\ddot{y} = e^{rx}(r^2 z + 2r\acute{z} + \ddot{z})$, by transferring into the equation we obtain : $e^{rx} a \left[(r^2 z + 2r\acute{z} + \ddot{z}) + b(rz + \acute{z}) + cz\right] = 0$, therefore $e^{rx}\left[ar^2 + br + cz + (2ar + b)\acute{z} + a\ddot{z}\right] = 0$, then $\ddot{z} = 0$, hence $z = \lambda_1 x + \lambda_2$, he general solution is given by $y = e^{rx}(\lambda_1 x + \lambda_2)$.

   Example 3.4.2 $\ddot{y} + 4\acute{y} + 4y = 0$, *on a* $r^2 + 4r + 4 = 0 \implies r_1 = r_2 = -2$, *then* $y = e^{-2x}(\lambda_1 x + \lambda_2)$.

3.4.2 Integration of the full equation

The solution of the equation without a second member being assumed to be known, we can :

- Either use the constant variation method.

- Either look for a particular solution of degree n of (3.10).

1. <u>$f(x) = P_n(x)$ **where** P_n **is a polynomial of degree** n</u>. It is natural to look for a particular solution in the form of a polynomial :

1) of degree n if $c \neq 0$.

2) of degree $n+1$ if $c = 0$ and $b \neq 0$.

3) of degree $n+2$ if $c = 0$ and $b = 0$.

Example 3.4.3 *$\acute{\acute{y}}$–$2\acute{y} - 3y = 3x^2 + 1$, we look for the particular solution in the form $y = \alpha x^2 + \beta x + \gamma$, then $\acute{y} = 2\alpha x + \beta$ and $\acute{\acute{y}} = 2\alpha$, we find $\alpha = -1, \beta = \frac{4}{3}, \gamma = \frac{-17}{3}$, the general solution is $y = \lambda_1 e^{-x} + \lambda_2 e^{3x} - x^2 + \frac{4}{3}x - \frac{-17}{3}$.*

2. <u>$f(x) = e^{mx}P_n(x)$**, with** $(m \in \mathbb{C}$</u>), we look for a particular integral in the form $y = e^{mx}z(x)$, then $\acute{y} = e^{mx}(mz + \acute{z})$, and $\acute{\acute{y}} = e^{mx}(m^2z^2 + 2m\acute{z} + \acute{\acute{z}})$, or after simplification by e^{mx}, $a\acute{\acute{z}} + (2am + b)\acute{z} + (am^2 + bm + c)z = P_n(x)$. We find ourselves brought back to the previous case, we will therefore take for $z(x)$ a polynomial :

   - of degree n if $am^2 + bm + c \neq 0$ that is to say if m is not the root of the characteristic equation.

   - of degree $n+1$ if, $am^2 + bm + c = 0$ and $2am + b \neq 0$ (m simple root).

   - of degree $n+2$ if, $am^2 + bm + c = 0$ and $2am + b = 0$ (m double root).

Example 3.4.4 *$\acute{\acute{y}}$–$\acute{y} = 2xe^x$, here $m = 1$, we have $m = 1$ is a simple root of the characteristic equation $r^2 - 1$. We are looking for a particular solution in the form $y = e^x z(x)$ where $z(x)$ is a polynomial of degree 2, n here equal to 1 since $P_n(x) = 2x$, it is the case $(n+1)$. $z(x) = \alpha x^2 + \beta x + \gamma$, therefore $\acute{z} = 2\alpha x + \beta$ and $\acute{\acute{z}} = 2\alpha$, by entering into the equation, and after calculation we find $\alpha = \frac{1}{2}, \beta = -\frac{1}{2}$, the particular solution is $y = e^x(\frac{x^2}{2} - \frac{x}{2})$. The general solution is therefore $y = \lambda_1 e^x + \lambda_2 e^{-x} + \frac{e^x}{2}(x^2 - x)$.*