Series

Chapitre 4

NUMERICAL SERIES

The notion of sequence is closely linked to that of "series", that is to say linked to the problem of the summation of an infinity of terms. It was the Greeks in the 5^{th} century BC who began to glimpse the notion of infinity. It was only in the 16^{th} century that infinity took on its full meaning. The notion of series comes from the fact that for an infinite list of real numbers, that is to say for a sequence $(U_n)_{n\in\mathbb{N}}$, we pose the problem of considering "the sum" of all the elements of this sequence of numbers : $U_0 + U_1 + ... + U_n + ...$How can we give meaning to a sum of an infinite number of terms.

4.1 General information on numerical series

4.1.1 Definitions

Definition 4.1.1 *To any sequence $(U_n)_{n\in\mathbb{N}}$ of real or complex numbers, it is possible to associate a sequence $(S_n)_{n\in\mathbb{N}}$ defined by : $\forall n \in \mathbb{N}, S_n = \sum_{k=0}^{n} U_k$. Conversely, to any sequence $(S_n)_{n\in\mathbb{N}}$, it is possible to associate a sequence $(U_n)_{n\in\mathbb{N}}$ defined by : $U_0 = Sl$ and $\forall n \in \mathbb{N}, U_{n+1} = S_{n+1} - S_n$.*

We call series the sequence $(S_n)_{n\in\mathbb{N}}$ and we note : $\sum_{n\in\mathbb{N}} U_n$ or by $\sum_n U_n$ or by $\sum U_n$.

Definition 4.1.2 *Let $(U_n)_{n\in\mathbb{N}}$ be a sequence of real or complex numbers. The sum $S_n = \sum_{k=0}^{n} U_k$ is called the partial sum of rank n of the series $\sum U_n$.*

- The sequence $(S_n)_{n\in\mathbb{N}}$ is called the sequence of partial sums of the series $\sum U_n$.

- The sequence $(U_n)_{n\in\mathbb{N}}$ is called a sequence associated with the series $\sum U_n$. U_n is said to be a general term of the series $\sum U_n$.

4.1.2 Convergence of a series

Definition 4.1.3 - *If the sequence of partial sums $(S_n)_{n\in\mathbb{N}}$ of a series $\sum U_n$ is convergent with finite limit S, we will say that the series is convergent and we will note $\sum_{n\in\mathbb{N}} U_n = S$.*

- If the sequence $(S_n)_{n\in\mathbb{N}}$ tends towards $+\infty$ or towards $-\infty$, we will say that the series is divergent of the first kind and we will note $\sum_{n\in\mathbb{N}} U_n = \pm\infty$.

- If the sequence $(S_n)_{n\in\mathbb{N}}$ does not converge towards a finite limit, nor towards $\pm\infty$, owe will say that the series is divergent of the second kind.

Remark 4.1.1 *Any finite series is convergent since the sequence of partial sums is constant from a certain rank. $S_n = U_0 + U_1 + ... + U_p \qquad \forall n \geq p$.*

Remark 4.1.2 *(a) We say that $\sum_{n\in\mathbb{N}} U_n$ and $\sum_{n\geq n_0} U_n$, where $n \in \mathbb{N}$, are of the same nature.*

(b) In the case where the series is convergent, the symbol $S = \sum_{n\in\mathbb{N}} U_n$ designates both the series and also the result of this series.

(c) The series $\sum_{n\in\mathbb{N}} U_n$ converges to S if : $\forall \epsilon > 0, \exists N_0 \in \mathbb{N}/, n \geq N_0 \Longrightarrow |S - S_n| \leq \epsilon$.

We use the definition of the convergence of a sequence because the partial sum $(S_n)_{n\in\mathbb{N}}$ is none other than a sequence, and we can write $\forall \epsilon > 0, \exists N_0 \in \mathbb{N}/, n \geq N_0 \Longrightarrow |\sum_{k=n}^{+\infty}| \leq \epsilon$.

Remark 4.1.3 *The sequence of partial sums (S_n) can be divergent by not having a limit when $n \longrightarrow \infty$ or by not having a finite limit for example $(S_n) = \exp(in)$ and $S_n = \ln n$.*

The nature of a series (convergence or divergence) does not depend on the first terms of the series, which means that the series $\sum_{n>0} u_n$ and $\sum_{n>N} u_n$ converge (or diverge) at the same time.

Proposition 4.1.1 *For the series $\sum_{n\in\mathbb{N}} U_n$ to be convergent, it is necessary that the associated sequence $(U_n)_{n\in\mathbb{N}}$ be convergent with zero limit. However, this condition is not sufficient, there exist divergent series whose associated sequence $(U_n)_{n\in\mathbb{N}}$ is convergent to 0.*

Example 4.1.1 *We consider the series $\sum_{n>0} e^{\frac{1}{n}}$, we know that $\lim\limits_{n\longrightarrow+\infty} e^{\frac{1}{n}} = 1 \neq 0$, therefore the given series is divergent.*

Definition 4.1.4 *a) We call the sum of two numerical series $\sum_{n\in\mathbb{N}} U_n$ et $\sum_{n\in\mathbb{N}} V_n$, the general term series $\forall n \in \mathbb{N}, W_n = U_n + V_n$ from where $\sum_{n\in\mathbb{N}} W_n = \sum_{n\in\mathbb{N}} U_n + \sum_{n\in\mathbb{N}} V_n$.*

b) We call multiplication of a numerical series $\sum_{n\in\mathbb{N}} U_n$ by a non-zero scalar λ, ($\lambda \in \mathbb{R}^*$ or $\lambda \in \mathbb{C}^*$) the general term series $\forall n \in \mathbb{N}, W_n = \lambda U_n$ hence $\sum_{n\in\mathbb{N}} W_n = \sum_{n\in\mathbb{N}} (\lambda U_n)$.

c) We call the product of two series $(U_n)_{n\in\mathbb{N}}$ and $(V_n)_{n\in\mathbb{N}}$ the general term series ; $\forall n \in \mathbb{N}, W_n = \sum_{k=0}^{n} U_k V_{n-k}$ hence $\sum_{n\in\mathbb{N}} W_n = \sum_{n\in\mathbb{N}} \left(\sum_{k=0}^{n} U_k V_{n-k}\right)$.

Proposition 4.1.2 (Vector space of convergent series) *Let $(U_n)_{n\in\mathbb{N}}$ et $(V_n)_{n\in\mathbb{N}}$ be two real sequences. If the series $\sum_{n\in\mathbb{N}} U_n$ and $\sum_{n\in\mathbb{N}} V_n$ are convergent with respective results U and V, then :*

- The series $\sum_{n\in\mathbb{N}} (U_n + V_n)$ is convergent with sum $U + V$.

- The series $\sum_{n\in\mathbb{N}} (\lambda U_n + \mu V_n)$ with ($\lambda, \mu \in \mathbb{R}$ or $\mathbb{C}$) is convergent with sum $\lambda U + \mu V$.

Consequence : We note that the complex series $\sum_{n\in\mathbb{N}}(a_n + ib_n)$ converges if and only if the two real series $\sum_{n\in\mathbb{N}} a_n$ and $\sum_{n\in\mathbb{N}} b_n$ converge. Then $\sum_{n\in\mathbb{N}}(a_n + ib_n) = \sum_{n\in\mathbb{N}} a_n + i \sum_{n\in\mathbb{N}} b_n$.

Proposition 4.1.3 *- The sum of two series, one of which is convergent and the other divergent of the first kind, is divergent of the first kind.*

- The sum of two series, one of which is convergent and the other is divergent of the second kind, is divergent of the second kind.

- The series $\sum_{n\in\mathbb{N}} \lambda U_n$ with ($\lambda \in \mathbb{R}$ or $\mathbb{C}$) is of the same nature as the series $\sum_{n\in\mathbb{N}} U_n$.

- The product of two convergent series is not necessarily a convergent series.

4.1.3 Séries de Cauchy

Proposition 4.1.4 *A series $\sum_{n\in\mathbb{N}} U_n$ is convergent if and only if*

$$\forall \epsilon > 0, \exists N_0 \in \mathbb{N}, /p \geq n \geq N_0 \Longrightarrow |\sum_{k=n}^{p} U_k| \leq \epsilon$$

or :

$$\forall \epsilon > 0, \forall p > 0, \exists N_0 \in \mathbb{N}, /n \geq N_0 \Longrightarrow |\sum_{k=n}^{n+p} U_k| \leq \epsilon$$

4.2 Positive term series

The interest in studying series with positive terms (i.e. with terms in $\mathbb{R}^+$) is that the sequence $(S_n)_{n\in\mathbb{N}}$ of partial sums defined by its general term : $\forall n \in \mathbb{N}, \sum_{k=0}^{n} U_k$ is real and increasing.

Indeed ; $S_{n+1} - S_n = \sum_{k=0}^{n+1} U_k - \sum_{k=0}^{n} U_k = U_k = U_n \geq 0$.

Proposition 4.2.1 *A series with positive terms $\sum_{n\in\mathbb{N}} U_n$ is either convergent or divergent of the first kind (of limit $+\infty$). Furthermore, for this series to be convergent, it is necessary and sufficient that the sequence $(S_n)_{n\in\mathbb{N}}$ of partial sums defined by its general term : $\forall n \in \mathbb{N}, S_n = \sum_{k=0}^{n} U_k$ is bounded.*

4.2.1 Comparison criterion

Proposition 4.2.2 *Let $\sum_{n\in\mathbb{N}} U_n$ and $\sum_{n\in\mathbb{N}} V_n$ be two series with positive terms, satisfying, $\forall n \in \mathbb{N}, U_n \leq V_n$.*

a) If $\sum_{n\in\mathbb{N}} V_n$, is convergent, then $\sum_{n\in\mathbb{N}} U_n$ is convergent.

b) If $\sum_{n\in\mathbb{N}} U_n$, is divergent of the first kind, then $\sum_{n\in\mathbb{N}} V_n$ is divergent of the first kind.

Proof. a) If we have $U_n \leq V_n, \forall n \in \mathbb{N}$, we also have : $U_0 + U_1 + ... + U_n = S_n \leq V_0 + V_1 + ... + V_n = \acute{S}_n \leq \sum_{n=0}^{+\infty} V_n = \acute{S}$. If $\sum_{n\in\mathbb{N}} V_n$ converges, then $\acute{S}$ exists and $\sum_{n\in\mathbb{N}} U_n$ is increased and therefore convergent.

b) obvious. ■

Proposition 4.2.3 (Use of equivalences) *Let $\sum_{n\in\mathbb{N}} U_n$ and $\sum_{n\in\mathbb{N}} V_n$ be two series with positive terms. If $U_n \underset{+\infty}{\sim} V_n$ then $\sum_{n\in\mathbb{N}} U_n$ and $\sum_{n\in\mathbb{N}} V_n$ are of the same nature.*

Example 4.2.1 (Harmonic series) *The harmonic series is given by its general term $U_n = \frac{1}{n}$. The general term tends towards zero, but the series diverges.*

Example 4.2.2 $U_n = \frac{1}{n \cos^2 n}$ *and $V_n = \frac{1}{n}$ we notice that $U_n \leq V_n, \forall n \in \mathbb{N}$ therefore $\sum_{n\in\mathbb{N}} V_n$ diverges hence the divergence of $\sum_{n\in\mathbb{N}} U_n$.*

Example 4.2.3 $Un = arc\sin \frac{2n}{4n^2+1}, Vn = \frac{2n}{4n^2+1}$. *We have $U_n \sim V_n$ and we set $\acute{V}_n = \frac{1}{2n}$ then $V_n \sim \acute{V}_n$ we see that $\sum_{n\in\mathbb{N}} \acute{V}_n$ diverges, so $\sum_{n\in\mathbb{N}} V_n$ diverges and finally $\sum_{n\in\mathbb{N}} U_n$ diverges.*

Example 4.2.4 $U_n = \frac{\sin^2 n}{n^2}$. *On a $U_n < \frac{1}{n^2} = Vn$, we know that $\sum_{n\in\mathbb{N}} V_n$ converges (Riemann serie's) so $\sum_{n\in\mathbb{N}} U_n$ converges.*

Example 4.2.5 $Un = \frac{\cos h\frac{1}{n}}{n}$, *and $V_n = \frac{1}{n}$. We see that $U_n \sim V_n$, but $\sum_{n\in\mathbb{N}} V_n$ diverges, therefore $\sum_{n\in\mathbb{N}} U_n$ diverges.*

4.2.2 Cauchy criterion

Proposition 4.2.4 *Let $\sum_{n\in\mathbb{N}} U_n$ be a series with positive terms. We consider $\lim\limits_{n\longrightarrow+\infty} \sqrt[n]{U_n} = l$ where $l \in \mathbb{R}_+ \cup \{+\infty\}$.*

a) If $0 \leq l \leq 1$, then the series $\sum_{n\in\mathbb{N}} U_n$ is convergent.

b) If $l > 1$ then the series $\sum_{n\in\mathbb{N}} U_n$ is convergent.

c) If $l = 1$ we cannot conclude

We note that if $\lim\limits_{n\longrightarrow+\infty} \sqrt[n]{U_n} \xrightarrow{>1} 1$, then we can conclude that $\sum_{n\in\mathbb{N}} U_n$ diverges.

Example 4.2.6 *Let the series be $\sum_{n\in\mathbb{N}} U_n = \sum_{n\in\mathbb{N}} \left(\frac{n+a}{n+b}\right)^{n^2}$. We have $\sqrt[n]{U_n} = \left(\frac{n+a}{n+b}\right)^{n} = \left(\frac{1+\frac{a}{n}}{1+\frac{b}{n}}\right)^{n} \longrightarrow \frac{e^a}{e^b} = e^{a-b} = l$*

Discussion : If $b > a$, then $l < 1$, the series converges.

If $b < a$, then $l > 1$, the series diverges.

If $a = b$, then $U_n = 1 \nrightarrow 0$ the series diverges.

4.2.3 D'Alembert criterion

Lemma 4.2.1 *Let $\sum_{n\in\mathbb{N}} U_n$ and $\sum_{n\in\mathbb{N}} V_n$ be two real series with positive terms. If we have $\forall n \in \mathbb{N}, \frac{U_{n+1}}{U_n} \leq \frac{V_{n+1}}{V_n}$, such that; $\exists \alpha \in \mathbb{R}$, then $\forall n \in \mathbb{N}, U_n \leq \alpha V_n$.*

Theorem 4.2.1 *Let $\sum_{n\in\mathbb{N}} U_n$ and $\sum_{n\in\mathbb{N}} V_n$ be two real series with positive term such that : $\lim\limits_{n\longrightarrow+\infty} \frac{U_n}{V_n} = l \neq 0$. Then the two series converge or diverge at the same time (of the same nature).*

Theorem 4.2.2 *Let $\sum_{n\in\mathbb{N}} U_n$ be a series with positive terms. We consider the series $\sum_{n\in\mathbb{N}} V_n$ defined such that, $\forall n \in \mathbb{N}, V_n = \frac{U_{n+1}}{U_n}$ moreover we consider $\lim\limits_{n\longrightarrow+\infty} V_n = \lim\limits_{n\longrightarrow+\infty} \frac{U_{n+1}}{U_n} = l$ où $l \in \mathbb{R}_+ \cup \{+\infty\}$.*

a) If $0l1$, then the series $\sum_{n\in\mathbb{N}} U_n$ is convergent.

b) If $l > 1$ then the series $\sum_{n\in\mathbb{N}} U_n$ is divergent.

c) If $l = 1$ we cannot conclude.

We assume that $\lim\limits_{n\longrightarrow+\infty} \frac{U_{n+1}}{U_n} \xrightarrow{>1} 1$, then we can conclude that $\sum_{n\in\mathbb{N}} U_n$ diverges.

Example 4.2.7 *Let the series* $\sum_{n\geq 1}\frac{n^3}{n!}$. *We set* $\forall n \geq 1, U_n = \frac{n^3}{n!}$, *then* $\lim\limits_{n\longrightarrow+\infty}\frac{U_{n+1}}{U_n} = \lim\limits_{n\longrightarrow+\infty}\frac{(n+1)^3}{(n+1)!}.\frac{n!}{n^3} = 0 < 1$. *The series converges.*

Proposition 4.2.5 *Let* $(U_n)_{n\in\mathbb{N}}$ *be a sequence of positive real numbers. Suppose that* $\lim\limits_{n\longrightarrow+\infty}\sqrt[n]{U_n} = l$, $\lim\limits_{n\longrightarrow+\infty}\frac{U_{n+1}}{U_n} = \acute{l}$, *then* $l = \acute{l}$.

4.2.4 Series, Integrals and Riemann criterion

Proposition 4.2.6 *Let* f *be a decreasing function, defined from* $\mathbb{R}^+$ *into* $\mathbb{R}^+$ *such that :* $U_n = f(n)$. *Then the series* $\sum_{n\in\mathbb{N}} U_n$ *and* $\int_0^{+\infty} f(x)dx$ *are of the same nature.*

Definition 4.2.1 *We call a Riemann series a series of the form* $\sum_{n>1}\frac{1}{n^a}$ *where* $a \in \mathbb{R}_+^*$.

Proposition 4.2.7 *a) If* $0 < a < 1$, *the Riemann series diverges.*

b) If $a > 1$, *the Riemann series converges.*

4.3 Series with any terms

In this paragraph we will study the case of series $\sum_{n\in\mathbb{N}} U_n$ where U_n s complex or real of any sign.

4.3.1 Definition et proposition

Definition 4.3.1 *We will say that the general term series* $U_n \in \mathbb{C}$ *is absolutely convergent if the series with positive terms of general term* $|U_n|$ *converges.*

Proposition 4.3.1 *Any absolutely convergent series is convergent.*

Remark 4.3.1 *There exist convergent series which are not absolutely convergent.*

Example 4.3.1 *Let* $\sum_{n\in\mathbb{N}} U_n$ *such that* $U_{2n-1} = \frac{1}{2n}$ *and* $U_{2n} = \frac{-1}{2n}$ *with* $n > 1$. *Then* $|U_n| \sim \frac{1}{n}$ *therefore* $\sum_{n\in\mathbb{N}} |U_n|$ *diverges, the series* $\sum_{n\in\mathbb{N}} U_n$ *is not absolutely convergent. However,* $S_{2n} = 0$ *and* $S_{2n-1} = \frac{1}{2n}$ *therefore the series* $\sum_{n\in\mathbb{N}} U_n$ *cconverges with zero sum. We will say that a series which converges without being absolutely convergent is semi-convergent.*

4.3.2 Abel's Sum

Theorem 4.3.1 *Let $\sum_{n\in\mathbb{N}} U_n$ be a series whose general term, real or complex, is written in the form $U_n = a_n b_n$; Suppose that :*

1) The sequence $(a_n)_{n\in\mathbb{N}}$ is in positive terms decreasing and tends to 0 when $n \longrightarrow +\infty$.

2) There exists $M \in \mathbb{R}$ such that $\forall x \in \mathbb{N} |\sum_{k=1}^{n} b_k| \leq M$.

Then the series $\sum_{n\in\mathbb{N}} U_n$ is convergent.

4.3.3 Alternating series

Definition 4.3.2 *We call an alternating series a numerical series whose general term U_n is of the form $U_n = (-1)^n V_n$, where $(V_n)_{n\in\mathbb{N}}$ denotes a sequence of constant sign.*

Example 4.3.2 *The series $\sum_{n=0}^{\infty} \frac{(-1)^n}{n+1}$ is an alternating series. It is called alternating harmonic series.*

Theorem 4.3.2 *If the sequence $(V_n)_{n\in\mathbb{N}}, V_n > 0$, is decreasing and converges to zero, then the series $\sum_{n\in\mathbb{N}} U_n = \sum_{n\in\mathbb{N}} (-1)^n V_n$ is convergent and its sum S satisfies the inequality : $S_{2p+1} \leq S \leq S_{2p}$.*

Chapitre 5

SEQUENCES AND SERIES OF FUNCTIONS

Many functions appear as boundaries of other simpler functions. This is the case, for example, of the exponential function, which can be defined by one of the following two formulas. $e^x = \lim\limits_{n\longrightarrow+\infty}(1+x^n)^n$ ou $e^x = \sum_{n=0}^{+\infty}\dfrac{x^n}{n!}$.

This is also the case for more theoretical problems, such as when we construct solutions to equations (for example differential), we often construct by induction towards an approximate solution which converge towards an exact solution.

5.1 Function suites

5.1.1 General notions

In the following, $\mathbb{K}$ designates one of the fields $\mathbb{R}$ or $\mathbb{C}$. Let E be a $\mathbb{K}$-vector space and I be any set, the set $\mathcal{F}_E(I)$ of maps defined on I, with values in E, is equipped with the following two scalar operations : addition $(f,g) \longrightarrow f+g$ and multiplication by a scalar $(\lambda, f) \longrightarrow \lambda f$, defined by : if $f, g \in_{FE(I)}$ and $\lambda \in K, (f+g)(x) = f(x)+g(x)$ and $(\lambda f)(x) = \lambda f(x).(\mathcal{F}_E(I), +, \times)$ is a K-vector space.

Definition 5.1.1 *A sequence in $\mathcal{F}_E(I)$ is a map of $\mathbb{N}$ in $\mathcal{F}_E(I)$ which associates with each natural number n a function f_n. It is noted $(f_n)_{n>0}$ or simply $(f_n)_n$.*

Remark 5.1.1 • *$(f_{n+n_0})_n$ is also denoted $(f_n)_{n>n_0}$.*

• In a sequence $(f_n)_n$, the f_n are assumed to have the same definition set.

• The sequence $(f_n)_n$ can be seen as a numerical sequence $(f_n(x))_n$ dependent on the parameter x, traversing a given set.

5.1.2 Simple convergence

The notion of convergence of a sequence of real or complex numbers naturally leads to that of convergence at each point for the sequences of functions defined as follows.

Definition 5.1.2 *A sequence $(f_n)_n$ of maps $f_n : I \longrightarrow \mathbb{K}$ is said to simply converge on I if there exists a map $f : I \longrightarrow \mathbb{K}$ such that $\forall x \in I; \lim\limits_{n\longrightarrow+\infty} f_n(x) = f(x)$*

- f is called simple limit of $(f_n)_n$

- If f exists it is unique.

We write :

$$\left(\forall x \in I; \lim_{n\longrightarrow+\infty} f_n(x) = f(x)\right) \iff (\forall x \in I; \forall \epsilon > 0; \exists N(x,\epsilon)/n \geq N) \implies |f_n(x) - f(x)| \leq \epsilon).$$

Remark 5.1.2 *It should be noted that N depends on x and ϵ.*

Example 5.1.1 *A numerical sequence is a very particular case of sequences of functions, here the functions are constants.*

Example 5.1.2 *Let $I =]0,1[$ and $\mathbb{K} = \mathbb{R}, f_n(x) = \dfrac{1}{(n+1)x}$. Then $\lim\limits_{n\longrightarrow+\infty} f_n(x) = f(x) = 0, \forall x \in I$.*

Example 5.1.3 *We consider $\forall x \in \mathbb{R}, f_n(x) = \left(1 + \dfrac{x}{n}\right)^n = e^{n\ln\left(1+\frac{x}{n}\right)}$. Then $\forall x \in \mathbb{R}; f_n(x) = e^x$.*

Theorem 5.1.1 *Let $(f_n)_n$ et $(g_n)_n$ two sequences in $\mathcal{F}_E(I)$ simply convergent to f and g respectively, and $\lambda \in \mathbb{K}$, Then*

1. *The sequence $(f_n + g_n)_n$ simply converges to $f + g$.*
2. *The sequence $(\lambda f_n)_n$ simply converges to λf.*
3. *The sequence $(f_n g_n)_n$ simply converges to fg.*

Theorem 5.1.2 (Cauchy criterion) *For a sequence $(f_n)_n$ of applications $f_n : I \longrightarrow \mathbb{K}$ to simply converge, it is necessary and sufficient that :*

$$\forall x \in I, \forall \epsilon > 0, \exists N(x,\epsilon)/m \geq N(x,\epsilon); n \geq N(x,\epsilon) \implies |f_m(x) - f_n(x)| \leq \epsilon.$$

5.1.3 Uniform convergence

Example 5.1.4 *Let $I = [0,1]$ and $\forall n \in \mathbb{N}, f_n(x) = x^n$, it is clear that $\lim_{n \longrightarrow +\infty} f_n(x) = f(x)$ such as,*

$$f(x) = \begin{cases} 0 & if\ x \in [0,1[\\ 1 & if\ x = 1 \end{cases}$$

We conclude that :

- Each function f_n is continuous whatever n.

- Each $(f_n)_n$ simply converges to f.

- f is not continuous.

This is why it is necessary to use a more precise notion which preserves continuity by passing to the limit, this is uniform convergence.

Definitions

Definition 5.1.3 *We call the norm of uniform convergence the norm for $(f_n)_n$ and f of $\mathcal{F}_E(I)$; $\|f_n - f\| = \sup_{x \in I} |f_n(x) - f(x)|$.*

Definition 5.1.4 *A sequence of maps $f_n : I \longrightarrow \mathbb{K}$ is said to be uniformly convergent on I if there exists a map $f : I \longrightarrow \mathbb{K}$ such that,*

$$\lim_{n \longrightarrow +\infty} \left(\sup_{x \in I} |f_n(x) = f(x)| \right) = 0.$$

or,

$$\forall \epsilon > 0; \exists N; \forall x; \exists n[n > N \implies |f_n(x) - f(x)| < \epsilon].$$

or

$$\lim_{n \longrightarrow +\infty} \|f_n - f\| = 0.$$

Interpretation : We have $|f_n(x) - f(x)| < \epsilon \implies f(x) - \epsilon < fn(x) < f(x) + \epsilon$. We say that for $n > N$, the graph of f_n is contained in a band of width 2ϵ symmetrical with respect to the graph of f.

Proposition 5.1.1 *Uniform convergence implies simple convergence. Indeed; $\forall x, \forall n : |f_n(x) - f(x)| \leq |f_n - f| \underset{+\infty}{\longrightarrow} 0$. The converse is false*

Example 5.1.5 *Let $f_n(x) = \dfrac{nx}{1+nx}$ with $x \in [0, +\infty[$. We have shown that this sequence simply converges to*

$$f(x) = \begin{cases} 0 & if \ x \in [0,1[\\ 1 & if \ x = 1 \end{cases}.$$

as; $\|f_n - f\| = \sup_{x\in I} |f_n(x) - f(x)| = \sup_{x\in I} \dfrac{1}{1+nx} = \lim_{x\longrightarrow 0} \dfrac{1}{1+nx} = 1$ i.e. $\|f_n - f\| \nrightarrow 0$, hence the convergence is not uniform.

Cauchy criterion for uniform convergence

Theorem 5.1.3 *Let $(f_n)_n$ be a sequence of functions on I. For the sequence $(f_n)_n$ to be uniformly convergent on I towards a function f it is necessary and sufficient that :*

$$\forall \epsilon > 0; \exists N; \forall n, \forall m; \forall x \subset I[n, m > N \implies |f_n(x) - f_m(x)| < \epsilon].$$

or

$$\forall \epsilon > 0; \exists N; \forall n, \forall m; \forall x \subset I[n, m > N \implies \|f_n(x) - f_m(x)\| < \epsilon].$$

Example 5.1.6 *Let $f_n(x) = \dfrac{2xn}{1+n^2x^2}$ sur $I = [0,1]$. We have $\lim_{n\longrightarrow+\infty} f_n = f = 0$. As the upper bound of the function $y \longrightarrow \dfrac{2y}{1+y^2}$ on $[0, +\infty[$ is equal to $\dfrac{1}{2}$ for $y = 1$, we have $\|f_n\| = \sup_I |f| = \dfrac{1}{2}$. Then $f_n \longrightarrow 0$ simply but not uniformly.*

How to show that a sequence of functions converges uniformly

To show that a sequence of functions $(f_n)_n$ is uniformly convergent :

1. We show that it is simply convergent, which allows us to define f.

2. We seek to increase $|f_n(x) - f(x)|$ by a sequence $(\varepsilon_n)_n$ of positive real numbers which converges to 0 such that $(\varepsilon_n)_n$ does not depend on x.

To determine $(\varepsilon_n)_n$, we have two methods.

1. Majoer $|f_n(x) - f(x)|$ independently of x.
2. Calculate $\sup_{x\in I} |f_n(x) = f(x)|$ using the study of the function $|f_n(x) = f(x)|$.

Operations and uniform convergence

Theorem 5.1.4 *Let $(f_n)_n$ and $(g_n)_n$ be two sequences in $\mathcal{F}_E(I)$ uniformly convergent to f and g respectively, and $\lambda \in \mathbb{K}$. So*

(i) The sequence $(f_n + g_n)_n$ converges uniformly $f + g$.

(ii) The sequence $(\lambda f_n)_n$ converges uniformly λf.

(iii) If the maps f and g are bounded, the sequence $(f_n g_n)_n$ converges uniformly to fg.

Remark 5.1.3 *In (iii), the assumption (f and g bounded) cannot be omitted, otherwise the theorem is false.*

5.1.4 Properties of uniform convergence

Continuity

Theorem 5.1.5 *Let $(f_n)_n \in \mathcal{F}_E(I)$, be a sequence of functions uniformly convergent to f. If all functions f_n are continuous at x_0, then f is continuous at x_0.*

Proof. Either

$$\epsilon > 0; \exists N / \forall n > N, \forall x \in I; |f_n(x) - f(x)| < \frac{\epsilon}{3} \tag{5.1}$$

Let us fix an integer $n > N$, f_n being continuous in x_0, then It exists

$$\delta > 0 / \forall x \in I; |x - x_0| < \delta \implies |f_n(x) - f_n(x_0)| < \frac{\epsilon}{3} \tag{5.2}$$

From (5.1) and (5.2) we obtain :

$$\forall x \in I; |x - x_0| < \delta \implies |f_n(x) - f_n(x_0)| \leq |f(x) - f_n(x)| + |f_n(x) - f_n(x_0)| + |f_n(x_0) - f(x_0)| < \epsilon.$$

■

corollary 5.1.1 *If $f_n \longrightarrow f$ uniformly and if all functions f_n are continuous on I, then f is continuous on I.*

Remark 5.1.4 *1) If a sequence of continuous functions converges to a non-continuous function, the convergence is not uniform.*

2) The theorem gives a sufficient condition for $f = \lim f_n$ to be continuous, but this condition is not necessary. It can happen that the functions f_n being continuous, f is continuous, without the convergence being uniform.

Example 5.1.7 *Let $(f_n)_n$ be such that $I = [0, 2]$.*

$$f_n(x) = \begin{array}{ll} nx^2 & \text{if } 0 \leq x \leq \frac{1}{n} \\ nx^2 + 2n & \text{if } \frac{1}{n} \leq x \leq \frac{2}{n} \\ 0 & \text{if } x \leq \frac{2}{n} \end{array} .$$

The functions f_n are continuous, and $\lim\limits_{n \longrightarrow +\infty} f_n = f = 0$ simply. Because :

If $x = 0$, $f_n(0) = 0$.

If $x > 0$, $f_n(x) = 0$ for all $n > \frac{2}{x}$.

However the convergence is not uniform since : $\|f_n(x) - f(x)\| = f_n(\frac{1}{n}) = n \longrightarrow +\infty$ when $n \longrightarrow +\infty$.

Example 5.1.8 *If a sequence of functions $(f_n)_n$ simply converges to f, then f is not necessarily continuous. Let $I = [0, 1]$, and let $f_n(x) = x^n$. It is clear that $f_n(x) \longrightarrow f(x)$ such that*

$$f(x) = \begin{cases} 0 & \text{if } x \in [0, 1[\\ 1 & \text{if } x = 1 \end{cases} .$$

f is not continuous.

Example 5.1.9 *There exist sequences* $(f_n)_n$ *which converge to* f *continues. Either*

$$f_n(x) = \begin{cases} 0 & \text{if } x \leq 0 \\ nx & \text{if } 0 \leq x \leq \frac{1}{n} \\ nx+2 & \text{if } \frac{1}{n} \leq x \leq \frac{2}{n} \\ 0 & \text{if } x \geq \frac{2}{n} \end{cases}.$$

- For $x < 0$*, we have* $\lim_{n \longrightarrow +\infty} f_n = 0$

- For $x > 0$*, we have* $n > \dfrac{2}{n} \implies f_n(x) = 0$ *therefore* $\lim_{n \longrightarrow +\infty} f_n = 0$*, so the limit is continuous on R although the convergence is not uniform there since :* $\|f_n\| = \sup_{x \in \mathbb{R}} |f_n(x)| = 1$ *does not tend towards zero.*

Integration

Theorem 5.1.6 *Let* $(f_n)_n$ *be a sequence of integrable functions on* $I = [a, b]$ *converging uniformly to* f*. So :*

a) f *is integrable on* $[a, b]$

b) $\lim_{n \longrightarrow +\infty} \int_a^b f_n(x)dx = \int_a^b f(x)dx.$

Proof. a) Let $\epsilon > 0$, there exists n such that :

$$\forall x_i \in \mathbb{R}, n \in [a, b], f_n(x) - \frac{\epsilon}{2(b-a)} \leq f(x) \leq f_n(x) + \frac{\epsilon}{2(b-a)} \tag{5.3}$$

because $f_n \longrightarrow f$ uniformly. The function f_n being integrable, there exists a subdivision $d = \{x_0, x_l, ..., x_k\}$ of $[a, b]$ such that

$$\sum_{i=1}^{k} (M_{n_i} - m_{ni})(x_i - x_{i-1}) < \frac{\epsilon}{2} \tag{5.4}$$

or

$$M_{n_i} = \sup_{[x_{i-1}, x_i]} f_n \text{ et } m_{n_i} = \inf[x_{i-1}, x_i] f_n.$$

Noting $M_i = \sup_{[x_{i-1}, x_i]} f$ and $m_i = \inf_{[x_{i-1}, x_i]} f$.

Then (5.3) implies for $1 \leq i \leq k$ that

$$m_{n_i} - \frac{\epsilon}{2(b-a)} \leq m_i \leq M_{n_i} + \frac{\epsilon}{2(b-a)}$$

Which gives taking into account (5.4

$$\sum_{i=1}^{k}(M_i - m_i)(x_i - x_{i-1}) \leq \sum_{i=1}^{k}(M_{n_i} - m_{n_i})(x_i - x_{i-1}) + \frac{\epsilon}{2} < \epsilon.$$

Thus, f is integrable. ■

b) We have for all n ;

$$|\int_a^b f_n(x)dx - \int_a^b f(x)dx| \leq \int_a^b |f_n - f|dx \leq \|f_n - f\| (b-a) \longrightarrow 0.$$

Remark 5.1.5 *The condition $f_n \longrightarrow f$ uniformly is sufficient but not necessary for* $\lim_{n \longrightarrow +\infty} \int_a^b f_n(x)dx = \int_a^b f(x)dx$.

Example 5.1.10 *Either*

$$f_n(x) = \begin{cases} 0 & \text{if } x = 0 \\ 0 & \text{if } \frac{1}{n} \leq x \leq 1 \\ 1 & \text{if } 0 < x < \frac{1}{n} \end{cases}$$

For $n > \dfrac{1}{x}$ we have $f_n(x) = 0$ therefore $\lim_{n \longrightarrow +\infty} f_n(x) = f(x) = 0$. But this convergence is not uniform because : $\sup_{[0,1]} f_n = 1 \forall n$, however

$\int_0^1 f_n(x)dx = \dfrac{1}{n} \longrightarrow 0 = \int_0^1 f(x)dx$ that is to say, although $\int_0^1 f_n(x)dx = \int_0^1 f(x)dx$, but the convergence is not uniform.

Derivation

We would like to be able to give a theorem similar to that of continuity or that of integration but unfortunately, this is not possible.

Theorem 5.1.7 *Let $(f_n)_n$ be a sequence of continuously differentiable functions on $[a, b]$($f_n \in C^1[a, b], \mathbb{R}$) satisfying the following properties :*

a) the sequence $(\acute{f_n})_n$ converges uniformly to g on $[a, b]$.

b) There exists a point $x_0 \in [a,b]$ such that the sequence $(f_n(x_0))$ converges to a limit l. Then $(f_n)_n$ converges uniformly to f on $[a,b]$ with $(f \in C^1[a,b],\mathbb{R})$ and we have $\forall x \in [a,b] \acute{f}(x) = g(x)$. In other words : $\lim_{n\longrightarrow+\infty} \acute{f_n}(x) = \left(\lim_{n\longrightarrow+\infty} f_n(x)\right)'$.

Proof. We have $f_n(x) = f_n(x_0) + \int_{x_0}^{x} \acute{f_n}(t)dt$. According to (3.1.24,a) we have $\lim_{n\longrightarrow+\infty} \int_{x_0}^{x} \acute{f_n}(t)dt = \int_{x_0}^{x} g(t)dt$ uniformly on $[a,b]$. Thus $(f_n)_n$ converges uniformly to f defined by :

$$f(x) = \lim_{n\longrightarrow+\infty} f_n(x_0) + \int_{x_0}^{x} g(t)dt = l + \int_{x_0}^{x} g(t)dt$$

as f continuously differentiable, then $\acute{f} = g$. ∎

5.2 Series of functions

5.2.1 Definitions and properties

Definition 5.2.1 *Let $(f_n)_n$ be a sequence of functions with real or complex values defined on a non-empty set $I \subset \mathbb{R}$, let us associate with this sequence the sequence of functions $(S_n)_n$ defined by : $\sum_{k=0}^{n} f_k$.*

We call a series of functions on (I) with general term f_n the pair $((f_n)_n, (S_n)_n)$. The sequence $(S_n)_n$ is called n^{th} partial sum of the series of functions $((f_n)_n, (S_n)_n)$ and will be denoted, $\sum f_n$ or $\sum_{n=0}^{+\infty} f_n$ or $f_0 + f_1 + ...f_n + ...$

Definition 5.2.2 *The series $\sum_{k=0}^{+\infty} f_k$ is called the remainder of order n of the series $\sum_{n=0}^{+\infty} f_n$.*

Definition 5.2.3 *We will say that $\sum_{k=0}^{+\infty} f_k$ is convergent in $x_0 \in I$ if the numerical series $\sum_{k=0}^{+\infty} f_k(x_0)$ is convergent. We will say that $\sum_{k=0}^{+\infty} f_k$ is convergent on I (or on a part $A \subset I$) if the series $\sum_{k=0}^{+\infty} f_k(x)$ is convergent at any point $x \in I$ (respectively $x \in A$), in this case we will say that the series $\sum_{k=0}^{+\infty} f_k(x)$ is simply convergent on I (respectively on A).*

We therefore have the following equivalences :

$$\sum f_n \text{ converges at } x_0 \iff \sum f_n(x_0) \text{ converges} \iff (S_n(x_0))_n \text{ converges}$$

$$\sum f_n \text{ converges on } I \iff \forall x \in I; \sum f_n(x) \text{ converges} \iff (S_n)_n \text{ converges on } I.$$

Thus : The simple convergence of a series on I is equivalent to the simple convergence of the sequence of its partial sums $(S_n)_n$ on I.

Definition 5.2.4 *Let* $\sum_{n=0}^{+\infty} f_n$ *be a convergent series on* I *and* $(S_n)_n$ *the sequence of its partial sums. The function* $S : I \longrightarrow \mathbb{C}$ *defined by* $S = \lim_{n \longrightarrow +\infty} S_n$ *is called the sum of the series and is denoted* $S = \sum_{n=0}^{+\infty} U_n$.

Remark 5.2.1 *The general theorems relating to numerical series remain true, with necessary modifications, for series of functions.*

5.2.2 Uniform convergence

Definition and example

Definition 5.2.5 *Let* $\sum_{n=0}^{+\infty} f_n$ *be a series of functions, we say that this series is uniformly convergent on* I *if the sequence* $(S_n)_n$ *of partial sums is uniformly convergent where* $S_n(x) = \sum_{n=0}^{+\infty} f_n(x)$. *In case of convergence, the limit* S *such that* $S_n(x) = \sum_{n=0}^{+\infty} f_n(x)$ *is called the sum of the series.*

Remark 5.2.2 *Thus the uniform convergence of* $\sum f_n$ *on a set* E *means :*

$$\forall \epsilon > 0; \exists N; / \forall n, \forall x \in E \left[n > N \implies \left| \sum_{k=n+1}^{+\infty} f_n(x) \right| < \epsilon \right].$$

Then the simple convergence is expressed by :

$$\forall x \in E; \forall \epsilon > 0; \exists N; / \forall n, \left[n > N \implies \left| \sum_{k=n+1}^{+\infty} f_n(x) \right| < \epsilon \right]$$

We will note : $\sum_{k=n+1}^{+\infty} f_n(x) = S - S_n$

Example 5.2.1 *Let* $f_n(x) = x^n$. *Then* $\forall x \neq 1; S_n(x) = \sum_{k=0}^{n} f_k(x) = 1 + x + ... + x^n = \dfrac{x^{n+1} - 1}{x - 1}$, *this series therefore simply converges in* $I =]-1, +1[$ *to the function* $S(x) = \dfrac{1}{1-x}$. *However the convergence is not uniform on* $I =]-1, +1[$. *Indeed,* $\lim_{x \overset{<1}{\longrightarrow} 1} (S_n(x) - S(x)) = +\infty, \forall n$, *so* $\|S_n(x) - S(x)\|$ *does not tend towards zero. On the other hand, in each interval* $[-\delta, \delta]$ *with* $0 < \delta < 1$, *the convergence is uniform.*

Abel's criterion for uniform convergence

Theorem 5.2.1 *We consider the series of general term functions* f_n. *If* f_n *is written in the form* $\forall x \in I, f_n(x) = \varepsilon_n(x) g_n(x)$ *with :*

(1) $\forall x \in I, \varepsilon_n(x)$ *is decreasing towards* 0.

(2) $\lim\limits_{n \longrightarrow +\infty} \left(\sup\limits_{x \in I} |\varepsilon_n(x)| \right) = 0.$

(3) $\exists M > 0$ *such that* $\forall x \in I; \forall n \in \mathbb{N} : \sum_{i=0}^{n} g_i(x) < M$.

Then the general term series f_n *converges uniformly on* I.

Example 5.2.2 *We show that this series converges uniformly on* I*, using the Abel criterion. We pose*

$$\forall n \in \mathbb{N}; \forall x \in I_\delta, [\delta, 2\pi - \delta] \text{ with } \delta \in]0, \pi[\text{ and } \alpha \in [0, 1[, f_n(x) = \frac{e^{inx}}{(n+1)^\alpha}$$

We show that this series converges uniformly on I*, using the Abel criterion. We set*

$$\forall x \in I_\delta; f_n(x) = \varepsilon_n(x) g_n(x), \text{ with } \varepsilon_n(x) = \frac{1}{(n+1)^\alpha} \text{ and } g_n(x) = e^{inx}.$$

We have

(1) $\varepsilon_n(x)$ *decreasing towards* 0 *because* $\alpha \in]0, 1[$.

(2) We have

$$\forall n \in \mathbb{N}, \sup_{x \in I_\delta} |\varepsilon_n(x)| = \frac{1}{(n+1)^\alpha} \text{ and } \lim_{n \longrightarrow +\infty} \left(\sup_{x \in I_\delta} |\varepsilon_n(x)| \right) = 0$$

(3) $\sum_{i=0}^{n} g_i(x) = 1 + r^{ix} + (e^{ix})^2 + ... + (e^{ix})^n$ *which is a sum of a geometric series with first term* $g_0(x) = 1$ *and reason* $q = e^{ix}$*, we deduce that,* $\sum_{i=0}^{n} g_i(x) = \dfrac{1 - (e^{ix})^{n+1}}{1 - e^{ix}}$ *because* $e^{ix} \neq 1, \forall x \in I_\delta$. *We must find an increase of* $|\sum_{i=0}^{n} g_i(x)|$ *which is independent of the variable* x *and* n*, since* $\left| e^{ix(n+1)} \right| = 1$ *We have*

$$\left| \sum_{i=0}^{n} g_i(x) \right| \leq \frac{1 + |e^{ix(n+1)}|}{|1 - e^{ix}|} \leq \frac{2}{|1 - e^{ix}|}.$$

We have

$$\begin{aligned} |1 - e^{ix}| &= \sqrt{(1 - \cos x)^2 + \sin^2 x} = \sqrt{1 - 2\cos x + \cos^2 x + \sin^2 x} \\ &= \sqrt{2 - 2\cos x} = \sqrt{4 \sin^2 \frac{x}{2}} = 2|sin^2 \frac{x}{2}| \\ &= 2 \sin \frac{x}{2} \text{ because } 0 < \frac{\delta}{2} \leq \frac{x}{2} \leq \pi - \frac{\delta}{2} < \pi \end{aligned}$$

First case : $0 < \frac{\delta}{2} \leq \frac{x}{2} \leq \frac{\pi}{2}$

The sine function is increasing on $]0, \frac{\pi}{2}]$*, which allows us to deduce :* $|1-e^{ix}| = 2\sin\frac{x}{2} \geq 2\sin\frac{\delta}{2}$.

Second case : $\frac{\pi}{2} \leq \frac{x}{2} \leq \pi - \frac{\delta}{2} < \pi$

The sine function is decreasing on $\left[\frac{\pi}{2}, \pi\right]$*, which allows us to deduce :* $\sin\frac{x}{2} \geq \sin(\pi - \frac{\delta}{2}) > \sin\pi$*, or we have* $\sin(\pi - \frac{\delta}{2}) = \sin\frac{\delta}{2}$*. Thus we obtain :* $|1-e^{ix}| = 2\sin\frac{x}{2} \geq 2\sin\frac{\delta}{2}$*, since the inequalities are the same in both cases, we deduce that :* $\sum_{i=0}^{n} g_i(x) \leq \frac{2}{2\sin\frac{\delta}{2}} = \frac{1}{\sin\frac{\delta}{2}} = M$*. Then according to Abel's criterion, we deduce that the series of functions converges uniformly on* I*.*

5.2.3 Normal convergence

Definition 5.2.6 *Let* $\sum_{n=0}^{+\infty} f_n, f_n(x) : I \longrightarrow \mathbb{K}$ *(*$\mathbb{K} = \mathbb{R}$ *or* $\mathbb{C}$*) be a series of functions, we say that this series is normally convergent on* I *if the numerical series :* $\sum_n \|f_n\|$ *where* $\|f_n\| = \sup_{x\in I} |f_n(x)|$*, is convergent.*

Proposition 5.2.1 *Let* $f_1, f_2 : I \longrightarrow \mathbb{K}$*, then* $\|f_1 + f_2\| \leq \|f_1\| + \|f_2\|$ *(triangular inequality).*

Theorem 5.2.2 *If the series* $\sum_{n=0}^{+\infty} f_n$ *converges normally on* I *; Then it is uniformly convergent on* I*.*

corollary 5.2.1 *Let* $\sum_{n=0}^{+\infty} f_n; f_n : I \longrightarrow \mathbb{K}$ *be a series of functions ; if there exists a convergent positive-term numerical series* $\sum_{n=0}^{+\infty} a_n$*, such that :* $\forall n, \forall x \in I, |f_n(x)| \leq a_n$*, then the series* $\sum_{n=0}^{+\infty} f_n$ *is normally convergent on* I*.*

Example 5.2.3 *Let* $\sum_{n=0}^{+\infty} f_n$ *such that ;* $f_n(x) = \frac{\sin(x^2+n^2)}{x^2+n^2}$*, we have* $\forall x \in \mathbb{R}; |f_n(x)| \leq \frac{1}{n^2}$*. And we have* $a_n = \frac{1}{n^2}$ *we know that* $\sum_{n=0}^{+\infty} a_n$ *is convergent, therefore the series* $\sum_{n=0}^{+\infty} f_n$ *is normally convergent, therefore uniformly convergent in* $\mathbb{R}$*.*

Remark 5.2.3 *Normal convergence is stronger than uniform convergence. There exist uniformly convergent series which do not converge normally.*

5.2.4 Uniform convergence and properties of a series of functions

Continuity

Theorem 5.2.3 *Let* $\sum_{n=0}^{+\infty} f_n, f_n(x) : I \longrightarrow \mathbb{K}$ *be a series of functions uniformly convergent on* I*. If all functions* f_n *are continuous in* $x_0 \in I$*, the sum* S *of the series is continuous in* $x_0 \in I$*.*

corollary 5.2.2 *If all functions f_n are continuous on I and if $\sum_{n=0}^{+\infty} f_n$ is uniformly convergent on I, then the sum is continuous on I.*

Remark 5.2.4 *The hypotheses are those of Theorem (3.2.16), we can therefore write :*

$$\lim_{x \longrightarrow x_0} \sum_{n=0}^{+\infty} f_n(x) = \sum_{n=0}^{+\infty} \lim_{x \longrightarrow x_0} f_n(x) + \sum_{n=0}^{+\infty} f_n(x_0).$$

<u>Generalization :</u> If $\sum_{n=0}^{+\infty} f_n$ is uniformly convergent on I, and if the finite limit $\lim_{x \longrightarrow x_0} f_n(x) = a_n(x_0 \in I)$ exists whatever n, the series $\sum_{n=0}^{+\infty} a_n$ is convergent and we have :

$$\lim_{x \longrightarrow x_0} \sum_{n=0}^{+\infty} f_n(x) = \sum_{n=0}^{+\infty} \lim_{x \longrightarrow x_0} f_n(x) + \sum_{n=0}^{+\infty} a_n(x_0).$$

Remark 5.2.5 *This result is often used to demonstrate that a given series is not uniformly convergent by demonstrating that the sum function $S_n(x)$ is discontinuous at a point.*

Example 5.2.4 *Study the uniform convergence of the series :*

$$x^2 + \frac{x^2}{1+x^2} + \frac{x^2}{(1+x^2)^2} + ... + \frac{x^2}{(1+x^2)^n} + ...$$

We have $\sum_{n=0}^{+\infty} \frac{x^2}{(1+x^2)^n}$, suppose $x \neq 0$, then the series is a geometric series of ratio $\frac{1}{1+x^2}$ and first term x^2. Then $S(x) = x^2 \frac{1}{1 + \frac{1}{1+x^2}} = 1+x^2$, If $x = 0, S_n(0) = 0$, hence $\lim_{x \longrightarrow 0} S_n(0) = S(0)$, on the other hand we have $\lim_{x \longrightarrow 0} S(x) = 1 \neq S(0)$ so S is discontinuous at the point $x = 0$. So the convergence is not uniform.

Integration

Theorem 5.2.4 *Let $\sum_{n=0}^{+\infty} f_n, f_n : I = [a,b] \longrightarrow \mathbb{K}$ be a series of functions uniformly convergent on $[a,b]$. If the functions f_n are integrable in $[a,b]$, then it is the same for the sum of the series and we have,*

$$\int_a^b S(x)dx = \int_a^b \left(\sum_{n=0}^{+\infty} f_n(x) \right) dx = \sum_{n=0}^{+\infty} \int_a^b f_n(x)dx.$$

Furthermore, the series $\sum_{n=0}^{+\infty} \int_a^b f_n(x)dx$ converges uniformly on $[a,b]$ towards $\int_a^x S(t)dt$.

Example 5.2.5 *We have $\frac{1}{1+x} = \sum_{n=0}^{+\infty} (-1)^n x^n$ and $\frac{1}{x^2} = \sum_{n=0}^{+\infty} (-1)^n x^{2n}$, converge uniformly on each interval $[a,b] \subset]-1,1[$, we can therefore integrate them term by term from 0 to x with*

$|x| < 1$. *So,*

$$\ln(x) = \sum_{n=0}^{+\infty} \frac{(-1)^n x^{n+1}}{n+1} = \sum_{n=0}^{+\infty} \frac{(-1)^n x^n}{n},$$
$$\arctan x = \sum_{n=0}^{+\infty} \frac{(-1)^n x^{2n+1}}{2n+1}.$$

Derivation

Theorem 5.2.5 *Let* $\sum_{n=0}^{+\infty} f_n, f_n : I \longrightarrow \mathbb{K}$ *be a series of functions whose general term* f_n *are continuously differentiable on* $[a,b](f_n \in C^1([a,b],\mathbb{K})$. *If*

(a) $\sum_{n=0}^{+\infty} f_n$ *is convergent at a point* $x_0 \subset [a,b]$,

(b) $\sum_{n=0}^{+\infty} \acute{f}_n(x)$ *is uniformly convergent on* $[a,b]$. *Then, the series* $\sum_{n=0}^{+\infty} \acute{f}_n(x)$ *is uniformly convergent on* $[a,b]$ *and we have* $\acute{S} = \left(\sum_{n=0}^{+\infty} f_n\right)' = \sum_{n=0}^{+\infty} \acute{f}_n(x)$.

Example 5.2.6 *Let* $\sum_{n=0}^{+\infty} f_n$ *such that* $f_n(x) = \dfrac{x^n}{n^3(1+x^n)}, I = [0,1]$ *; we have* $|f_n(x)| \leq \dfrac{1}{n^3}$ *because* $x^n \leq 1$ *therefore* $\sum \dfrac{1}{n^3}$ *is convergent, We have normal convergence hence uniform convergence therefore simple convergence on* $[0,1]$. *Is the sum derivable ?*

(i) The functions f_n *are differentiable with continuous derivatives on* $[0,1]$. *Indeed;* $\acute{f}_n(x) = \dfrac{x^{n-1}}{n^2(1+x^n)^2}$.

(ii) The series $\sum_{n=0}^{+\infty} f_n$ *converges at least at a point of* $[0,1]$.

(iii) The series of derivatives converges uniformly on $[0,1]$. *Indeed;* $\left|\acute{f}_n(x)\right| = \left|\dfrac{x^{n-1}}{n^2(1+x^n)^2}\right| \leq \dfrac{1}{n^2}$. *Then the sum is therefore differentiable on* $[0,1]$ *and we have : For*

$$x \in [0,1], \acute{S}(x) = \left(\sum_{n=0}^{+\infty} f_n(x)\right)' = \sum_{n=0}^{+\infty} \acute{f}_n(x).$$

Chapitre 6

ENTIGER SERIES

The theory of integer series allows the majority of usual functions to be expressed as sums of series. We say that an analytic function is a series which can be expressed locally as a convergent integer series. This makes it possible to demonstrate properties of these functions, to calculate complicated sums and also to solve differential equations.

6.1 Definitions and properties

Definition 6.1.1 *An integer series is a series of functions with* $U_n(z) = a_n z^n$ *where* $a_n \in \mathbb{C}$ *and* $z \in \mathbb{C}$. a_n *is the coefficient of order* n, a_0 *the constant term. By convention, we set* $z^0 = 1 \forall z \in \mathbb{C}$. *If* $U_n(x) = a_n x^n$ *where* $a_n \in \mathbb{C}$ *and* $x \in \mathbb{R}$, *we speak of an integer series with a real variable.*

Proposition 6.1.1 - *If there exists* $R \in [0, +\infty[$ *such that* $|z| < R$, *the general term series* $U_n(z) = a_n z^n$ *converges.*

- *If* $|z| > R$, *the series diverges.*

- *Moreover* $0 \leq |r| < R$, *the series converges normally on the closed disk* $\overline{D_r} = \{z \in \mathbb{C}/|z| \leq r\}$.

Remark 6.1.1 *We consider the entire series of general term* $U_n(x) = a_n x^n$

- *If there exists* $R \in [0, +\infty[\cup\{+\infty\},)$ *that is to say that* R *can take the infinite value, such that* $x \in]-R, +R[$, *the general term series* $U_n(x) = a_n x^n$ *converges.*

- *If* $|x| > R$, *the series diverges roughly.*

- *For normal convergence, it is enough to take* $r \in [0, R[$ *and* $x \in [-r, +r]$.

Definition 6.1.2 R *is the radius of convergence of the series.* $\overline{D_r} = \{z \in \mathbb{C}/|z| \leq r\}$ *is the* *convergence disk. By convention, we have* $D_0 = \emptyset$ *and* $D_{+\infty} = \mathbb{C}$. *In the real case* $]-R, R[$ *is the convergence interval.*

Proposition 6.1.2 - *if* $\lim_{n\longrightarrow+\infty} \sqrt[n]{|a_n|} = L \in [0, +\infty[$ *then* $R = \dfrac{1}{L}$.

- *if* $\lim_{n\longrightarrow+\infty} \left|\frac{a_{n+1}}{a_n}\right| = L \in [0, +\infty[$ *then* $R = \dfrac{1}{L}$.

Example 6.1.1 *Consider the entire series of general term* $U_n(z) = n!z^n$, *with* $z \neq 0$. *To find out if this series converges, we use the D'Alembert criterion. Thus, we have :* $\left|\dfrac{a^{n+1}(z)}{a^n(z)}\right| = \left|\dfrac{(n+1)!z^{(n+1)}}{n!z^n}\right| = (n+1)|z|$. *Or* $\lim_{n\longrightarrow+\infty}(n+1)|z| = +\infty$ *therefore for all* $z \neq 0$ *the series diverges, we say that the radius of convergence of this entire series is* 0.

Example 6.1.2 *Let the entire series of general term* $U_n(z) = \dfrac{z^n}{n^\alpha}$, *with* $\alpha \in \mathbb{R}$. *We have* $\left|\dfrac{a^{n+1}(z)}{a^n(z)}\right| = \dfrac{n^\alpha}{(n+1)^\alpha}|z| = \left(\frac{n}{n+1}\right)|z| \underset{n\longrightarrow+\infty}{\longrightarrow} |z|$. *So the convergence radius is* $R = 1, (|z| < 1)$.

Example 6.1.3 *Let the entire series of general term* $U_n(z) = \dfrac{z^n}{n^{3n}}$. *We have* $\sqrt[n]{|U_n(z)|} = \dfrac{|z|}{n^3} \underset{n\longrightarrow+\infty}{\longrightarrow} 0$ *; Then the radius of convergence is* $R = +\infty$ *(convergence for all* z*).*

Example 6.1.4 *Let the entire series of general term* $U_n(z) = a^{n\sin(\frac{2n\pi}{3}+\frac{\pi}{6})} z^n$, *with* $(a > 1)$. *We have* $\sqrt[n]{|U_n(z)|} = a^{\sin(\frac{2n\pi}{3}+\frac{\pi}{6})}|z|$. *When* $n \in \mathbb{N}$ *the function* $n \longrightarrow \sin(\frac{2n\pi}{3} + \frac{\pi}{6})$ *takes the values* $\dfrac{1}{2}$ *and* -1, *Now we have* $\sup_{k\geq n} \sqrt[n]{|U_n(z)|} = a^{\frac{1}{2}}|z|$. *so* $a^{\frac{1}{2}}|z| < \dfrac{1}{a^{\frac{1}{2}}} = a^{-\frac{1}{2}}$. *Then* $R = a^{-\frac{1}{2}}$.

Remark 6.1.2 (Study on the edge of the convergence disk) *For an entire series* $\sum_{n=0}^{+\infty} a_n z^n =$ *whose radius of convergence* R *is distinct from* 0 *and* $+\infty$, *la Proposition (4.1.5) eaves in doubt, the nature of the entire series at a point* z *on the edge of the convergence disk, that is to say for* $z \in \mathbb{C}$ *such that* $z = R$. *All cases can occur : absolute convergence, semi-convergence, divergence. For example*

(1) $U_n(z) = n^\alpha z^n$ *with* $(\alpha \in \mathbb{R}_+^*), R = 1$. *We have* $|U_n(z)| = n^\alpha$, *for all* z *such that* $|z| = 1$ *does not tend to* 0 *when* n *tends to* ∞. *The series diverges at any point from the edge of the convergence disk.*

(2) $U_n(z) = \dfrac{z^n}{n^\alpha}$ *with* $(\alpha \in \mathbb{R}_+^*)$, $R = 1$. *If* $\alpha > 1$, *we notice that* $|U_n(z)| = \dfrac{1}{n^\alpha}$ *for* $|z|$ *the Riemann series* $\left(\dfrac{1}{n^\alpha}\right)$ *is convergent, therefore the entire series is absolutely convergent at every point on the edge of the convergence disk. If* $0 < \alpha \leq 1$, *let us study the cases* $z = \pm 1$, *pour*

$z = 1$ *we have the divergent Riemann series* $\left(\frac{1}{n^\alpha}\right)$. *If* $z = -1$ *we have the alternating series* $\left(\frac{(-1)^n}{n^\alpha}\right)$ *convergent.*

6.2 Operations on entire series

We consider the entire series of general term $U_n(z) = a_n z^n$ and $V_n(z) = b_n z^n$ with convergence radius R_a and R_b respectively.

- If we consider the entire series of general term $W_n(z) = c_n z^n = U_n(z) + V_n(z) = (a_n + b_n)z^n$. Then $R_c \geq \min(R_a, R_b)$, moreover if $R_a \neq R_b, R_c = \min(R_a, R_b)$.
- If we consider the entire series of general term $W_n(z) = d_n z^n = \sum_{n=0}^{+\infty} U_n(z) + V_{n-k}(z) = \sum_{n=0}^{+\infty} \left(a_k z^k\right)\left(b_{n-k} z^{n-k}\right) = z^n \sum_{n=0}^{+\infty}(a - kb_{n-k})$, then $R_c \geq \min(R_a, R_b)$

Example 6.2.1 *We consider the series* $S_1 = \sum_{n=0}^{+\infty} z^n$ *and* $S_2 = \sum_{n=0}^{+\infty} -z^n$*, We have* $U_n(z) = z^n, V_n(z) = -z^n, (\forall n, a_n = 1, b_n = -1)$*, then* $R_a = 1$ *and* $R_b = 1$*. So* $S = \sum_{n=0}^{+\infty} c_n z^n = \sum_{n=0}^{+\infty}(a_n + b_n)z^n = \sum_{n=0}^{+\infty}(1 + (-1))z^n = 0$*, has convergence radius* $R_c = +\infty \geq \min(R_a, R_b)$.

Example 6.2.2 *We consider the series* $S_1 = \sum_{n=0}^{+\infty} z^n, R_a = 1$*. We consider the entire series defined as follows :* $b_0 = 1, b_1 = -1$*, and* $\forall n \geq 2, b_n = 0$*, we deduce that :* $S_2 = 1 - z$ *which has convergence radius* $R_b = +\infty$*. If we calculate* $S = S_1 \times S_2$ *we obtain* $W_n(z) = d_n z^n = z^n \sum_{n=0}^{+\infty}(a_k b_{n-k})$*, we deduces that :* $d_0 = a_0 b_0 = 1, d_1 = a_0 b_1 + a_1 b_0 = -1 + 1 = 0$ *and therefore* $\forall n \geq 1, d_n = 0, R_d = +\infty$.

6.3 Derivation and integration of integer series

6.3.1 Properties

Theorem 6.3.1 *Let* $\sum_{n=0}^{+\infty} a_n z^n$ *be an integer series with sum* $f(z)$*. The function* $z \longrightarrow f(z)$ *is continuous in the convergence disk of the series.*

Proposition 6.3.1 *Let the entire series be defined by :* $S(x) = \sum_{n=0}^{+\infty} a_n x^n$ *such that* $x \in]-R, +R[$*, where* R *is the radius of convergence. If the function* S *is* $C^{+\infty}$ *then* $]-R, +R[$ *alors* $\acute{S}(x) = \sum_{n=1}^{+\infty} n a_n x^{n-1} = \sum_{n=0}^{+\infty} (n + 1)\, a_{n+1} x^n$*. So, this new series also has a radius of convergence* R.

corollary 6.3.1 *Let the entire series be defined by :* $S(x) = \sum_{n=0}^{+\infty} a_n x^n$ *such that* $x \in]-R, +R[$, *where R is the radius of convergence. We set* $T(x) = \sum_{n=1}^{+\infty} a_{n-1}\frac{x^n}{n} = \sum_{n=0}^{+\infty} a_n \frac{x^{n+1}}{n+1}$. *The radius of convergence of this series is also equal to R, and* $\forall x \in]-R, +R[, \acute{T}(x) = S(x)$.

6.3.2 Applications

1. We know the geometric series $\forall x \in]-1, +1[, \sum_{n=0}^{+\infty} x^n = \frac{1}{1-x}$, and we know an antiderivative $\int_{-1}^{1} \frac{1}{1-x} dx = -\ln(1-x)$. The primitive of $\sum_{n=0}^{+\infty} x^n$ is equal to $\sum_{n=0}^{+\infty} \frac{x^{n+1}}{n+1}$. Then we deduce that $\ln(1-x) = -\sum_{n=1}^{+\infty} \frac{x^n}{n}$. What's more

$$\forall x \in]-1, +1[, \ln(1+x) = \ln(1-(1-x)) = -\sum_{n=1}^{+\infty} \frac{(-x)^n}{n} = \sum_{n=1}^{+\infty} (-1)^{n-1} \frac{x^n}{n}.$$

2. What's more

$$\forall x \in]-1, +1[, \arctan(x) = \frac{1}{2}[\ln(1+x) - \ln(1-x)]$$

$$\arctan(x) = \frac{1}{2}\left[\sum_{n=1}^{+\infty} (-1)^{n-1}\frac{x^n}{n} + \sum_{n=1}^{+\infty} \frac{x^n}{n}\right] = \left[\sum_{n=1}^{+\infty} \frac{(-1)^{n-1}+1}{2}\right]\frac{x^n}{n}.$$

For all odd n we have $n = 2p+1, \forall p \in \mathbb{N}$

$$\arctan(x) = \sum_{p=0}^{+\infty}\left[\frac{(-1)^{(2p+1)-1}+1}{2}\frac{x^{2p+1}}{2p+1}\right] = \sum_{p=0}^{+\infty}\frac{x^{2p+1}}{2p+1}, \forall x \in]-1, +1[.$$

3. Consider the function $\arctan x$ which is the primitive of $\frac{1}{1+x^2}$ which vanishes at 0. To obtain its expansion into an integer series $\forall x \in]-1, +1[$ we use the previous result , we obtain

$$\frac{1}{1+x^2} = \frac{1}{1+(-x^2)} = \sum_{n=0}^{+\infty} (-1)^n x^{2n}$$

By integrating this series with $\arctan 0 = 0$ to determine the constant, we obtain

$$\forall x \in]-1, +1[, \arctan x = \sum_{n=0}^{+\infty} (-1)^n \frac{x^{2n+1}}{2n+1}.$$

4. We seek the expansion in integer series of the function $f(x) = \frac{1}{1-3x+2x^2}$, since $1-3x+2x^2 = 2(x-1)(x-\frac{1}{2})$, then $f(x) = \frac{1}{x-1} + \frac{1}{(x-\frac{1}{2})}$, but we have

$\frac{1}{x-1} = -\frac{1}{x-1} = -\sum_{n=0}^{+\infty} x^n$ et $\frac{1}{(x-\frac{1}{2})} = \frac{2}{2(x-\frac{1}{2})} = -\frac{2}{1-2x} = -2\sum_{n=0}^{+\infty}(2x)^n$, with $-1 < x < 1$. So $f(x) = -\sum_{n=0}^{+\infty} x^n = 2\sum_{n=0}^{+\infty}(2x)^n = \sum_{n=0}^{+\infty}(-1-2^{n+1})x^n$ we set $a_n = (-1-2^{n+1})$, so $f(x) = \sum_{n=0}^{+\infty} a_n x^n, R = \frac{1}{2}$.

Chapitre 7

FOURIER SERIES

L Fourier series are series of functions of a particular type, which are used to study periodic functions. The idea is to express any 2π-periodic function as a linear combination of simple 2π-periodic functions, of the form $\cos(nx)$ or $\sin(nx)$, with $n \in \mathbb{N}$. This "linear combination" will, in general, be an infinite sum, that is to say a series :

7.1 Definitions et proprieties

Definition 7.1.1 (trigonometric series) *We call a trigonometric series a series of functions* $\sum f_n$ *whose general term is of the form* $f_n(x) = a_n \cos(nx) + b_n \sin(nx)$ *with* $x \in \mathbb{R}$ *and, for all* $n \in \mathbb{N}, a_n \in \mathbb{C}$ *and* $b_n \in \mathbb{C}$.

Proprietie 7.1.1 (Convergence 1) *If* $\sum a_n$ *and* $\sum b_n$ *converge absolutely, then the trigonometric series* $\sum (a_n \cos(nx) + b_n \sin(nx))$ *converges normally on* $\mathbb{R}$.

Proprietie 7.1.2 (Convergence 2) *If the sequences* $(a_n)_{n\in\mathbb{N}}$ *and* $(b_n)_{n\in\mathbb{N}}$ *are real, decreasing, and tend towards* 0 *then, for all* $x_0 \in \mathbb{R}/2\pi\mathbb{Z}$ *fixed* $\sum (a_n \cos(nx_0) + b_n \sin(nx_0))$ *converges. Moreover for all* $\varepsilon > 0$, $\sum (a_n \cos(nx) + b_n \sin(nx))$ *converges uniformly on each interval of the form* $[2n\pi + \varepsilon, 2(n+1)\pi\varepsilon]$ *with* $n \in \mathbb{Z}$.

The proof of this property is an application of the uniform Abel rule. We then have :

Proprietie 7.1.3 (Complex writing) *Any trigonometric series :*

$$\sum_{n\in\mathbb{N}} (a_n \cos(nx) + b_n \sin(nx))$$

can be rewritten in the form $\sum_{n\in\mathbb{Z}} c_n e^{inx}$ *with* $c_0 = a_0$ *and* $\forall n \in \mathbb{N}, c_n = \dfrac{a_n - ib_n}{2}$ *and* $c_{-n} = \dfrac{a_n + ib_n}{2}$. *Then,* $\forall n \in \mathbb{N}, a_n = c_n + c_{-n}$ *and* $bn = i(c_n - c_{-n})$.

When a trigonometric series converges uniformly on $[-\pi, \pi]$, we can find its coefficients according to its sum

Propretie 7.1.4 (Evaluation of the coefficients) *Let* $\sum (a_n \cos(nx) + b_n \sin(nx))$ *be a trigonometric series uniformly convergent on* $[-\pi, \pi]$. *Note,*

$$S(x) = \sum_{n\in\mathbb{N}} (a_n \cos(nx) + b_n \sin(nx)),$$

for $x \in \mathbb{R}$. *Then* $a_0 = \dfrac{1}{2\pi}\int_{-\pi}^{\pi} S(x)dx$ *and for all* $n \in \mathbb{N}^*$, $a_n = \dfrac{1}{\pi}\int_{-\pi}^{\pi} S(x)\cos(nx)dx$ *and* $b_n = \dfrac{1}{\pi}\int_{-\pi}^{\pi} S(x)\sin(nx)dx$.

Remark 7.1.1 *1. S is an* $\mathbb{R} \longrightarrow \mathbb{C}$ *function. We therefore have here integrals of functions* $\mathbb{R} \longrightarrow \mathbb{C}$ *to which we must give meaning. By definition, for* $f : \mathbb{R} \longrightarrow \mathbb{C}$, $\int_a^b f(x)dx = \int_a^b \mathcal{R}e(f(x))dx + i\int_a^b \mathcal{I}m(f(x))dx$.

2. We have no expression for b_0. *In fact, since* b_0 *is the coefficient of* $\sin(0x) = 0$, *it has no importance, we can choose for example* $b_0 = 0$.

If the trigonometric series is given by its complex writing, the expressions simplify :

Propretie 7.1.5 (Trigo-complexe serie) *Let* $\sum_{n\in\mathbb{Z}} c_n e^{inx}$ *be a trigonometric series written in complex form which converges uniformly on* $[-\pi; \pi]$. *Let us note, for all* $x \in \mathbb{R}, S(x) = \sum_{n=-\infty}^{+\infty} c_n e^{inx}$. *Then for all* $n \in \mathbb{Z}, c_n = \dfrac{1}{2\pi}\int_{-\pi}^{\pi} S(x)e^{-inx}dx$.

Remark 7.1.2 *Since* $\cos(nx)$ *and* $\sin(nx)$ *are* 2π*-periodic, so* $S(x)$ *is* 2π*-periodic. Because of this, we can change the integration interval : for all* $\alpha \in \mathbb{R}$, *for all* $n \in \mathbb{Z}, c_n = \dfrac{1}{2\pi}\int_{\alpha}^{\alpha+2\pi} S(x)e^{-inx}dx$. *The same is true for* a_n *and* b_n.

Now that we have studied trigonometric series, we can return to the initial program : given any 2π-periodic function, can we rewrite it as the sum of a trigonometric series ?

Definition 7.1.2 (Fourier series) *Let* f *is* 2π*-periodic, Its Fourier series is by definition the trigonometric series* $\sum_{n\in\mathbb{N}} (a_n \cos(nx) + b_n \sin(nx))$ *defined by* $a_0 = \dfrac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx$ *and for all*

$n \in \mathbb{N}, a_n = \dfrac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)dx$ and $b_n = \dfrac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)dx$, if these integrals are defined. Or, equivalently, it is the trigonometric series written in complex form $\sum_{n\in\mathbb{Z}} c_n e^{inx}$ where, for all $n \in \mathbb{Z}, c_n = \dfrac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}dx$. The coefficients a_n and b_n (or, equivalently c_n) are called ***Fourier coefficients*** *of f.*

Propretie 7.1.6 (Parity) *1. Since f is 2π-periodic function, we can change the integration interval to $[\alpha, \alpha + 2\pi]$, for all $\alpha \in \mathbb{R}$.*

2. If f is even, for all $n \in \mathbb{N}, b_n = 0$.

3. If f is odd, for all $n \in \mathbb{N}, a_n = 0$.

Analogously to what happens when we develop a function in integer series, given a function f 2[redacted]-periodic whose Fourier coefficients are defined, two questions arise :

1. Does the Fourier series of f converge ?
2. If yes, does it converge to f f ?

Unfortunately, as with entire series, the answer may be no to each of these questions. There is a whole theory describing the convergence of the Fourier series under various assumptions about f. Among this theory, we will retain for this course the following result :

Theorem 7.1.1 (Dirichlet Jordan) *Let f be a 2π-periodic function continuous on $[-\pi, \pi]$ sexcept possibly at a finite number of points. We assume that at these points of discontinuity, f admits a finite right limit and a left limit. Finally, we suppose that f admits at every point of $[-\pi, \pi]$ a right derivative and a left derivative (finite). Then for all $x \in \mathbb{R}$, the Fourier series of f is convergent at x and has the sum $\dfrac{1}{2}\left(\lim\limits_{y\longrightarrow x^+} f(y) + \lim\limits_{y\longrightarrow x^-} f(y)\right)$. In particular, at any point x where f is continuous, the sum of its Fourier series is $f(x)$.*

It is convenient to reinterpret the theory of Fourier series using the notions of vector space and dot product. We can then retain certain aspects of the Fourier series by keeping in mind the analogy with the simple vector space that is $\mathbb{R}^2$, which is equipped with the scalar product $\vec{x}.\vec{y} = x_1y_1 + x_2y_2$. This analogy is written in a more natural way when we use the complex writing of Fourier series. The space which, for Fourier series, plays the role of the vector space $\mathbb{R}^2$ is the set of periodic functions $\mathcal{F} = \{f : R \longrightarrow \mathbb{C}; 2\pi - \text{and whose squareis integrable on } [-\pi, \pi]\}$. We can define a product on $\mathcal{F}$ (a function $\mathcal{F} \times \mathcal{F} \longrightarrow \mathcal{C}$) which will play the role of the scalar product of $\mathbb{R}^2$:

Definition 7.1.3 (Scalar product) *For $f, g \in \mathcal{F}$, we call the scalar product of f and g, and we note (f, g) the complex number $(f, g) = \dfrac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\bar{g}(x)dx$ where $\bar{g}(x)$ denotes the conjugate complex number of $g(x)$.*

When we have a scalar product, we can define a norm :

Definition 7.1.4 (Norm) *Let $f \in \mathcal{F}$. We call the norm of f and we note $\|f\|$ the positive real number $\|f\| = \sqrt{(f, f)}$.*

Remark 7.1.3 *The norm of $\mathbb{R}^2$ is constructed this way from the scalar product : $\|\vec{x}\| = \sqrt{\vec{x}.\vec{x}} = \sqrt{x_1^2 + x_2^2}$.*

Propretie 7.1.7 (Orthonormal basis) *The (infinite) set of functions $\{x \mapsto e^{inx}, n \in \mathbb{Z}\}$ forms an orthonormal basis (infinite) of $\mathcal{F}$ provided with the scalar product. Indeed we have already seen that for all $n_0 \in \mathbb{Z}$,*

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{inx}e^{-in_0x}dx = \begin{cases} 1 \text{ if } n = n_0 \\ 0 \text{ otherwise} \end{cases},$$

which translates to :

$$\left(e^{in_0x}, e^{inx}\right) = \begin{cases} 1 \text{ if } n = n_0 \\ 0 \text{ otherwise} \end{cases},$$

which is the definition of an orthonormal family. The fact that this family contains enough elements to be considered a base requires further development :

The difference between $\mathbb{R}^2$ and $\mathcal{F}$ is that an orthonormal basis of $\mathbb{R}^2$ contains only 2 elements while an orthonormal basis of F contains infinitely many elements. We say that $\mathcal{F}$ is of infinite dimension. By analogy with $\mathbb{R}^2$, we say that we have decomposed $f \in \mathcal{F}$ according to the orthonormal basis $\{x \mapsto e^{inx}, n \in \mathbb{Z}\}$ if we found coefficients $c_n \in \mathbb{Z}$ such that $\lim\limits_{N \longrightarrow +\infty} \left\|f(x) - \sum_{n=-N}^{+N} c_n e^{inx}\right\| = 0$. The previous proposition asserts that this decomposition is possible for all $f \in \mathcal{F}$. Then we get the following interpretation.

7.2 Geometric interpretation of Fourier series

Let $f \in \mathcal{F}$. Its Fourier series is nothing other than its decomposition according to the orthonormal basis $\{x \mapsto e^{inx}, n \in \mathbb{Z}\}$. This interpretation allows us to retain the expression of the Fourier coefficients of f :

Propretie 7.2.1 (Orthogonal projection) *Let $f \in \mathcal{F}$. For all $n \in \mathbb{Z}$ its Fourier coefficient c_n is the orthogonal projection of f on e^{inx}, i.e. $c_n = (f(x), e^{inx}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}dx$.*

Finally, this interpretation makes it possible to connect the norm of f with its Fourier coefficients :

Theorem 7.2.1 (Parseval-Bessel) *Let $f \in \mathcal{F}$ and $\{c_n, n \in \mathbb{Z}\}$ be its Fourier coefficients in complex writing, $\{(a_n, b_n), n \in \mathbb{N}\}$ be its Fourier coefficients in real writing. Then the norm of f verifies :*

*1. **Bessel inegality** : for all $N \in \mathbb{N}$,*

$$\begin{aligned}\|f\|^2 = (f, f) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\bar{f}(x)dx &\geq \sum_{n=-N}^{N} |c_n|^2 \\ &= |a_0|^2 + \frac{1}{2}\sum_{n=1}^{N}(|a_n|^2 + |b_n|^2).\end{aligned}$$

*2. **Parseval egality :***

$$\|f\|^2 = (f, f) = \sum_{n=-\infty}^{+\infty} |c_n|^2 = |a_0|^2 + \frac{1}{2}\sum_{n=1}^{+\infty}(|a_n|^2 + |b_n|^2).$$