Exercise series N°1

Exercise 1 *:* Consider the following assertions:

A_1 : $\exists x \in \mathbb{R}$, $\forall y \in \mathbb{R}$: $x + y > 0$.

A_2 : $\forall x \in \mathbb{R}$, $\exists y \in \mathbb{R}$: $x + y > 0$.

A_3 : $\forall x \in \mathbb{R}$, $\forall y \in \mathbb{R}$: $x + y > 0$.

A_4 : $\exists x \in \mathbb{R}$, $\forall y \in \mathbb{R}$: $y^2 > x$.

1. Are assertions A_1, A_2, A_3 and A_4 true or false?
2. Give their negation.

Exercise 2 *:*

- If a and b are two positive or zero real numbers, show that:

$$\sqrt{a} + \sqrt{b} \leq 2\sqrt{a+b}.$$

- Prove by induction the following equalities:

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2} \quad and \quad \sum_{k=0}^{n-1} 2^k = 2^n - 1, \quad with \ \ n \in \mathbb{N}^*$$

- Show that $\sqrt{2}$ is not a rational number.

Exercise 3 Let x and $y \in \mathbb{R}$.

1. Show that the following relationships are always true:
   (a) If $|x| < y$ *then* $-y < x < y$
   (b) $|x + y| \leq |x| + |y|$.
   (c) $||x| - |y|| \leq |x - y|$.
2. Solve the following inequalities:
   (a) $|x - 2| > 5$.
   (b) $|x + 2| > |x|$.
   (c) $|2x - 1| < |x - 1|$.

Exercise 4 Determine (if they exist): the all upper and lower bounds, supremum, infimum, maximum, and minimum, of the following sets:

$$E_1 = \left\{1, \frac{1}{3}, \frac{1}{5}, ..., \frac{1}{2n+1}, ...; \quad n \in \mathbb{N}\right\}, \qquad E_2 =]0, 5], \qquad E_3 = \left\{4 - \frac{1}{n}; n \in \mathbb{N}^*\right\},$$

$$E_4 = \left\{\frac{1}{2} + \frac{n}{2n+1}, \ \frac{1}{2} - \frac{n}{2n+1}; \quad n \in \mathbb{N}^*\right\}$$

Exercise 5 Show that the following relationships are true.

- $x - 1 < E(x) \leq x$,
- $E(x) + E(y) \leq E(x + y)$,
- $E(x) - E(y) \geq E(x - y)$,
- $E\left(\frac{E(nx)}{n}\right) = E(x)$,

with $x,\ y \in \mathbb{R}$, $n \in \mathbb{N}^*$ and $E(.)$ is the integral part function.

Solution

Solution of the Exercise 1 *:*

A_1 : is false, because we can find an y in $\mathbb{R}$ such that for any x in $\mathbb{R}$ we have $x+y$ less or equal to zero $(x+y \leq 0.)$
For example, if we take $y=0$, then for all x negative $(x \leq 0)$ we have $x+y=x \leq 0$
The negation: $\forall x \in \mathbb{R}$, $\exists y \in \mathbb{R}$: $x+y \leq 0$.

A_2 : is true, the fact that for any x we can find an $y \in \mathbb{R}$ for which the inequality $x+y>0$ is verified.
For exemple, if we take $y=-x+1$ then $x+y=1>0$.
The negation: $\exists x \in \mathbb{R}, \quad \forall y \in \mathbb{R}: \quad x+y \leq 0$.

A_3 : is false, because if we choose, for example, $y \leq 0$ and $x \leq 0$ then $x+y<0$.
The negation: $\exists x \in \mathbb{R}, \quad \exists y \in \mathbb{R}: \quad x+y \leq 0$.

A_4 : is true, and it is the fact that for all $y \in \mathbb{R}$, it is enough to take an x in the interval $]-\infty, y^2[$ for the inequality $y^2>x$ to be verified.
The negation: $\forall x \in \mathbb{R}$, $\exists y \in \mathbb{R}$: $y^2 \leq x$.

Solution of the Exercise 2 *:*

- For two positive or zero real numbers a and b, we have:

  $$\left\{\begin{array}{l} a \leq a+b \\ b \leq a+b \end{array}\right. \Longrightarrow \left\{\begin{array}{l} \sqrt{a} \leq \sqrt{a+b}.....(*) \\ \sqrt{b} \leq \sqrt{a+b}.....(**) \end{array}\right. \text{ (the fact that the root function is an increasing function)}$$

  by adding the two sides of the inequalities (*) and (**), we will have:

  $$\sqrt{a}+\sqrt{b} \leq 2\sqrt{a+b}.$$

- Recall that the proof by induction is based on the following three steps:

  Step 1: Verify that the desired result holds for $n=n_0$

  Step 2: Assume that the desired result holds for n.

  Step 3: Use the assumption from step 2 to show that the result holds for $(n+1)$.

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}, \quad with \quad n \in \mathbb{N}^* \tag{1}$$

$$\sum_{k=0}^{n-1} 2^k = 2^n-1, \quad with \quad n \in \mathbb{N}^* \tag{2}$$

for $n=1$:

$$\left\{\begin{array}{lllll} \sum\limits_{k=1}^{n} k & = & \sum\limits_{k=1}^{1} k & = & 1 \\ \frac{n(n+1)}{2} & = & \frac{1(1+1)}{2} & = & \frac{2}{2}=1 \end{array}\right. \tag{3}$$

for n**:** We assume that the following equality is true for n.

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2} \tag{4}$$

for $n+1$**:** On the one hand, using the assumption (4), we have:

$$\sum_{k=1}^{n+1} k = \sum_{k=1}^{n} k + (n+1) = \frac{n(n+1)}{2} + (n+1) = \frac{(n+1)(n+2)}{2}$$

On the other hand we have:

$$\sum_{k=1}^{n+1} k = \frac{(n+1)((n+1)+1)}{2} = \frac{(n+1)(n+2)}{2}$$

Consequently, the equality (1) holds for $n+1$. From the above three steps we conclude that (1) holds for all $n \in \mathbb{N}^*$.

for $n = 1$**:**

$$\begin{cases} \sum_{k=0}^{n-1} 2^k & = & \sum_{k=0}^{0} 2^k & = & 2^0 = 1 \\ 2^n - 1 & = & 2^1 - 1 & = & 2^1 - 1 = 1 \end{cases} \tag{5}$$

So the equality holds for $n = 1$.

for n**:** We assume that the following equality is true for n.

$$\sum_{k=0}^{n-1} 2^k = 2^n - 1 \tag{6}$$

for $n+1$**:** On the one hand, using the assumption (6), we have:

$$\sum_{k=0}^{n+1-1} 2^k = \sum_{k=0}^{n} 2^k = \sum_{k=1}^{n-1} 2^k + 2^n = 2^n - 1 + 2^n = 2 \times 2^n - 1 = 2^{n+1} - 1$$

On the other hand we have:

$$\sum_{k=0}^{n} 2^k = 2^{n+1} - 1$$

Consequently, the equality (2) holdsF for $n+1$. From the above three steps we conclude that (2) holds for all $n \in \mathbb{N}^*$.

- Proof By Contradiction that $\sqrt{2}$ is irrational
  Recall that for $n \in \mathbb{N}$, we have:

$$n \text{ is an odd natural number } \Leftrightarrow n^2 \text{ is an odd natural number.}$$

$$n \text{ is an even natural number } \Leftrightarrow n^2 \text{ is an even natural number.}$$

Note: The demonstration of the two equivalences above is an additional exercise to be left for the student. Assume that $\sqrt{2}$ is rational.
Then, let $\sqrt{2} = \frac{p}{q}$, where $p \in \mathbb{Z}$ and $q \in \mathbb{Z}^*$, and p and q are relatively prime i.e $gcd(p,q) = 1$.

$$\sqrt{2} = \frac{p}{q} \Rightarrow 2 = \frac{p^2}{q^2} \Rightarrow p^2 = 2q^2 \Rightarrow p^2 \text{ is even } \Rightarrow p \textbf{ is even}, \text{ say } p = 2m$$
$$\Rightarrow 4m^2 = 2q^2 \Rightarrow 2m = q^2 \Rightarrow q \textbf{ is even}.$$

Thus, both p and q are even and have 2 as a common factor. But we assumed that p and q are relatively prime. This is a contradiction. Thus, $\sqrt{2}$ cannot be written as $\frac{p}{q}$ for $p \in \mathbb{Z}$ and $q \in \mathbb{Z}^*$ Thus $\sqrt{2}$ is irrational.

Solution of the Exercise 3 Let x and $y \in \mathbb{R}$.

1. From the definition of the absolute value we have:

$$\begin{cases} x < y, & \text{if } x \geq 0; \\ -x < y, & \text{if } x < 0. \end{cases} \Longrightarrow \begin{cases} x < y, & \text{if } x \geq 0; \\ x > -y, & \text{if } x < 0. \end{cases} \Longrightarrow -y < x < y. \tag{7}$$

2. We have

$$\begin{cases} -|x| \leq x \leq |x| \\ -|y| \leq y \leq |y| \end{cases} \Longrightarrow -|x| - |y| \leq x + y \leq |x| + |y| \Longrightarrow -(|x| + |y|) \leq x + y \leq (|x| + |y|) \tag{8}$$

As $|x| + |y| \geq 0$, then from (7) and (8) we can conclude that :

$$|x + y| \leq |x| + |y|. \tag{9}$$

3. $||x| - |y|| \leq |x - y|$?
We have

$$\begin{cases} |x| \leq |(x - y) + y| \\ |y| \leq |(y - x) + x| \end{cases} \text{using the inequality (9)} \Rightarrow \begin{cases} |x| \leq |(x - y) + y| \leq |(x - y)| + |y| \\ |y| \leq |(y - x) + x| \leq |(y - x)| + |x| \end{cases}$$

$$\Rightarrow \begin{cases} |x| \leq |(x - y)| + |y| \\ |y| \leq |(y - x)| + |x| \end{cases} \Rightarrow \begin{cases} |x| \leq |(x - y)| + |y| \\ |y| \leq |(y - x)| + |x| \end{cases} \Rightarrow \begin{cases} |x| - |y| \leq |x - y| \\ |y| - |x| \leq |x - y| \end{cases} \Rightarrow \begin{cases} |x| - |y| \leq |x - y| \\ |x| - |y| \geq -|x - y| \end{cases}$$

Finally,

$$-|x - y| \leq |x| - |y| \leq |x - y|.$$

Thus, from the result proven at the beginning of the exercise, we conclude that

$$||x| - |y|| \leq |x - y|.$$

Resolution of inequalities:

1. $|x - 2| > 5$. we have the inequality $|x - 2| > 5$, then using the absolute value definition, we can be rewritten the inequality as follows:

$$\begin{cases} (x - 2) > 5, & \text{if } x - 2 \geq 0; \\ -(x - 2) > 5, & \text{if } x - 2 < 0. \end{cases} \Rightarrow \begin{cases} (x - 2) > 5, & \text{if } x \geq 2; \\ -(x - 2) > 5, & \text{if } x < 2. \end{cases} \Rightarrow \begin{cases} x > 7, & \text{if } x \geq 2; \\ x < -3, & \text{if } x < 2. \end{cases} \tag{10}$$

Thus, the solutions of the inequality $|x - 2| > 5$ are:

$$x \in]-\infty, -3[\cup]7, +\infty[.$$

2. $|x + 2| > |x|$.

x	-2	0	
$\lvert x\rvert$	$-x$	$-x$	x
$\lvert x+2\rvert$	$-x-2$	$x+2$	$x+2$
	A	B	C

We notice that three situations are possible:

Case A:

$$for\ x \in]-\infty, -2[, \quad -x - 2 > -x \Rightarrow x + 2 < x \Rightarrow 2 < 0$$

Thus the set of solution in this case is empty i.e. $E_A = \{\} = \varnothing$

Case B:

$$for\ x \in [-2,0],\ x+2 > -x \Rightarrow x > -1$$

Thus, the set of solution in this case $x \in [-2,0]$ and $x > -1$ i.e. $E_B =]-1,0]$

Case C:

$$for\ x \in]0,+\infty[,\ x+2 > x \Rightarrow 2 > 0. \text{ This latest inequality is always true}, x \in \mathbb{R}$$

Thus, the set of solution in this case $x \in]0,+\infty[$ and $x \in \mathbb{R}$ i.e. $E_C =]0,+\infty[$

From the three cases above, we conclude that the set of solutions to the inequality $|x+2| > |x|$ is:

$$E = E_A \cup E_B \cup E_C = \varnothing \cup]-1,0] \cup]0,+\infty[=]-1,+\infty[.$$

3. $|2x-1| < |x-1|$. Note that:

x	$1/2$	1	
$\lvert 2x-1\rvert$	$-2x+1$	$2x-1$	$2x-1$
$\lvert x-1\rvert$	$-x+1$	$-x+1$	$x-1$
	A	B	C

With the same reasoning as in Example 2, we can show the following:

$$E_A = \left]0, \frac{1}{2}\right[, \quad E_B = \left[\frac{1}{2}, \frac{2}{3}\right[, \quad and \quad E_C = \varnothing$$

$$\Rightarrow E = \left]0, \frac{2}{3}\right[.$$

Solution of the Exercise 4

1. max, min, sup, inf, lb, ub of E_1 we have

$$n \in \mathbb{N} \Leftrightarrow 0 \leq n < \infty \Leftrightarrow 1 \leq n+1 < \infty \Leftrightarrow 0 < \frac{1}{n+1} \leq 1 \Leftrightarrow E_1 =]0,1]. \tag{11}$$

From (11), we conclude that

lb: $lb =]-\infty;0]$.

inf: $inf = max(]-\infty;0]) = 0$.

min: the minimum of E_1 does not exist, because E_1 is an open interval on the left side.

ub: $ub = [1;+\infty[$

sup: $sup = min([1;+\infty[) = 1$.

min: max=1 (because $1 \in E_1$).

2. max, min, sup, inf, lb, ub of E_2

lb: $lb =]-\infty;0]$.

inf: $inf = max(]-\infty;0]) = 0$.

min: the minimum of E_2 does not exist, because E_2 is an open interval on the left side.

ub: $ub = [5;+\infty[$

sup: $sup = min([5;+\infty[) = 5$.

min: max=5 (because $5 \in E_2$).

3. max, min, sup, inf, lb, ub of E_3

$$n \in \mathbb{N}^* \Leftrightarrow 1 \leq n < \infty \Leftrightarrow 0 < \frac{1}{n} \leq 1 \Leftrightarrow -1 \leq \frac{-1}{n} < 0 \Leftrightarrow 3 \leq 4 - \frac{1}{n} < 4 \Leftrightarrow E_1 = [3, 4]. \qquad (12)$$

From (12), we conclude that

lb: $lb =]-\infty; 3]$.

inf: $inf = max(]-\infty; 3]) = 3$.

min: min=3;

ub: $ub = [4; +\infty[$

sup: $sup = min([4; +\infty[) = 4$.

min: the maximum of E_3 does not exist, because E_3 is an open interval on the right side.

4. max, min, sup, inf, lb, ub of E_3 Let's define the following subsets:

$$u_n = \frac{1}{2} + \frac{n}{2n+1}, \quad n \in \mathbb{N}^*$$

$$v_n = \frac{1}{2} - \frac{n}{2n+1}; \quad n \in \mathbb{N}^*$$

It is easy to show that u_n is an increasing sequence while v_n is a decreasing sequence. Indeed,

$$\begin{aligned} u_{n+1} - u_n &= \left(\frac{1}{2} + \frac{n+1}{2n+3}\right) - \left(\frac{1}{2} + \frac{n}{2n+1}\right) \\ &= \frac{(2n^2 + n + 2n + 1) - (2n^2 + 3n)}{(2n+3)(2n+1)} \\ &= \frac{1}{(2n+3)(2n+1) \geq 0} > 0 \\ &\Leftrightarrow \ u_n \text{ is an increasing sequence.} \end{aligned}$$

$$\begin{aligned} v_{n+1} - v_n &= \left(\frac{1}{2} - \frac{n+1}{2n+3}\right) - \left(\frac{1}{2} - \frac{n}{2n+1}\right) \\ &= \frac{-(2n^2 + n + 2n + 1) + (2n^2 + 3n)}{(2n+3)(2n+1)} \\ &= \frac{-1}{(2n+3)(2n+1)} < 0 \\ &\Leftrightarrow \ v_n \text{ is a decreasing sequence.} \end{aligned}$$

so,

$$\begin{cases} u_1 \leq u_n < \lim\limits_{n\to\infty} u_n, \\ \lim\limits_{n\to+\infty} v_n < v_n \leq v_1, \end{cases} \quad \begin{cases} \frac{5}{6} \leq u_n < 1, \\ 0 < v_n \leq \frac{1}{6}, \end{cases} \qquad (13)$$

At this level, to answer the main question of the exercise we can proceed in two ways:
First way:

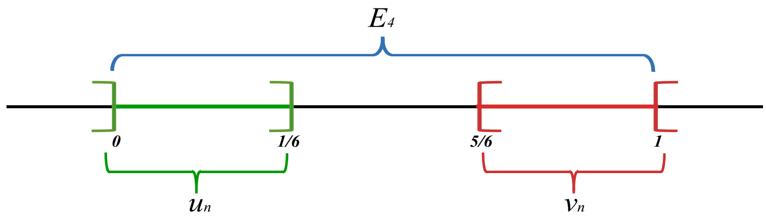
lb: we have $lb_u =]-\infty; \frac{5}{6}]$ and $lb_v =]-\infty; 0] \ \Rightarrow \ lb_{E_4} = lb_u \cap lb_u =]-\infty; 0]$.

inf: we have $inf_u = \frac{5}{6}]$ and $inf_v = 0 \ \Rightarrow \ inf_{E_4} = min(inf_u, inf_v) = 0$.

min: we have $min_u = \frac{5}{6}]$ and min_v does not exist $\Rightarrow lb_{E_4}$ does not exist.;

ub: we have $ub_u = [1; +\infty[$ and $ub_v = [\frac{1}{6}; +\infty[\ \Rightarrow \ lb_{E_4} = ub_u \cap ub_u = [1; +\infty[$.

sup: we have $sup_u = 1$ and $sup_v = \frac{1}{6} \ \Rightarrow \ sup_{E_4} = max(sup_u, sup_v) = 1$.



min: we have max_u does not exist and $max_v = \frac{1}{6} \Rightarrow lb_{E_4}$ does not exist.;

Second way: From (13), we note that

$$u_n \in \left[\frac{5}{6};1\right[\text{ and } v_n \in \left]0;\frac{1}{6}\right] \Rightarrow E_5 = \left]0;\frac{1}{6}\right] \cup \left[\frac{5}{6};1\right[,$$

thus,

lb: $lb =]-\infty;0]$.

inf: $inf = max(]-\infty;0]) = 0$.

min: minimum does not exist;

ub: $ub = [1;+\infty[$

sup: $sup = min([1;+\infty[) = 1$.

min: the maximum does not exist.

Solution of the Exercise 5

- $x - 1 < E(x) \leq x$?
  According to the definition of the integer part of a real number, we have

$$\begin{aligned} E(x) \leq x < E(x)+1 &\Leftrightarrow 0 \leq x - E(x) < 1 \\ &\Leftrightarrow 0 \leq x - E(x) < 1 \\ &\Leftrightarrow -x \leq -E(x) < -x+1 \\ &\Leftrightarrow x \geq E(x) > x-1. \end{aligned}$$

- $E(x) + E(y) \leq E(x+y)$?
  Let x and y two real numbers. We have

$$\begin{cases} x = E(x) + R_x, & \text{with } R_x \in [0,1[; \\ y = E(y) + R_y, & \text{with } R_y \in [0,1[; \end{cases}$$

On the one hand, as $R_x + R_y < 2$ then

$$R_x + R_y = \begin{cases} 0, & \text{if } R_x + R_y \in [0;1[; \\ 1, & \text{if } R_x + R_y \in [1;2[; \end{cases}$$

On the other hand,

$$\begin{aligned} E(x+y) &= E(E(x) + R_x + E(y) + R_y) \\ &= E\left((E(x)+E(y)) + (R_y + R_x)\right) \\ &= E(x) + E(y) + E(R_x + R_y) \end{aligned}$$

Consequently,

$$\begin{aligned} & \begin{cases} E(x+y) = E(x) + E(y), & \text{if } R_x + R_y \in [0;1[; \\ E(x+y) = E(x) + E(y) + 1, & \text{if } R_x + R_y \in [1;2[; \end{cases} \\ \Rightarrow \quad & \begin{cases} E(x+y) = E(x) + E(y), & \text{if } R_x + R_y \in [0;1[; \\ E(x+y) > E(x) + E(y), & \text{if } R_x + R_y \in [1;2[; \end{cases} \\ \Rightarrow \quad & E(x+y) \geq E(x) + E(y). \end{aligned}$$

- $E(x) - E(y) \geq E(x-y)$?
  Let $x, y \in \mathbb{R}$.

$$E(x) = E((x-y)+y) \geq E(x-y) + E(y) \Rightarrow E(x) - E(y) \geq E(x-y).$$

- $E\left(\frac{E(nx)}{n}\right) = E(x)$?
  According to the definition of the integer part of a real number, we have

$$\begin{aligned} E(x) \leq x < E(x)+1 \quad & \Leftrightarrow \quad nE(x) \leq nx < nE(x) + n \\ & \Leftrightarrow \quad E\left(nE(x)\right) \leq E(nx) < E\left(nE(x)+n\right), \ (E(.) \text{ is an increasing function}) \\ & \Leftrightarrow \quad nE(x) \leq E(nx) < nE(x) + n \text{ (integer part of an integer number)} \\ & \Leftrightarrow \quad E(x) \leq \frac{E(nx)}{n} < E(x) + 1 \text{ (definition of E(.))} \\ & \Leftrightarrow \quad E\left(\frac{E(nx)}{n}\right) = E(x). \end{aligned}$$

Exo1:

1.a) $z = 6 \Rightarrow |z| = 6.$

$z = 6\left(\cos(\overset{?}{\theta}) + i\sin(\overset{?}{\theta})\right)$

$= 6(1 + 0i)$

$\Rightarrow \begin{cases} \cos(\theta) = 1 \\ \sin(\theta) = 0 \end{cases}$

$\Rightarrow \theta = 0.$

$\Rightarrow z = 6e^{0i}$

$= 6\left(\cos(0) + i\sin(0)\right)$

1.b) $z = 6i \Rightarrow |z| = 6.$

$\Rightarrow z = 6\left(\cos(\overset{?}{\theta}) + i\sin(\overset{?}{\theta})\right)$

$\Rightarrow \begin{cases} \cos(\theta) = 0 \\ \sin(\theta) = 1 \end{cases} \Rightarrow \theta = \frac{\pi}{2}.$

$\Rightarrow z = 6\left(\cos\left(\frac{\pi}{2}\right) + i\sin\left(\frac{\pi}{2}\right)\right)$

$= 6e^{\frac{\pi}{2}i}$

1.c) $z = 2 + 2i$

$|z| = \sqrt{2^2 + 2^2} = \sqrt{4 \times 2} = 2\sqrt{2}.$

$\Rightarrow z = 2\sqrt{2}\left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i\right)$

$= 2\sqrt{2}\left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i\right)$

Recall that

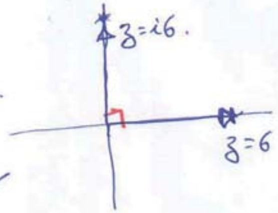
① $z = x + iy = re^{\theta i}$

$= r\left(\cos(\theta) + i\sin(\theta)\right)$

with $\begin{cases} r = |z| \\ \theta = \arg(z). \end{cases}$

② $z^n = r^n e^{n\theta i}$

$= r^n\left(\cos(n\theta) + i\sin(n\theta)\right)$



$= 2\sqrt{2}\left(\cos(\theta) + i\sin(\theta)\right)$

$\Rightarrow \begin{cases} \cos(\theta) = \frac{\sqrt{2}}{2} \\ \sin(\theta) = \frac{\sqrt{2}}{2}. \end{cases}$

$\Rightarrow \boxed{\theta = \frac{\pi}{4}}$

thus $z = 2\sqrt{2}\left(\cos\left(\frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{4}\right)\right)$

$= 2\sqrt{2}e^{\frac{\pi}{4}i}$

1.d) $z = -2 + 2i$

$\Rightarrow |z| = \sqrt{(-2)^2 + 2^2}$

$= \sqrt{2 \times 4} = 2\sqrt{2}.$

$z = 2\sqrt{2}\left(-\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\right)$

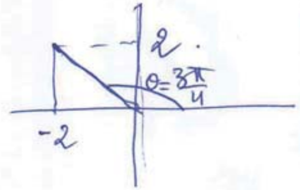
$= 2\sqrt{2}\left(\cos(\overset{?}{\theta}) + i\sin(\overset{?}{\theta})\right)$

$\Rightarrow \begin{cases} \cos(\theta) = -\frac{\sqrt{2}}{2}. \\ \sin(\theta) = \frac{\sqrt{2}}{2}. \end{cases}$

$\Rightarrow \theta = \frac{3\pi}{4}$

$\Rightarrow z = 2\sqrt{2}\left(\cos\left(\frac{3\pi}{4}\right) + i\sin\left(\frac{3\pi}{4}\right)\right)$

$= 2\sqrt{2}\,e^{\frac{3\pi}{4}i}$



1.e) $z = 3 + \sqrt{3}i$

$|z| = \sqrt{3^2 + \sqrt{3}^2} = \sqrt{12}$

$= 2\sqrt{3}.$

$\Rightarrow z = 2\sqrt{3}\left(\frac{3}{2\sqrt{3}} + \frac{\sqrt{3}}{2\sqrt{3}}i\right)$

$= 2\sqrt{3}\left(\frac{\sqrt{3}}{2} + \frac{1}{2}i\right)$

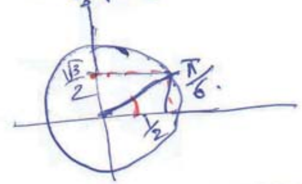
$= 2\sqrt{3}\left(\cos(\theta) + i\sin(\theta)\right).$

$\Rightarrow \begin{cases} \cos(\theta) = \frac{\sqrt{3}}{2} \\ \sin(\theta) = \frac{1}{2} \end{cases}$

$\Rightarrow \boxed{\theta = \frac{\pi}{6}}$

$\Rightarrow z = 2\sqrt{3}\left(\cos\left(\frac{\pi}{6}\right) + i\sin\left(\frac{\pi}{6}\right)\right)$

$= 2\sqrt{3}\,e^{\frac{\pi}{6}i}$



2.a: $z = 3e^{\frac{5\pi}{4}i}$

$\Rightarrow z = 3\left(\cos\left(\frac{5\pi}{4}\right) + i\sin\left(\frac{5\pi}{4}\right)\right)$

$= 3\left(\cos\left(\pi + \frac{\pi}{4}\right) + i\sin\left(\pi + \frac{\pi}{4}\right)\right)$

$= 3\left(-\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}\right)$

$= -\frac{3\sqrt{2}}{2} - i\frac{3\sqrt{2}}{2}.$

$\Rightarrow \begin{cases} a = -\frac{3\sqrt{2}}{2} \\ b = -\frac{3\sqrt{2}}{2}. \end{cases}$

2.b) $z = 5e^{\frac{7\pi}{4}i}$

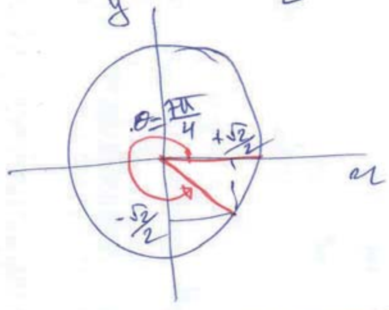
$\Rightarrow z = 5\left(\cos\left(\frac{7\pi}{4}\right) + i\sin\left(\frac{7\pi}{4}\right)\right).$

$= 5\left(\cos\left(2\pi - \frac{\pi}{4}\right) + i\sin\left(2\pi - \frac{\pi}{4}\right)\right).$

$= 5\left(\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}\right)$

$$\Rightarrow z = \frac{5\sqrt{2}}{2} - \frac{5\sqrt{2}}{2} i$$

$$\Rightarrow \begin{cases} a = \frac{5\sqrt{2}}{2} \\ b = -\frac{5\sqrt{2}}{2} \end{cases}$$



2.c

$$z = 2\, e^{\frac{\pi}{12} i}$$

$$= 2\left(\cos\left(\tfrac{\pi}{12}\right) + i \sin\left(\tfrac{\pi}{12}\right)\right)$$

$\cos\left(\frac{\pi}{12}\right) = ?$ $\quad \sin\left(\frac{\pi}{12}\right) = ?$

we have $\begin{cases} \cos^2(\theta) + \sin^2(\theta) = 1 & (*) \\ \cos(2\theta) = 2\cos^2(\theta) - 1 & (**) \end{cases}$

from (**) we deduce that

$$\cos(\theta) = \sqrt{\frac{\cos(2\theta) + 1}{2}}$$

thus $\cos\left(\frac{\pi}{12}\right) = \sqrt{\frac{\cos\left(\frac{\pi}{6}\right) + 1}{2}}$

$$= \sqrt{\frac{\frac{\sqrt{3}}{2} + 1}{2}}$$

$$\boxed{\cos\left(\tfrac{\pi}{12}\right) = \frac{\sqrt{\sqrt{3}+2}}{2}} = \alpha$$

And from (*) we obtain

$$\sin\left(\tfrac{\pi}{12}\right) = \sqrt{1 - \cos^2\left(\tfrac{\pi}{12}\right)}$$

$$= \sqrt{1 - \frac{\sqrt{3}+2}{4}}$$

$$\boxed{\sin\left(\tfrac{\pi}{12}\right) = \frac{\sqrt{2-\sqrt{3}}}{2}} = \beta$$

thus,

$$z = 2(\alpha + i\beta).$$

$$= (2\alpha) + (2\beta) i$$

$$\Rightarrow \begin{cases} a = 2\alpha \\ b = 2\beta. \end{cases}$$

2.d). $z = \sqrt{16 e^{\frac{2\pi}{3} i}}$

$$\Rightarrow z = \left(16\, e^{\frac{2\pi}{3} i}\right)^{1/2}$$

$$= 4 \times e^{\frac{\pi}{3} i}$$

$$= 4\left(\cos\left(\tfrac{\pi}{3}\right) + i \sin\left(\tfrac{\pi}{3}\right)\right)$$

$$= 4\left(\tfrac{1}{2} + i \tfrac{\sqrt{3}}{2}\right).$$

$$\boxed{z = 2 + 2\sqrt{3} i}$$

$$\Rightarrow \begin{cases} a = 2 \\ b = 2\sqrt{3} \end{cases}$$

In general (~~particular~~ when the power "n" is large enought, in order to avoid tedious calculations of $z = w^n$ we first write w in it polar form and subsequently use the DeMoivre formula.

3.a) $z = (2+2i)^8$

from the example (1.c) we have

$$(2+2i) = 2\sqrt{2}\left(\cos\left(\tfrac{\pi}{4}\right) + i \sin\left(\tfrac{\pi}{4}\right)\right).$$

So, $(2+2i)^8 = (2\sqrt{2})^8 \left(\underbrace{\cos\left(\tfrac{8\pi}{4}\right)}_{1} + \underbrace{i \sin\left(\tfrac{8\pi}{4}\right)}_{0}\right).$

$$= (2\sqrt{2})^8 = 4096.$$

3.b) $z = ~~\sqrt[3]{3+\sqrt{3}}~~ = (3+\sqrt{3} i)^{3/2}.$

From l'example (1.e) we have

$$3 + \sqrt{3} i = 2\sqrt{3}\left(\cos\left(\tfrac{\pi}{6}\right) + i \sin\left(\tfrac{\pi}{6}\right)\right)$$

So, $(3+\sqrt{3} i)^{3/2} = (2\sqrt{3})^{3/2}\left(\cos\left(\frac{\pi}{6} \times \frac{3}{2}\right) + i \sin\left(\frac{3}{2} \times \frac{\pi}{6}\right)\right)$

$$= (2\sqrt{3})^{\frac{3}{2}}\left(\cos\left(\tfrac{\pi}{4}\right) + i \sin\left(\tfrac{\pi}{4}\right)\right).$$

$$= (2\sqrt{3})^{\frac{3}{2}}\left(\tfrac{\sqrt{2}}{2} + i \tfrac{\sqrt{2}}{2}\right)$$

4.a) Let $w = \frac{1+i}{\sqrt{2}} = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} i = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} i$ ie $z^2 = w$.

Suppose that $z = x + iy$ with $x, y \in \mathbb{R}$.

$$\begin{cases} z^2 = w \\ |z^2| = |z|^2 = |w| \end{cases} \Rightarrow \begin{cases} (x^2 - y^2) + (2xy)\, i = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} i \\ x^2 + y^2 = 1 \end{cases}$$

$$\Rightarrow \begin{cases} x^2 - y^2 = \frac{\sqrt{2}}{2} & \text{—— (1)} \\ x^2 + y^2 = 1 & \text{—— (2)} \\ 2xy = \frac{\sqrt{2}}{2} & \text{—— (3)} \end{cases}$$

then

$\Rightarrow$ (1) + (2) $\Rightarrow 2x^2 = 1 + \frac{\sqrt{2}}{2}$

$\Rightarrow x = \pm \frac{\sqrt{2+\sqrt{2}}}{2}$ —— (4)

(2) − (1) $\Rightarrow 2y^2 = 1 - \frac{\sqrt{2}}{2}$.

$\Rightarrow y = \pm \frac{\sqrt{2-\sqrt{2}}}{2}$ —— (5)

From (3), (4) and (5) we deduce that

$$z = \begin{cases} \frac{\sqrt{2+\sqrt{2}}}{2} + \frac{\sqrt{2-\sqrt{2}}}{2} i \\ -\frac{\sqrt{2+\sqrt{2}}}{2} - \frac{\sqrt{2-\sqrt{2}}}{2} i \end{cases} \text{—— } (*)$$

4.b) From Example (1.c) we have

$2 + 2i = 2\sqrt{2}\left(\cos\left(\frac{\pi}{4}\right) + i \sin\left(\frac{\pi}{4}\right)\right)$.

$\Rightarrow \frac{1+i}{\sqrt{2}} = \cos\left(\frac{\pi}{4}\right) + i \sin\left(\frac{\pi}{4}\right)$

thus, $z = \sqrt{\frac{1+i}{\sqrt{2}}} = \left(\frac{1+i}{\sqrt{2}}\right)^{\frac{1}{2}} = \cos\left(\frac{\pi}{4} \times \frac{1}{2}\right) + i \sin\left(\frac{\pi}{4}\frac{1}{2}\right)$

$= \cos\left(\frac{\pi}{8}\right) + i \sin\left(\frac{\pi}{8}\right)$ —— $(**)$

So, by superposition of $(*)$ with $(**)$ we will have:

$$\begin{cases} \cos\left(\frac{\pi}{8}\right) = \frac{\sqrt{2+\sqrt{2}}}{2} \\ \sin\left(\frac{\pi}{8}\right) = \frac{\sqrt{2-\sqrt{2}}}{2} \end{cases}$$

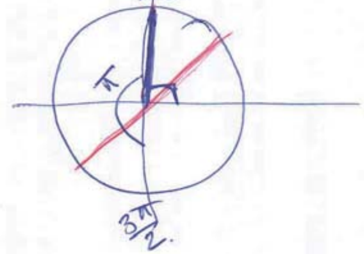
EXO2: (2.1) $z = 1 + i \Rightarrow z = \sqrt{2}\left(\cos\left(\frac{\pi}{4}\right) + i \sin\left(\frac{\pi}{4}\right)\right)$

$\Rightarrow z^n = (\sqrt{2})^n \left(\cos\left(\frac{n\pi}{4}\right) + i \sin\left(\frac{n\pi}{4}\right)\right)$

z^n is pure imaginary $\Rightarrow \frac{n\pi}{4} = \frac{\pi}{2} + k\pi \ / k \in \mathbb{Z}$

$\cos\left(\frac{n\pi}{4}\right) = 0$

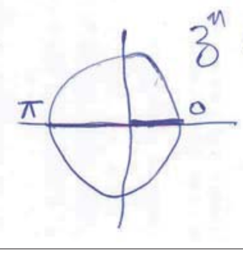
$\Rightarrow \boxed{n = 4k + 2 / k \in \mathbb{Z}}$



z^n is a real number $\Rightarrow \sin\left(\frac{n\pi}{4}\right) = 0$

$\Rightarrow \frac{n\pi}{4} = k\pi \ / k \in \mathbb{Z}$

$\Rightarrow \boxed{n = 4k / k \in \mathbb{Z}}$



(2.a) let $z = x+iy$ / $x, y \in \mathbb{R}$ then

As $z \neq -1 \Rightarrow x \neq -1$, $y \neq 0$

$$w = \frac{z-i}{z+1}$$

$$= \frac{x+iy-i}{x+iy+1} = \frac{x+(y-1)i}{(x+1)+iy} = \frac{[x+(y-1)i]*[(x+1)-iy]}{[(x+1)+iy]*[(x+1)-iy]}$$

$$= \frac{x(x+1) - ixy + (y-1)(x+1)i + (y-1)y}{(x+1)^2+y^2}$$

$$= \frac{x(x+1)+(y-1)y}{|w|^2} + \frac{[(y-1)(x+1) - xy]}{|w|^2} i$$

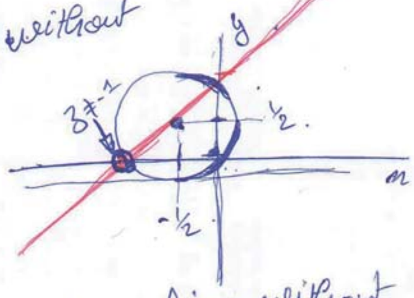
w is ~~pure~~ imaginary $\Rightarrow$

$$x(x+1) + (y-1)y = 0 \Rightarrow x^2 + x + y^2 - y = 0$$

$$\Rightarrow \left(x^2 + 2\times\tfrac{1}{2}x + \left(\tfrac{1}{2}\right)^2\right) + \left(y^2 - 2\times\tfrac{1}{2}y + \left(\tfrac{1}{2}\right)^2\right) = \frac{1}{2}$$

$$\Rightarrow \left(x+\tfrac{1}{2}\right)^2 + \left(y - \tfrac{1}{2}\right)^2 = \frac{1}{2} \quad — (*)$$

$\Rightarrow$ (*) is a circle with center $(-\frac{1}{2}, \frac{1}{2})$ and Radius $\frac{\sqrt{2}}{2}$ without a point $(-1, 0)$



(2.b) w is a real number

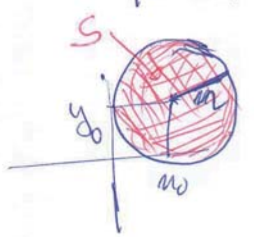
$$\Rightarrow (y-1)(x+1) - xy = 0$$

$$\Rightarrow \cancel{yx} + y - x - 1 - \cancel{xy} = 0$$

$\boxed{y = x+1} \Rightarrow (x, y)$ form ligne without the point $(-1, 0)$.

Exo3: suppose that $z_0 = x_0 + iy_0$ and $z = x+iy$

$$|z - z_0| \leq r \Rightarrow |z - z_0|^2 \leq r^2$$



$$\Rightarrow |(x-x_0) + i(y-y_0)|^2 \leq r^2$$

$$\Rightarrow (x-x_0)^2 + (y-y_0)^2 \leq r^2$$

$\Rightarrow$ the solution is the disk delimited by the circle with center (x_0, y_0) and radius r.

2. $|2z+i| \leq |\bar{z}+1|$ / we put $z = x+iy$.

$$\Rightarrow |2(x+iy)+i|^2 \leq |(x+iy)+1|^2$$

$$\Rightarrow |2x + 2iy + i|^2 \leq |(x+1)+iy|^2.$$

$$\Rightarrow 4x^2 + (2y+1)^2 \leq (x+1)^2 + y^2.$$

$$\Rightarrow 4x^2 + 4y^2 + 4y + \cancel{1} \leq x^2 + 2x + \cancel{1} + y^2.$$

$$\Rightarrow 3x^2 + 3y^2 + 4y - 2x \leq 0$$

$$\Rightarrow x^2 + y^2 + \tfrac{4}{3}y - \tfrac{2}{3}x \leq 0$$

$$\Rightarrow \left(x^2 - 2\times\tfrac{1}{3}x + \tfrac{1}{9}\right) + \left(y^2 + 2\,\tfrac{2}{3}y + \tfrac{4}{9}\right) \leq \frac{5}{9}.$$

$$\boxed{\begin{array}{c} \left(x - \frac{1}{3}\right)^2 + \left(y + \frac{2}{3}\right)^2 \leq \frac{5}{9} \\ (x - x_0)^2 + (y - y_0)^2 \leq r^2 \end{array}}$$

the same thing as the first exeple

EXO4 :

a/ $5z + 2i = (i+1)z - 3 \Rightarrow 5z - (i+1)z = -3 - 2i$

$\Rightarrow z = \dfrac{-3-2i}{4-i}$

$\Rightarrow z = \dfrac{(-3-2i)*(4+i)}{4^2 + 1^2}$

$\Rightarrow \boxed{z = \left(\dfrac{-10}{17}\right) + \left(\dfrac{-11}{17}\right)i}$

b) $\dfrac{z-i}{z+1} = 4i \Rightarrow z - i = 4i(z+1)$

$\Rightarrow (1-4i)z = 4i + i$

$\Rightarrow z = \dfrac{5i}{1-4i} = \boxed{\dfrac{-20}{17} + \dfrac{5}{17}i = z}$

c) $2z + i\bar{z} = 3$ / we put $z = x + iy$

So $2z + i\bar{z} = 3 \Leftrightarrow 2(x+iy) + i(x-iy) = 3$

$\Leftrightarrow 2x + 2iy + ix + y = 3$

$\Leftrightarrow 2x + (2y)i + ix + y = 3$

$\Leftrightarrow (2x+y) + (x+2y)i = 3$

$\Rightarrow \begin{cases} 2x + y = 3 \\ x + 2y = 0 \end{cases} \Rightarrow \begin{cases} x = 2 \\ y = -1 \end{cases}$

$\Rightarrow \boxed{z = 2 - i}$

Corrigé de l'Exercice n°5

1) En réécrivant autrement le polynôme P, à savoir : $P(z) = z^3 - 22z - 36 + i(9z^2 + 12z - 12) = z^3 - 22z - 36 + 3i(3z^2 + 4z - 4)$, on s'apperçoit que si z_1 est une racine réelle de P, alors on doit avoir nécessairement $z_1^3 - 22z_1 - 36 = 0$ et $3z_1^2 + 4z_1 - 4 = 0$. Cherchons donc les racines **réelles** du polynôme $R(z) = 3z^2 + 4z - 4$ en calculant son discriminant : $\Delta = 4^2 - 4\times3\times(-4) = 16 + 48 = 64 = 8^2$ d'où l'existence de deux racines réelles $\dfrac{-4-\sqrt{64}}{2\times3} = -2$ et $\dfrac{-4+\sqrt{64}}{2\times3} = \dfrac{2}{3}$. Sur ces deux racines, seule –2 est racine du polynôme $S(z) = z^3 - 22z - 36$. Ainsi la seule racine réelle de P est $\boxed{z_1 = -2}$

2) Il existe donc un polynôme $Q(z)$ tel que $P(z) = (z-(-2))Q(z) \Leftrightarrow P(z) = (z+2)Q(z)$, avec $\deg Q = \deg P - 1 = 2$, donc de la forme $Q(z) = az^2 + bz + c$.

Pour trouver Q, effectuons la division euclidienne du polynôme P par z-2 (puisque l'égalité ci-dessus entraîne $Q(z) = \dfrac{P(z)}{z+2}$, pour tout $z \neq -2$)

On obtient :

$$\begin{array}{l|l} z^3 + 9iz^2 + 2(6i-11)z - 3(4i+12) & z+2 \\ \hline z^3 + 2z^2 & z^2 + (9i-2)z\ -6i-18 \\ \quad (9i-2)z^2 + 2(6i-11)z - 3(4i+12) & \\ \quad (9i-2)z^2 + 2(9i-2)z & \\ \qquad (-6i-18)z - 3(4i+12) & \\ \qquad (-6i-18)z - 12i - 36 & \\ \qquad\qquad 0 & \end{array}$$

Le polynôme Q est donc :

$\boxed{Q(z) = z^2 + (9i-2)z - 6(i+3)}$

3) On calcule le discriminant du polynôme Q :

$\Delta = (9i-2)^2 - 4\times1\times(-6(i+3)) = -81 - 36i + 4 + 24i + 72 = -5 - 12i$. L'astuce est de remarquer que $-5 - 12i = (2-3i)^2$, ce qui permet de calculer les deux racines complexes de Q : L'une vaut $\dfrac{-(9i-2)-(2-3i)}{2} = \dfrac{-6i}{2} = -3i$ et l'autre vaut $\dfrac{-(9i-2)+(2-3i)}{2} = \dfrac{-12i+4}{2} = -6i + 2$. L'équation Q(z)=0 admet donc une solution imaginaire pure : $\boxed{z_2 = -3i}$

4) L'autre solution de l'équation Q(z)=0 ayant été calculée ci-dessus, et par application de la règle du produit nul, $P(z) = 0 \Leftrightarrow (z+2)Q(z) = 0 \Leftrightarrow z + 2 = 0$ ou $Q(z) = 0$ et ainsi $\boxed{S = \{-2; -3i; -6i+2\}}$

5) Notons A le point d'affixe $z_1 = -2$, B le point d'affixe $z_2 = -3i$ et C le point d'affixe $z_3 = -6i + 2$

L'affixe du vecteur $\overrightarrow{AB}$ vaut $z_2 - z_1 = -3i + 2$. Celle du vecteur $\overrightarrow{AC}$ vaut $z_3 - z_1 = -6i + 4$

Puisque $z_3 - z_1 = 2(z_2 - z_1)$, on en déduit que $\overrightarrow{AC} = 2\overrightarrow{AB}$, c'est à dire que les vecteurs $\overrightarrow{AB}$ et $\overrightarrow{AC}$ sont colinéaires, donc que les points A,B et C sont alignés

Exercice n°6

1) La suite $\left(z_n\right)_{n\in\mathbb{N}}$ est une suite géométrique de raison $q=\dfrac{1+i\sqrt{3}}{4}$

Forme exponentielle de $q=\dfrac{1+i\sqrt{3}}{4}$: $|q|=\sqrt{\left(\dfrac{1}{4}\right)^2+\left(\dfrac{\sqrt{3}}{4}\right)^2}=\sqrt{\dfrac{4}{16}}=\sqrt{\dfrac{1}{4}}=\dfrac{1}{2}$, et si on note θ un argument de q, à 2π près, on a $\cos(\theta)=\dfrac{\frac{1}{4}}{\frac{1}{2}}=\dfrac{1}{2}$ et $\sin(\theta)=\dfrac{\frac{\sqrt{3}}{4}}{\frac{1}{2}}=\dfrac{\sqrt{3}}{2}$ d'où on reconnaît $\theta=\dfrac{\pi}{3}[2\pi]$ et ainsi $q=\dfrac{1}{2}e^{i\frac{\pi}{3}}$.

(on pouvait aussi directement remarquer que $q=\dfrac{1}{2}\left(\dfrac{1}{2}+\dfrac{i\sqrt{3}}{2}\right)=\dfrac{1}{2}\left(\cos\dfrac{\pi}{3}+i\sin\dfrac{\pi}{3}\right)=\dfrac{1}{2}e^{i\frac{\pi}{3}}$)

Ainsi, pour tout entier $n\in\mathbb{N}$ $z_n=z_0\times q^n=8\left(\dfrac{1}{2}\right)^n\left(e^{i\frac{\pi}{3}}\right)^n=8\left(\dfrac{1}{2}\right)^n e^{i\frac{n\pi}{3}}$

2) Pour tout entier $n\in\mathbb{N}$, puisque $z_n\neq 0$,

$$\frac{z_{n+1}-z_n}{z_{n+1}}=\frac{\frac{1+i\sqrt{3}}{4}z_n-z_n}{\frac{1+i\sqrt{3}}{4}z_n}=\frac{\frac{1+i\sqrt{3}}{4}-1}{\frac{1+i\sqrt{3}}{4}}=\frac{\frac{1+i\sqrt{3}}{4}-\frac{4}{4}}{\frac{1+i\sqrt{3}}{4}}=\frac{-3+i\sqrt{3}}{4}\times\frac{4}{1+i\sqrt{3}}=\frac{\left(-3+i\sqrt{3}\right)\left(1-i\sqrt{3}\right)}{\left(1+i\sqrt{3}\right)\left(1-i\sqrt{3}\right)}$$

$$=\frac{-3+3\sqrt{3}i+\sqrt{3}i-i^2\left(\sqrt{3}\right)^2}{1-\left(i\sqrt{3}\right)^2}=\frac{4\sqrt{3}i-3+3}{4}=\boxed{\sqrt{3}i}$$

En calculant module et argument de ce dernier complexe, on obtient $\left|\dfrac{z_{n+1}-z_n}{z_{n+1}}\right|=\left|\sqrt{3}i\right|\Leftrightarrow\dfrac{M_nM_{n+1}}{OM_{n+1}}=\sqrt{3}\Leftrightarrow$ $\boxed{M_nM_{n+1}=\sqrt{3}OM_{n+1}}$ (le réel k dont parle l'énoncé est $\sqrt{3}$.). De plus, $\arg\left(\dfrac{z_{n+1}-z_n}{z_{n+1}}\right)=\arg\left(\sqrt{3}i\right)[2\pi]\Leftrightarrow$ $\boxed{\left(\overrightarrow{OM_{n+1}};\overrightarrow{M_nM_{n+1}}\right)=\dfrac{\pi}{2}[2\pi]}$ donc $\boxed{\text{le triangle } OM_nM_{n+1} \text{ est rectangle en } M_{n+1}}$

3) Nous avons calculé, dans la question1), que pour tout entier $n\in\mathbb{N}$ $z_n=8\left(\dfrac{1}{2}\right)^n e^{i\frac{n\pi}{3}}$. Ainsi $\boxed{r_n=8\left(\dfrac{1}{2}\right)^n}$, et puisque $0<\dfrac{1}{2}<1$, $\lim\limits_{n\to+\infty}\left(\dfrac{1}{2}\right)^n=0$, donc $\boxed{\lim\limits_{n\to+\infty}r_n=0}$, donc $\boxed{\text{le point } M_n \text{ a pour position limite le point O lorsque } n \text{ tend vers plus l'infini.}}$

EX01

- $U_{n+1} - U_n = \left(\frac{1}{n+2} + \frac{1}{n+3}\right) - \left(\frac{1}{n+1} + \frac{1}{n+2}\right) = \frac{-2}{(n+1)(n+3)} < 0$

$\Rightarrow U_n \searrow$

- note that $U_n = n^2\left(1 - \frac{1}{n+1}\right)$ can be rewrit as follow

$$U_n = \frac{n^2(n+1) - n^2}{n+1} = \frac{n^2(n+1-1)}{n+1} = \boxed{\frac{n^3}{n+1} = U_n}$$

so $U_{n+1} - U_n = \frac{(n+1)^3}{n+2} - \frac{n^3}{n+1} = \frac{(n+1)^4 - n^3(n+2)}{(n+1)(n+2)}$

$$= \frac{(n^4 + 4n^3 + 6n^2 + 4n + 1) - (n^4 + 2n^3)}{(n+1)(n+2)}$$

$$= \frac{2n^3 + 6n^2 + 4n + 1}{(n+1)(n+2)} = \frac{2n(n+1)(n+2) + 1}{(n+1)(n+2)} > 0$$

$\Rightarrow U_n \nearrow$

- $U_{n+1} - U_n = (n+1)\left((n+1) - (-1)^{n+1}\right) - n\left(n - (-1)^n\right)$

$$= (n+1)^2 - (n+1)(-1)^{n+1} - n^2 + n(-1)^n$$

$$= \left[(n+1)^2 - n^2\right] + (n+1)\underline{\underline{(-1)^n}} + n\underline{\underline{(-1)^n}}$$

$$= \left[2n+1\right) + (n+1+n)(-1)^n = (2n+1)\left(1 + (-1)^n\right)$$

$$= \begin{cases} 2(2n+1) > 0 & \text{if } n = 2p \\ 0 \geq 0 & \text{if } n = 2p+1 \end{cases} \Rightarrow U_n \nearrow.$$

(1)

- $U_n = \sqrt[n]{a} = a^{\frac{1}{n}}$, in this case it is preferable to compare U_{n+1} and U_n by $\frac{U_{n+1}}{U_n}$.

$$\frac{U_{n+1}}{U_n} = \frac{a^{\frac{1}{n+1}}}{a^{\frac{1}{n}}} = a^{\frac{1}{n+1}} \cdot a^{\frac{1}{n}} = a^{\left(\frac{1}{n+1} - \frac{1}{n}\right)} = a^{\frac{-1}{n(n+1)}}$$

$$= \frac{1}{a^{\frac{1}{n(n+1)}}} \quad \left(\text{as } a > 1 \Rightarrow a^{\frac{1}{n(n+1)}} > 1\right.$$

$$\Rightarrow \frac{U_{n+1}}{U_n} = \frac{1}{a^{\frac{1}{n(n+1)}}} < 1 \Rightarrow U_n \searrow.$$

(2) $V_n = \frac{1}{n} \sum_{i=1}^{n} U_n$

$$V_{n+1} - V_n = \frac{1}{n+1} \sum_{i=1}^{n+1} U_i - \frac{1}{n} \sum_{i=1}^{n} U_i$$

$$= \frac{n \sum_{i=1}^{n+1} U_i - (n+1) \sum_{i=1}^{n} U_i}{n(n+1)}$$

$$= \frac{n S_n + n U_{n+1} - (n+1) S_n}{n(n+1)} \quad / S_n = \sum_{i=1}^{n} U_i$$

$$= \frac{n U_{n+1} - S_n}{(n+1)n} = \frac{\sum_{i=1}^{n} U_{n+1} - \sum_{i=1}^{n} U_i}{n(n+1)}$$

$$= \sum_{i=1}^{n} (U_{n+1} - U_i) / n(n+1)$$

(2)

it is clear that for all $i \leq n$

if $U_n \nearrow \Rightarrow U_{n+1} - U_i \geq 0 \Rightarrow V_{n+1} - V_n \geq 0 \Rightarrow V_n \nearrow$

if $U_n \searrow \Rightarrow U_{n+1} - U_n \leq 0 \Rightarrow V_{n+1} - V_n \leq 0 \Rightarrow V_n \searrow$

indeed,

$$U_{n+1} - U_i \geq 0 \Rightarrow \sum_{i=1}^{n} (U_{n+1} - U_i) \geq 0 \Rightarrow \frac{\sum (U_{n+1} - U_i)}{n(n+1)} \geq 0$$

$$\Rightarrow V_{n+1} - V_n \geq 0 \Rightarrow V_n \nearrow$$

$$U_{n+1} - U_i \leq 0 \Rightarrow \sum_{i=1}^{n} (U_{n+1} - U_i) \leq 0 \Rightarrow V_{n+1} - V_n \leq 0$$

$$\Rightarrow V_n \searrow .$$

EXO2:

① yes is true the fact that if $\lim_{n \to \infty} U_n = \ell$ then all of it subsequence tend to the same limit ℓ.

② false, Exaple we have $U_n = (-1)^n$ is a non-convergent sequence although U_{2n} and U_{2n+1} are c.v sequences

$$\lim_{n \to \infty} U_{2n} = 1 \text{ and } \lim_{n \to \infty} U_{2n+1} = -1$$

$$\lim_{n \to \infty} U_n = \nexists .$$

③ false; we have $U_n = (-1)^n$ is a non-cvt sequence despeet $\lim_{n \to \infty} U_{4n} = \lim_{n \to \infty} U_{4n+2} = 1$.

③

• True, ~~uuss~~ U_n c.v if and only if its subsequence

① $U_{k_1(n)}$ and $U_{k_2(n)}$ c.v towards the same limit

② and $k_1(n) \cup U_{k_2(n)} = U_n$.

EXO3:

• $\lim_{n \to \infty} U_n = 0 \Leftrightarrow \forall \varepsilon > 0, \exists N \in \mathbb{N} : |U_n| < \varepsilon_1, n \geq N$

$$|U_n| < \varepsilon_1 \Rightarrow \left|\frac{1}{n+1}\right| < \varepsilon_1 \Rightarrow \frac{1}{n+1} < \varepsilon_1$$

$$\Rightarrow n+1 > \frac{1}{\varepsilon_1}$$

$$\Rightarrow \boxed{n > \frac{1}{\varepsilon_1} + 1}$$

$$\Rightarrow \boxed{N = E\left(\frac{1}{\varepsilon_1}\right) + 1}$$

• $\lim_{n \to \infty} \sqrt[n]{a} = 1 \Leftrightarrow |\sqrt[n]{a} - 1| < \varepsilon_1$

$\Rightarrow \sqrt[n]{a} - 1 < \varepsilon_1$ (the fact $U_n \nearrow$ see exo1) $a > 1 \Rightarrow a^{\frac{1}{n}} > 1^{\frac{1}{n}}$.

$$\Rightarrow \sqrt[n]{a} < \varepsilon_1 + 1$$

$$\Rightarrow a^{\frac{1}{n}} < \varepsilon_1 + 1$$

$\Rightarrow \ln(a^{\frac{1}{n}}) < \ln(1+\varepsilon_1)$ ($\ln(x) \nearrow$)

$\Rightarrow \frac{1}{n} < \frac{\ln(1+\varepsilon_1)}{\ln(a)}$ / $\ln(a) > 0$

$$\Rightarrow \boxed{n > \frac{\ln(a)}{\ln(1+\varepsilon_1)}}$$

④

$$\Rightarrow \boxed{N = E\left(\frac{\ln(a)}{\ln(1+\varepsilon_1)}\right) + 1}$$

- $\lim_{n\to\infty} w_n = b \Rightarrow \left|\frac{(-1)^n + bn}{(n+1)} - b\right| < \varepsilon_1$

$$\Rightarrow \left|\frac{(-1)^n + bn - bn - b}{(n+1)}\right| < \varepsilon_1$$

$$\Rightarrow \left|\frac{(-1)^n - b}{n+1}\right| < \varepsilon_1 \quad (*)$$

we have $-1-b \leq (-1)^n - b \leq 1-b$

$$\Rightarrow |(-1)^n - b| \leq \max(|1+b|;|1-b|)$$

$$\Rightarrow |(-1)^n - b| \leq \begin{cases} 1-b & \text{if } b<0 \text{ — 1st case} \\ 1+b & \text{if } b\geq 0 \text{ — 2nd case} \\ 1 & \text{if } b = 0 \end{cases}$$

1st case: $b \leq 0$

$$\left|\frac{(-1)^n - b}{n+1}\right| < \varepsilon_1 \Rightarrow \frac{1-b}{n+1} < \varepsilon_1$$

$$\Rightarrow n+1 > \frac{1-b}{\varepsilon_1} \Rightarrow n > \frac{1-b}{\varepsilon_1} - 1$$

$$\Rightarrow \boxed{N = E\left(\frac{1-b}{\varepsilon_1} - 1\right) + 1} \Rightarrow \boxed{E\left(\frac{1-b}{\varepsilon_1}\right) = N}$$

2nd case: $b > 0$

$$\left|\frac{(-1)^n - b}{n+1}\right| < \varepsilon_1 \Rightarrow \frac{1+b}{n+1} < \varepsilon_1 \Rightarrow n > \frac{1+b}{\varepsilon_1} - 1$$

$$\Rightarrow \boxed{N = E\left(\frac{1+b}{\varepsilon_1}\right)}.$$

(c)

$\lim_{n\to\infty} T_n = 0 \Rightarrow |T_n - 0| < \varepsilon_1 \Rightarrow |c^n| < \varepsilon_1$

Let consider the 02 possible cases separately $n = 2p$ and $n = 2p+1$.

1st case: $n = 2p$. / $c \neq 0$

$$|c^n| < \varepsilon_1 \Rightarrow c^{2p} < \varepsilon_1 \Rightarrow p\ln(c^2) < \varepsilon_1$$

$$\Rightarrow p > \frac{\varepsilon_1}{\ln(c^2)} \quad / \ln(c^2) < 0 \text{ the fact } c^2 \in [0,1]$$

$$\Rightarrow 2p > \frac{2\varepsilon_1}{\ln(c^2)}$$

$$\Rightarrow n > \frac{2\varepsilon_1}{\ln(c^2)}$$

$$\Rightarrow \boxed{N_1 = E\left(\frac{2\varepsilon_1}{\ln(c^2)}\right) + 1}$$

2nd case and $c \neq 0$ $|c^{2p+1}| = |c * c^{2p}|$

$$|c^n| < \varepsilon_1 \Rightarrow |c| * |c^{2p}| < \varepsilon_1$$

$$\Rightarrow c^{2p} < \varepsilon/|c| \Rightarrow p\ln(c^2) < \ln(\varepsilon/|c|)$$

$$\Rightarrow p > \ln(\varepsilon/|c|)/\ln(c^2)$$

$$\Rightarrow 2p+1 > 2\alpha + 1 \quad / \alpha = \frac{\ln(\varepsilon/|c|)}{\ln(c^2)}$$

$$\Rightarrow n > 2\alpha + 1$$

$$\Rightarrow N_2 = E(2\alpha+1) = E(2\alpha) + 2$$

so, if $c \neq 0 \Rightarrow N = \max(N_1, N_2)$

(c)

if $c=0 \Rightarrow T_n = 0,\ \forall n \in \mathbb{N} \Rightarrow N = 0.$

Exo3.2

indication: replace $\varepsilon = 0{,}001$ in each expression of N.

③. K_n divergent $\Leftrightarrow \forall A \in \mathbb{R},\ \exists N \in \mathbb{N} : K_n < A,\ n > N.$

$K_n < A \Rightarrow \dfrac{-n^2+n+1}{n+1} < A$

$\Rightarrow \dfrac{(-n^2-2n-1)+3n+2}{n+1} < A$

$\Rightarrow -(n+1) + \dfrac{3n+2}{n+1} < A$ —— (*)

we have $\dfrac{3n+2}{n+1} \leq \dfrac{3n+3}{n+1} \leq 3$

$\Rightarrow \dfrac{3n+2}{n+1} \leq 3$ —— (**)

so, from (*) and (**) we deduce that:

$-(n+1) + 3 \leq A$

$\Rightarrow -(n+1) \leq A - 3.$

$\Rightarrow n+1 \geq 3 - A.$

$\Rightarrow \boxed{n \geq 2 - A}$

$\Rightarrow N_0 = E(2-A) + 1.$

$\Rightarrow \boxed{N = \max(N_0, 1).}$

7

Exo4.

- $\lim\limits_{n\to+\infty} U_n = \lim\limits_{n\to+\infty} \dfrac{n+(-1)^n}{n-(-1)^n} = \lim\limits_{n\to\infty} \dfrac{\cancel{n}}{\cancel{n}} \times \dfrac{1+\frac{(-1)^n}{n}}{1-\frac{(-1)^n}{n}} = 1.$

- $\lim\limits_{n\to\infty} U_n = \lim\limits_{n\to\infty} \sqrt{n+a} - \sqrt{n+b} = \lim\limits_{n\to\infty} \dfrac{(n+a) - (n+b)}{\sqrt{n+a} + \sqrt{n+b}}$

$= \lim\limits_{n\to\infty} \dfrac{a-b}{\sqrt{n+a} + \sqrt{n+b}} = 0$

- $\lim\limits_{n\to\infty} U_n = \lim \dfrac{a^n - b^n}{a^n + b^n}$ ⟶ $a < b$ ①; $a = 0 \Rightarrow \boxed{\lim\limits_{n\to\infty} U_n = 0}$; $a > b$ ②

① $\lim\limits_{n\to\infty} U_n = \lim\limits_{n\to\infty} \dfrac{\cancel{b^n}\left(\left(\frac{a}{b}\right)^n - 1\right)}{\cancel{b^n}\left(\left(\frac{a}{b}\right)^n + 1\right)} = \boxed{-1.}$

② $\lim\limits_{n\to\infty} U_n = \lim\limits_{n\to+\infty} \dfrac{\cancel{a^n}\left(1 - \left(\frac{b}{a}\right)^n\right)}{\cancel{a^n}\left(1 + \left(\frac{b}{a}\right)^n\right)} = \boxed{+1.}$

- $\lim\limits_{n\to\infty} U_n = \lim\limits_{n\to\infty} 1 + \left(\frac{-1}{a}\right) + \left(\frac{-1}{a}\right)^2 + \dots + \left(\frac{-1}{a}\right)^n$

$= \lim\limits_{n\to\infty} 1 \times \dfrac{1 - R^n}{1 - R} \ /\ R = -\dfrac{1}{a}$

1st case: $a \in]0, 1]$

$R \geq 1 \Rightarrow \lim\limits_{n\to\infty} U_n = \begin{pmatrix} +\infty \\ -\infty \end{pmatrix}$

2nd case: $a \in]1, +\infty[\Rightarrow 0 < R < 1$

$\Rightarrow \lim\limits_{n\to\infty} U_n = \dfrac{1}{1-R}$ such that $\boxed{R = -\frac{1}{a}.}$

- $\lim\limits_{n\to\infty} U_n = \lim\limits_{n\to\infty} \dfrac{n^2}{3^n} - \left(\dfrac{2}{3}\right)^n = 0$

1

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty}\left(1+\frac{a}{n}\right)^n = e^a. \text{ Indeed,}$$

we compute the limit of $\ln\left(1+\frac{a}{n}\right)^n$

$$\lim_{n\to\infty} \ln\left(1+\frac{a}{n}\right)^n = \lim_{n\to\infty} n \ln\left(1+\frac{a}{n}\right)$$

let $y = \frac{a}{n} \Rightarrow n = \frac{a}{y}$.

$$\Rightarrow \lim_{n\to\infty} n \ln\left(1+\frac{a}{n}\right) = \lim_{y\to 0} a\,\frac{\ln(1+y)}{y}$$

$$= a \underbrace{\lim_{y\to 0}\left(\frac{\ln(1+y)}{y}\right)}_{\to 1} = \boxed{a}$$

$$\Rightarrow \lim_{n\to\infty}\left(1+\frac{a}{n}\right)^n = \lim_{n\to\infty} e^{n\ln\left(1+\frac{a}{n}\right)} = \boxed{e^a}$$

$$u_n = \sum_{k=1}^{n} \frac{1}{\sqrt{k}} - \frac{1}{\sqrt{1+k}} = \left(\frac{1}{\sqrt{1}} - \cancel{\frac{1}{\sqrt{2}}}\right) + \left(\cancel{\frac{1}{\sqrt{2}}} - \cancel{\frac{1}{\sqrt{3}}}\right)$$

$$\cdots + \left(\cancel{\frac{1}{\sqrt{n}}} - \frac{1}{\sqrt{n+1}}\right)$$

$$\boxed{\sum\left(\frac{1}{\sqrt{k}} - \frac{1}{\sqrt{k+1}}\right) = 1 - \frac{1}{\sqrt{n+1}}}$$

$$\Rightarrow \lim_{n\to\infty} u_n = \lim_{n\to\infty}\left(1 - \underbrace{\frac{1}{\sqrt{n+1}}}_{0}\right) = 1.$$

2

by definition we have:

$$kx - 1 < E(kx) \leq kx \quad (\text{see serie N}^{\circ}1)$$

$$\Rightarrow \sum(kx-1) < \sum E(kx) \leq \sum kx.$$

$$\frac{2}{n^2}\left(\sum kx - 1\right) < \frac{2}{n^2} \sum E(kx) \leq \frac{2}{n^2} \sum kx.$$

we have $\sum_{k=1}^{n} kx = x \sum_{k=1}^{n} k = \boxed{x\,\frac{n(n+1)}{2}}$

so,

$$x\left(-\frac{2}{n^2} + \frac{2}{n^2} \times \frac{n(n+1)}{2}\right) < u_n \leq x\left[\frac{n(n+1)}{2} \times \frac{2}{n^2}\right]$$

$$\lim \frac{-2}{n^2} + \frac{n^2+n}{n^2} = 1 \text{ and } \lim_{n\to\infty} \frac{n(n+1)}{n^2} = 1$$

$$\Rightarrow x < \lim_{n\to\infty} u_n \leq x$$

$$\Rightarrow \boxed{\lim_{n\to\infty} u_n = x}$$

3

Exo5:

① To show that $\forall n \geq 1$ we have $U_n \geq \sqrt{a}$.

it sufficient to show that $U_{n+1} - \sqrt{a} \geq 0$ $\forall n \in \mathbb{N}$

$$U_{n+1} - \sqrt{a} = \frac{1}{2}\left(U_n + \frac{a}{U_n}\right) - \sqrt{a}.$$

$$= \left(\frac{U_n^2 + a}{2U_n}\right) - \sqrt{a}.$$

$$= \frac{U_n^2 - 2U_n\sqrt{a} + a}{2U_n}.$$

$$= \frac{(U_n - \sqrt{a})^2}{2U_n}$$

As $\begin{cases} U_n > 0, \forall n \in \mathbb{N}^* \\ (U_n - \sqrt{a})^2 \geq 0 \end{cases} \Rightarrow U_{n+1} - \sqrt{a} \geq 0$

$\Rightarrow U_{n+1} \geq \sqrt{a}.$, $\forall n \in \mathbb{N}$.

$\Rightarrow \boxed{U_n \geq \sqrt{a}, \forall n \in \mathbb{N}^*}$

$$U_{n+1} - U_n = \frac{1}{2}\left(U_n + \frac{a}{U_n}\right) - U_n. = \frac{U_n^2 + a}{2U_n} - U_n.$$

$$= \frac{U_n^2 + a - 2U_n^2}{2U_n} = \frac{a - U_n^2}{2U_n} < 0$$ the

fact that $U_n > a \Rightarrow a - U_n^2 < 0 \Rightarrow U_n \searrow$ — (*)

we have from the first question that

U_n is lower bounded and U_n is $\searrow$

$\Rightarrow U_n$ converges.

let ℓ be the limits of U_n when $n \to +\infty$

$$\lim_{n\to+\infty} U_{n+1} = \lim_{n\to+\infty} U_n = \ell.$$

$$\Rightarrow \ell = \frac{1}{2}\left(\ell + \frac{a}{\ell}\right)$$

$$\Rightarrow \cancel{2}\ell = \cancel{\ell} + \frac{a}{\ell} \Rightarrow \ell^2 = a$$

$$\Rightarrow \ell = \begin{cases} +\sqrt{a} \\ -\sqrt{a} \text{ rejected} \end{cases}$$

$$\Rightarrow \boxed{\lim_{n\to\infty} U_n = \ell.}$$

Exo6: $\forall a, b \geq 0$. we have:

- $(\sqrt{a} - \sqrt{b})^2 \geq 0 \Rightarrow a - 2\sqrt{ab} + b \geq 0$

$\Rightarrow a + b \geq 2\sqrt{ab}.$

$\Rightarrow \boxed{\sqrt{ab} \leq \frac{a+b}{2}}$

- we have $a \leq a \leq b$ (*) and $a \leq b \leq b$ (**)

(*) + (**) $\Rightarrow 2a \leq a + b \leq 2b \Rightarrow \boxed{a \leq \frac{a+b}{2} \leq b}$.

(*) × (**) $\Rightarrow a^2 \leq ab \leq b^2 \Rightarrow \boxed{a \leq \sqrt{ab} \leq b}$

(2.a) $u_n < v_n$?

We prove the proposition by induction.

* $n=0$: $u_0 < v_0$ ✓ — (1)
* Suppose that $u_n < v_n$ — (2)

from the 1st question we have

$$a < b \Rightarrow \sqrt{ab} < \frac{a+b}{2}$$

as $u_n < v_n \Rightarrow \sqrt{u_n v_n} < \frac{u_n + v_n}{2}$

$$\Rightarrow u_{n+1} < v_{n+2} \quad \text{— (3)}$$

(1), (2) + (3) $\Rightarrow$ $\forall n \in \mathbb{N}$, $u_n < v_n$.

(2.b) $v_n \searrow$?

~~Let~~ As $u_n < v_n \Rightarrow u_n \leq \frac{u_n + v_n}{2} \leq v_n$. from Q1

$$\Rightarrow u_n \leq v_{n+1} \leq v_n$$

$$\Rightarrow v_{n+1} - v_n \leq 0 \Rightarrow \boxed{v_n \searrow}$$

(2.c) $u_n \nearrow$?

As $u_n < v_n \Rightarrow u_n < \sqrt{u_n v_n} < v_n$.

$$\Rightarrow u_n < u_{n+1} < v_n \Rightarrow u_n \nearrow.$$

the fact u_n is increasing and upper bounded the u_n c.v and v_n is decreasing and lower bounded $\Rightarrow v_n$ c.v

* $\lim_{n\to\infty} u_{n+1} = \lim_{n\to\infty} \sqrt{u_n v_n} \Rightarrow l_1 = \sqrt{l_1 l_2}$
  $\Rightarrow l_1^2 = l_1 l_2 \Rightarrow \boxed{l_1 = l_2}$
* $\lim_{n\to\infty} v_{n+1} = \lim_{n\to+\infty} \frac{u_n + v_n}{2} \Rightarrow l_2 = \frac{l_1 + l_2}{2} \Rightarrow \boxed{l_1 = l_2}$.

Ex07:

* $u_{n+1} - u_n = \left(1 + \frac{1}{2!} + \cdots + \frac{1}{n!} + \frac{1}{(n+1)!}\right) - \left(1 + \frac{1}{2!} + \cdots + \frac{1}{n!}\right)$

$$= \frac{1}{(n+1)!} > 0 \Rightarrow u_n \nearrow. \quad \text{— (1)}$$

* $v_{n+1} - v_n = (u_{n+1} - u_n) + \left(\frac{1}{(n+1)!} - \frac{1}{n!}\right)$

$$= \frac{1}{(n+1)!} + \frac{1}{(n+1)!} - \frac{n+1}{(n+1)!}$$

$$= \frac{-n+1}{(n+1)!} \leq 0 \quad \forall n \in \mathbb{N}^*.$$

$$\Rightarrow v_n \searrow \quad \text{— (2)}$$

* $\lim_{n\to\infty} v_n - u_n = \lim_{n\to\infty} \frac{1}{n!} = 0$ — (3)

from (1), (2) + (3) $\Rightarrow$ u_n and v_n converge to the same limit (because u_n and v_n are adjacent).

University of Biskra
Mathematics Department
Module: Analysis 1

First year license
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Exercise series N°4

Exercise 1 Prove that the derivative of an even differentiable function is odd, and the derivative of an odd differentiable function is even. What about the nth derivative of an even and an odd function?

Exercise 2 Consider the function f defined by:

$$f(x) = \left(\frac{sin(2x)}{2\sqrt{1-cos(x)}}\right)^m, \quad m \in \mathbb{N}^*.$$

1. Determine the domain of the function f.
2. Discuss the parity (even or odd) of f according to the values of the parameter m.
3. Verify that f is a 2π-periodic function, then discuss the limit of f at all bounds of its domain, according to the values of the parameter m.

Exercise 3

1. Show that the curves of the following functions are symmetrical with respect to a vertical axis $x = x_0$.
$$f(x) = \sqrt{(x-1)^2+1}, \quad g(x) = x^2 + 2x + 4.$$
2. For each of the following functions, determine the point of symmetry of their graphs.
$$f(x) = \frac{2x-1}{x+1}, \quad g(x) = \frac{x^2-1}{x-2}$$
3. Show that any function having the form
$$f(x) = \frac{ax+b}{x-c} \quad \text{with } a,\ b,\ c \in \mathbb{R}.$$
admits a point of symmetry.
4. Show that any function having the form
$$f(x) = \sqrt{(x-a)^2+b}, \quad g(x) = (x-a)^2 + b \quad \text{with } a,\ b \in \mathbb{R}.$$
admits a vertical axe of symmetry.

Exercise 4 In each of the following cases, determine the limit, if it exists:

$$\lim_{x\to 4} \tfrac{x^2-7x+12}{x^2-16}, \quad \lim_{x\to 1} \left(\tfrac{1}{x^2-3x+2} - \tfrac{1}{x-1}\right), \quad \lim_{x\to \frac{\pi}{2}} \tfrac{\sqrt[3]{sin(x)}}{x-\frac{\pi}{2}}$$

$$\lim_{x\to 0} x sin(\tfrac{1}{x}), \quad \lim_{x\to +\infty} x sin(\tfrac{1}{x}), \quad \lim_{x\to 0} \tfrac{ln(1-sin(x))}{x}, \quad \lim_{x\to 0} \tfrac{sin(ax)}{sin(bx)}.$$

$$\lim_{x\to \pm\infty} \tfrac{\sqrt{x^2-7}}{3x+5}, \quad \lim_{x\to \pm\infty} \sqrt{x^2+6x+1} - x, \quad \lim_{x\to 1} \tfrac{\sqrt{x^2-1}+\sqrt{x}-1}{\sqrt{x-1}}, \quad \lim_{x\to 1} \tfrac{\sqrt{x}-1}{\sqrt[4]{x}-1}, \quad \lim_{x\to 1} \tfrac{\sqrt[3]{x}-1}{\sqrt[4]{x}-1}$$

$$\lim_{x\to 0} (1+ax)^{1/x}, \quad \lim_{x\to \pm\infty} \left(\tfrac{x^2+x}{x^2+x+2}\right)^{x^2+x}, \quad \lim_{x\to \pm\infty} P_n(x)e^{-x}, \quad \lim_{x\to \pm\infty} \tfrac{ln(P_n(x))}{x}$$

Note: $a, b \in \mathbb{R}^*$, $n \in \mathbb{N}^*$ and $P_n(x)$ is a positive polynomial of degree n

Exercise 5

- Find all the possible values of the constants a, b and $c \in \mathbb{R}$ such that the following functions are continuous on their domains.

$$f(x) = \begin{cases} x^2+2x, & \text{if } x \geq 1; \\ -x+c, & \text{if } x < 1. \end{cases} \qquad g(x) = \begin{cases} x^2, & \text{if } x \leq 0; \\ a\ e^x + b, & \text{if } 0 < x < \pi; \\ 1 - cos(x), & \text{if } x \geq \pi; \end{cases}$$

$$h(x) = \begin{cases} 1, & \text{if } x \leq 0; \\ ae^{-x} + be^x + cx(e^x - e^{-x}), & \text{if } 0 < x < 1; \\ e^{2-x}, & \text{if } x \geq 1; \end{cases}$$

- Study the continuity of the following function on $\mathbb{R}$, $f(x) = E(x)$. What can we conclude?

Exercise 6 For each of the following functions determine their domains and subsequently check if they have a removable discontinuity.

$$f_1(x) = e^{\frac{-1}{x^2}}, \quad f_2(x) = e^{\frac{-1}{x}}, \quad f_3(x) = \frac{1+x}{1+x^3}, \quad f_4(x) = sin(x+1)ln(|x+1|),$$

$$f_5(x) = \left(\frac{sin(2x)}{2\sqrt{1-cos(x)}}\right)^{2m}, \quad m \in \mathbb{N}^* \quad f_6(x) = cos(x)cos(1/x).$$

Exercise 7

I) Let f and g two increasing continuous functions on an interval I. Show that:

$$\text{if } (f(I) \subset g(I)) \text{ or } (g(I) \subset f(I)) \text{ then } \exists c \in I \text{ such as } f(c) = g(c)$$

II) Show that the following equation has at least one solution on $]-\infty; 2[$.

$$sin(x) = \frac{2x+1}{x-2}.$$

III) We consider the equation (1), of unknown $x > 0$.

$$ln(x) = ax. \qquad (1)$$

1. Prove that if $a \leq 0$, the equation (1) admits a unique solution and that this solution belongs to $]0, 1]$
2. Show that if $a \in]0, 1/e[$, the equation (1) admits exactly two solutions.
3. Show that if $a = 1/e$, the equation admits a unique solution whose value will be specified. Prove that if $a > 1/e$, equation (1) has no solution.

Exercise 8

- Let $f : \mathbb{R} \to \mathbb{R}$. Suppose $c \in \mathbb{R}$ and that $f'(c)$ exists. Prove that f is continuous at c.
- Prove that:

1) $(e^x)' = e^x$ 2) $\left(\frac{f(x)}{g(x)}\right)' = \frac{f'(x)g(x)-f(x)g'(x)}{[g(x)]^2}$ 3) $arcsin(x)' = \frac{1}{\sqrt{(1-x^2)}}$

4) $arctan(x)' = \frac{1}{1+x^2}$ 5) $\left(f^{-1}(x)\right)' = \frac{1}{f' \circ f^{-1}(x)}$ 6) $f \circ g(x) = g'(x)\ f' \circ g(x)$

Exercise 9

- Return to the examples of the Exercise 5, and determine the domain of differentiability of the considered functions according to the parameters a, b, and c.
- Determine the two real numbers a and b, so that the function f, defined on $\mathbb{R}$ by:

$$f(x) = \begin{cases} \sqrt{x}, & if\ 0 \leq x \leq 1; \\ ax^2 + bx + c, & x>1, \end{cases}$$

  is differentiable on $\mathbb{R}_+.^*$.

- Study the differentiability of the following functions:

$$f_1(x) = \begin{cases} x^2 cos(1/x), & \text{if } x \neq 0; \\ 0, & \text{else.} \end{cases} \quad f_2(x) = \begin{cases} sin(x)sin(1/x), & \text{if } x \neq 0; \\ 0, & \text{else.} \end{cases}$$

$$f_3(x) = \begin{cases} \frac{|x|\sqrt{x^2-2x+1}}{x-1}, & \text{if } x \neq 1; \\ 1, & \text{else.} \end{cases}$$

- Study the differentiability of the following functions at x_0:

$$f_1(x) = \sqrt{x}, \ \ x_0 = 0, \quad f_2(x) = (1-x)\sqrt{1-x^2}, \ \ x_0 = -1, \quad f_3(x) = (1-x)\sqrt{1-x^2}, \ \ x_0 = 1.$$

  What can we conclude?

Exercise 10 Calculate the derivatives of the following functions.

1) $e^{sin(x^3)}$	2) $ln\left(x^2 + e^{-x^2}\right)$	3) $ln\left(\frac{x+1}{x-1}\right)$	4) $sin(2x^2 + cos(x))$
5) $arcsin(x^2 + x)$	6) $arctan(x^2 + x)$	7) $\sqrt[3]{x^2 + x}$	8) $a^{\left(\frac{x-1}{x+1}\right)}, \ a \in \mathbb{R}_+^*$
9) $e^{e^{x^2+1/x}}$	10) $log_a(arcsin(x)), \ a \in \mathbb{R}_+^*$	11) $\sqrt{\lvert x^2 - 4x + 3\rvert}$	12) $\frac{1-tan^2(x)}{(1+tan(x))^2}$

Exercise 11

1. In the application of mean value theorem's to the function

$$f(x) = \alpha x^2 + \beta x + \gamma, \quad \alpha, \ \beta, \ \gamma \in \mathbb{R}^*$$

   on the interval $[a; b]$ specify the number $c \in]a; b[$. Give a geometric interpretation.

2. Let x and y two reals with $0 < x < y$, show that

$$x < \frac{y-x}{ln(y) - ln(x)} < y.$$

Exercise 12 Let f and $g \to [a; b]$ be two continuous functions on $[a; b]$ $(a < b)$ and differentiable on $]a; b[$. We suppose that $g'(x) \neq 0$ for all $x \in]a; b[$.

1. Show that $g(x) \neq g(a)$, for all $x \in]a; b[$.

2. Let us set $\alpha = \frac{f(b)-f(a)}{g(b)-g(a)}$ and consider the function $h(x) = f(x) - \alpha g(x)$ for $x \in [a; b]$. Show that h satisfies the hypotheses of Rolle's theorem and deduce that there exists a real number $c \in]a; b[$ such that

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)}.$$

3. We assume that $\lim_{x \to b} \frac{f'(x)}{g'(x)} = l$, where l is a finite real number. Show that

$$\lim_{x \to b-} \frac{f(x) - f(b)}{g(x) - g(b)} = l.$$

4. Application. Calculate the following limit:

$$\lim_{x \to 1^-} \frac{arccos(x)}{\sqrt{1-x^2}}.$$

Exercise 13 Using the derivative notions, determine the following limits:

1) $\lim_{x \to 0} \frac{e^{3x-2} - e^2}{x}$	2) $\lim_{x \to 1} \frac{ln(2-x)}{x-1}$	3) $\lim_{x \to \pi} \frac{sin(x)}{x^2-\pi^2}$
4) $\lim_{x \to \frac{\pi}{2}} \frac{e^{cos(x)}}{x-\frac{\pi}{2}}$	5) $\lim_{x \to 0} \frac{ln(1-sin(x))}{x}$	6) $\lim_{x \to +\infty} (ln(x+1) - ln(x))$.

Exercise 14 Give the domain of differentiability of the following functions then calculate the nth-order derivative, by justifying its existence.

$$f(x) = 2x^k, \ k \in \mathbb{N}^*, \ \ f(x) = 1/x, \ \ f(x) = 1/x^2, \ \ f(x) = sin(2x), \ \ f(x) = sin(x)cos(x),$$

$$f(x) = \frac{1}{1-x^2}, \ \ f(x) = x^2 e^x.$$

Exo1:

* 1st case: f is an even function ie $f(-x) = f(x)$ — (*)

$f'(-x) = ?$ we have $(f\circ g(x))' = g'(x)\, f'\circ g(x)$

so for $g(x) = -x$ we have

$f'(-x) = (-x)'\, f'(-x)$
$= -f'(-x)$. — (1)

from (*) we have $f'(-x) = f'(x)$ so.

$f'(x) = -f'(-x) \Rightarrow f'(-x) = -f'(x) \Rightarrow f'$ is an odd f

* 2nd case: f is an odd function ie $f(-x) = -f(x)$.

$f'(x) = g'(x)\, f'(g(x))$ % we put $g(x) = -x$
$= (-x)'\, f'(-x)$
$= -f'(-x)$. $\Rightarrow -f'(-x) = -f'(x)$

we have $f'(-x) = -f(x)$ $\Rightarrow f'(-x) = f'(x) \Rightarrow f'$ ~~odd~~ even.

• with same raisoning as above we can show that:

f is even $\Rightarrow \begin{cases} f^{(n)} \text{ is even if } n = 2k \\ f^{(n)} \text{ is odd if } n = 2k+1 \end{cases}$ $/ k \in \mathbb{N}$

f is odd $\Rightarrow \begin{cases} f^{(n)} \text{ is odd if } n = 2k \\ f^{(n)} \text{ is even if } n = 2k+1 \end{cases}$

This latest remark can be justified by proof by induction.

Exo2

(1) $D_f = \{x \in \mathbb{R} : 1 - \cos(x) \geq 0 \wedge 1 - \cos(x) \neq 0\}$
$= \{x \in \mathbb{R} : 1 - \cos(x) > 0\}$.

$1 - \cos(x) > 0 \Rightarrow \cos(x) < 1 \Rightarrow \cos(x) \neq 1$.

$\cos(x) = 1 \Rightarrow x = 2\pi k \,/\, k \in \mathbb{Z}$

$\Rightarrow \boxed{D_f = \mathbb{R} / \{2\pi k / k \in \mathbb{Z}\}}$

(2) $\begin{cases} f \text{ is odd} \Rightarrow f(-x) = -f(x) \\ f \text{ is even} \Rightarrow f(-x) = +f(x) \end{cases}$

$$f(-x) = \left(\frac{\sin(-2x)}{2\sqrt{1-\cos(-x)}}\right)^m = \left(\frac{-\sin(2x)}{2\sqrt{1-\cos(x)}}\right)^m$$

$\boxed{f(-x) = (-1)^m f(x)}$ so $\begin{cases} f(x) = +f(x) \text{ if } m = 2k \\ f(-x) = -f(x) \text{ if } m = 2k+1 \end{cases}$

$\Rightarrow \begin{cases} f \text{ is odd if } m = 2k+1 \\ f \text{ is even if } m = 2k \end{cases}$ $/ k \in \mathbb{N}$.

(3) f is T-periodic $\Rightarrow f(x+T) = f(x)$.

$$f(x+2\pi) = \left(\frac{\sin(2(x+2\pi))}{2\sqrt{1-\cos(x+2\pi)}}\right)^m = f(x)$$ ✓

$\begin{cases} \cos(x + 2k\pi) = \cos(x) \\ \sin(x + 2k\pi) = \sin(x) \end{cases}$ $/ \forall k \in \mathbb{Z}$.

③ $D_f = \dots]-4\pi, -2\pi[\cup]-2\pi, 0[\cup]0, 2\pi[\cup]2\pi, 4\pi[\dots$

As f is periodic, it is enough to calculate the limit, for example, at $x_0 = 0$, and the rest will be the same at all the bounds of D_f.

$$\lim_{x\to 0} f(x) = \lim_{x\to 0}\left(\frac{\sin(2x)}{2\sqrt{1-\cos(x)}}\right)^m$$

$$= \lim_{x\to 0}\left(\frac{\sin(x)\cos(x)}{\sqrt{1-\cos(x)}}\right)^m = \lim_{x\to 0}\left(\frac{\sin(x)\cos(x)\sqrt{1+\cos(x)}}{\sqrt{1-\cos^2(x)}}\right)^m$$

$$= \lim_{x\to 0}\left(\frac{\sin(x)}{|\sin(x)|}\right)^m\left(\cos(x)\sqrt{1+\cos(x)}\right)^m.$$

$$\Rightarrow \begin{cases} \lim\limits_{x \overset{>}{\to} 0} f(x) = (\sqrt{2})^m. \\ \lim\limits_{x \overset{<}{\to} 0} f(x) = (-1)^m(\sqrt{2})^m. \end{cases}$$

$$\Rightarrow \lim_{x\to 0} f(x) = \begin{cases} (\sqrt{2})^m & \text{if } m = 2k \,/\, k\in\mathbb{N}^* \\ \nexists & \text{if } m = 2k+1 \,/\, k\in\mathbb{N} \end{cases}$$

Exo3 $f(x) = \sqrt{(x-1)^2+1} \Rightarrow D_f = \mathbb{R}$

f admet a vertical axis of symmetry noted $\cancel{x} = a$ $\Rightarrow$ $\forall x \in D_f: \begin{cases} \cancel{a}-\cancel{x} \in D_f \\ x+a \in D_f \end{cases}$ and $f(x-a) = f(x+a)$

As $D_f = \mathbb{R} \Rightarrow$ the first condition is checked $\forall a \in \mathbb{R}$

- let $a \in \mathbb{R}$, ~~if a is as~~ ~~at~~

$f(\cancel{a}-\cancel{x}) = f(x+a)$? $\exists a$?

$f(\cancel{a}-\cancel{x}) = f(x+a) \Rightarrow \sqrt{(a-x-1)^2+1} = \sqrt{(a+x-1)^2+1}$

$\Rightarrow (a-x-1)^2 \cancel{+1} = (a+x-1)^2 \cancel{+1}$

$$\Rightarrow \begin{cases} a-x-1 = a+x-1 & — ① \\ a-\cancel{x}-1 = -a-\cancel{x}+1 & — ② \end{cases}$$

from ① $\Rightarrow -x = x$ reject $(x \in D_f)$.

form ② $\Rightarrow a-1 = -a+1$

$\Rightarrow 2a = 2 \Rightarrow \boxed{a=1}$

$\Rightarrow$ the function f admits a vertical axis ~~[illegible]~~ ~~are~~ $x=1$ as an axis of symmetry.

- $g(x) = x^2+2x+4$ $D_g = \mathbb{R}$.

$\exists a? \Rightarrow g(a-x) = g(a+x)$.

$g(a-x) = g(a+x) \Rightarrow (a-x)^2 + 2(a-x) + \cancel{4} = (x+a)^2 + 2(a+x) + \cancel{4}$

$\Rightarrow$ ~~$a^2 - 2ax + x^2 + 2a - 2x = a^2 + 2ax + x^2 + 2a - 2x$~~.

$\Rightarrow \cancel{a^2} - 2ax + \cancel{x^2} + \cancel{2a} - 2x = \cancel{a^2} + 2ax + \cancel{x^2} + \cancel{2a} + 2x$

$\Rightarrow 4ax + 4x = 0$

$$4x(a+1)=0 \Rightarrow \begin{cases} x=0 \text{ (Rejected)} \\ \text{or} \\ a=-1 \end{cases}$$

$\Rightarrow$ the symetric axis is $\boxed{x=-1}$.

2

2.1 $f(x)=\dfrac{2x-1}{x+1} \Rightarrow D_f=]-\infty,-1[\cup]-1,+\infty[$

f admet the pt (a,b) as a symmetry point if:

$\forall x \in D_f$: $\underbrace{a-x,\ a+x \in D_f}_{1}$ and $\underbrace{f(a-x)+f(a+x)=2b}_{2}$.

to check 1 the value of a must be "$a=-1$".

$$f(-1-x)+f(-1+x)=\frac{-2(x+1)-1}{\cancel{-1}-x\cancel{+1}}+\frac{2(x-1)-1}{\cancel{-1}+x\cancel{-1}}$$

$$=4 = 2\times2 \Rightarrow \boxed{b=2}$$

then the pt is $(-1,2)$.

2.2 $g(x)=\dfrac{x^2-1}{x-2} \Rightarrow D_g=]-\infty,2[\cup]2,+\infty[$

$\Rightarrow \boxed{a=2}$

$$g(2-x)+g(2+x)=\frac{(2-x)^2-1}{\cancel{2}-x\cancel{-2}}+\frac{(2+x)^2-1}{\cancel{2}-x\cancel{+2}}$$

$$=4=2\times2$$

$\Rightarrow \boxed{b=2}$

$\Rightarrow (a,b)=(2,2)$.

4 det the ~~pt~~ (α,β) the coordinate of the pt of symmetry if it exist.

$f(x)=\dfrac{ax+b}{x-c} \Rightarrow D_f=]-\infty,c[\cup]c,+\infty[$

from $D_f \Rightarrow \boxed{\alpha=c}$

$$f(\alpha-x)+f(\alpha+x)=f(c-x)+f(c-x)$$

$$=\frac{a(c-x)+b}{\cancel{c}-x-\cancel{c}}+\frac{a(c+x)+b}{\cancel{c}+x-\cancel{c}}$$

$$=\frac{ac-ax+b}{-x}+\frac{ac+ax+b}{x}$$

$$=\frac{\cancel{-ac}+ax\cancel{-b}\cancel{+ac}+ax\cancel{+b}}{x}$$

$$=\frac{2ax}{x}=2a.$$

$\Rightarrow \boxed{\beta=a}$

then the pt in question is (c,a).

<u>Exaple:</u> if we take the example of 2° question $(\alpha,\beta)=(-1,2)$ this coincides with the results obtained previously.

4 let $x=x_0$ be the axis of the symetric $g(x)=(x-a)^2+b$

$$g(x_0-x)=g(x_0+x)\ ?\quad \exists x_0?$$

$$(x_0-x-\cancel{a})^2+\cancel{b} = (x_0+x-\cancel{a})^2+\cancel{b}$$

$$\Rightarrow \begin{cases} \cancel{x_0}-x-\cancel{a} = \cancel{x_0}+x-\cancel{a} \quad \text{—— (1)} \\ \cancel{x_0}-\cancel{x}-a = -x_0-\cancel{x}+a \quad \text{—— (2)} \end{cases}$$

(1) $\Rightarrow -x = x$ (Rejected).

(2) $\Rightarrow 2x_0 = 2a \Rightarrow \boxed{x_0 = a}$

$$f(x) = \sqrt{(x-a)^2+b} \Rightarrow \begin{cases} D_f = \mathbb{R} \text{ if } b \geq 0 \\ D_f =]-\infty; -\sqrt{-b}+a[\cup]\sqrt{-b}+a; +\infty[\text{ if } b<0 \end{cases}$$

- $\overset{?}{\exists} x_0 \in \mathbb{R} \,/\, f(x_0-x) = f(x_0+x)$.

$f(x_0-x) = f(x_0+x)$.

$\Rightarrow \sqrt{(x_0-x-a)^2+b} = \sqrt{(x_0+x-a)^2+b}$.

$\Rightarrow (x_0-x-a)^2 = (x_0+x-a)^2$.

$$\Rightarrow \begin{cases} \cancel{x_0}-x-\cancel{a} = \cancel{x_0}+x-\cancel{a} \text{ (Rejected)} \\ x_0-\cancel{x}-a = -x_0-\cancel{x}+a \; ✓ \end{cases}$$

$\Rightarrow 2x_0 = 2a \Rightarrow \boxed{x_0 = a}$ ✓

Ex04:

- $\lim\limits_{x\to 4} \dfrac{x^2-8x+16}{x^2-16} = \lim\limits_{x\to 4} \dfrac{\cancel{(x-4)}(x-3)}{\cancel{(x-4)}(x+4)}$

$= \lim\limits_{x\to 4} \dfrac{x-3}{x+4} = 1/8$

- $\lim\limits_{x\to 1} \left(\dfrac{1}{x^2-3x+2} - \dfrac{1}{x-1}\right) = \cancel{\lim \dfrac{x-1-x^2-3x+2}{}}$

$= \lim\limits_{x\to 1} \left(\dfrac{1}{(x-1)(x-2)} - \dfrac{1}{x-1}\right) = \lim\limits_{x\to 1} \dfrac{-(x-2)+1}{(x-1)(x-2)}$

$$= \lim_{x\to 1} \frac{3-x}{(x-1)(x-2)} = \begin{cases} \dfrac{2}{0^-} = -\infty \text{ when } x \xrightarrow{<} 1 \\ \dfrac{2}{0^+} = +\infty \text{ when } x \xrightarrow{>} 1 \end{cases}$$

- $$\lim_{x\to \pi/2} \frac{\sqrt[3]{\sin(x)}}{x-\frac{\pi}{2}} = \begin{cases} \dfrac{1}{0^-} = -\infty \text{ when } x \xrightarrow{<} \pi/2 \\ \dfrac{1}{0^+} = +\infty \text{ when } x \xrightarrow{>} \pi/2 \end{cases}$$

- $\lim\limits_{x\to 0} x \sin\left(\frac{1}{x}\right)$

we have $-1 \leq \sin\left(\frac{1}{x}\right) \leq 1$. —— (*)

$\Rightarrow x \to 0^+ \quad -x \leq x\sin\left(\frac{1}{x}\right) \leq x$.

$\lim\limits_{x\to 0^+}(-x) \leq \lim\limits_{x\to 0^+} x\sin\left(\frac{1}{x}\right) \leq \lim\limits_{x\to 0^+} x$

$0 \leq \lim\limits_{x\to 0^+} x\sin\left(\frac{1}{x}\right) \leq 0$

$\Rightarrow \boxed{\lim\limits_{x\to 0^+} x\sin\left(\frac{1}{x}\right) = 0}$ —— (1)

when $x \to 0^-$ (*) $\Rightarrow -x \geq x\sin\left(\frac{1}{x}\right) \geq x$.

$\Rightarrow \lim\limits_{x\to 0^+} x\sin\left(\frac{1}{x}\right) = 0$ —— (2)

from ① and ② $\Rightarrow \lim_{x\to 0} x\sin\left(\frac{1}{x}\right) = 0.$

- $\lim_{x\to+\infty} x\sin\left(\frac{1}{x}\right)$, we put $y=\frac{1}{x} \Rightarrow x\to+\infty \quad y\to 0$.

so $\lim_{x\to+\infty} x\sin\left(\frac{1}{x}\right) = \lim_{y\to 0}\frac{\sin(y)}{y} = 1.$

- $\lim_{x\to 0}\frac{\ln(1-\sin(x))}{x}$; ~~here we must note that~~

$\lim_{x\to 0}\frac{\ln(1-\sin(x))}{x} = \lim_{x\to 0}\frac{\sin(x)}{x}\cdot\frac{\ln(1-\sin(x))}{+\sin(x)}$

$= \lim_{x\to 0}\frac{\sin(x)}{x} * \lim_{x\to 0}\frac{\ln(1-\sin(x))}{\sin(x)}$ — (*)

$\lim_{x\to 0}\frac{\sin(x)}{x} = 1$ — ①

$\lim_{x\to 0}\frac{\ln(1-\sin(x))}{\sin(x)} = ?$ let $y=-\sin(x) \Rightarrow (x\to 0 \Rightarrow y\to 0)$

$\Rightarrow \lim_{x\to 0}\frac{\ln(1-\sin(x))}{\sin(x)} = \lim_{y\to 0}\frac{\ln(1+y)}{-y} = -\lim_{y\to 0}\underbrace{\frac{\ln(1+y)}{y}}_{1}$

$\Rightarrow \lim_{x\to 0}\frac{\ln(1-\sin(x))}{\sin(x)} = -1$ — ②

de (*), ① and ② $\Rightarrow \lim_{x\to 0}\frac{\ln(1-\sin(x))}{x} = 1\times(-1) = \boxed{-1}$

- $\lim_{x\to 0}\frac{\sin(ax)}{\sin(bx)} = \lim_{x\to 0}\frac{a * \frac{\sin(ax)}{ax}}{b * \frac{\sin(bx)}{bx}} =$ (*)

~~(*) $= \frac{a}{b}$ lim~~

(*) $= \frac{a}{b}\,\frac{\lim_{x\to 0}\frac{\sin(ax)}{ax}\to 1}{\lim_{x\to 0}\frac{\sin(bx)}{bx}\to 1} = \boxed{\frac{a}{b}}$

- $\lim_{x\to\pm\infty}\frac{\sqrt{x^2-7}}{3x+5}$

(*) $\lim_{x\to+\infty}\frac{\sqrt{x^2-7}}{3x+5} = \lim_{x\to+\infty}\frac{\cancel{x}\sqrt{1-7/x^2}}{3\cancel{x}\left(1+\frac{5}{3x}\right)} = \boxed{1/3}$

- $\lim_{x\to-\infty}\frac{\sqrt{x^2-7}}{3x+5} = \lim_{x\to-\infty}\frac{-x\sqrt{1-7/x^2}}{3x\left(1+5/3x\right)} = \boxed{-1/3}$

- $\lim_{x\to\pm\infty}\sqrt{x^2+6x+1}-x \quad (\infty-\infty?)$.

- $\lim_{x\to+\infty}\sqrt{x^2+6x+1}-x = \lim_{x\to+\infty}\frac{\cancel{x^2}+6x+1-\cancel{x^2}}{\sqrt{x^2+6x+1}+x}$.

$\lim_{x\to+\infty}\frac{6x+1}{x\left(\sqrt{1+6/x+1/x^2}+1\right)} = \boxed{3}$

- $\lim_{x\to-\infty}\sqrt{x^2+6x+1}-x = \infty+\infty = \boxed{+\infty}$.

- $\lim_{x\to 1}\frac{\sqrt{x^2-1}+\sqrt{x}-1}{\sqrt{x-1}}$; it should be noted that the limit in this case make sens only when $x\geq 1$ else $\sqrt{x-1}$ is not defined.

$$\lim_{x \to 1} \frac{\sqrt{x^2-1}+\sqrt{x}-1}{\sqrt{x-1}} = \lim_{x \to 1} \sqrt{x+1} + \frac{\sqrt{x}-1}{\sqrt{x-1}}$$

$$\lim_{x\to 1}\sqrt{x+1} = \sqrt{2} \quad \text{and} \quad \lim_{x\to 1}\frac{\sqrt{x}-1}{\sqrt{x-1}} = \lim_{x\to 1}\frac{\sqrt{x}-1}{\sqrt{\sqrt{x}-1}\,\sqrt{\sqrt{x}+1}}$$

$$\Rightarrow \lim_{x\to 1}\frac{\sqrt{x}-1}{\sqrt{x-1}} = \lim_{x\to 1}\frac{\sqrt{\sqrt{x}-1}}{\sqrt{\sqrt{x}+1}} = \frac{0}{\sqrt{2}} = 0$$

$$\Rightarrow \lim_{x\to 1}\frac{\sqrt{x^2-1}+\sqrt{x}-1}{\sqrt{x-1}} = \sqrt{2}+0 = \boxed{\sqrt{2}}$$

Another way

$$\frac{\sqrt{x}-1}{\sqrt{x-1}} = \frac{(\sqrt{x}-1)\sqrt{x-1}}{x-1} = \frac{(\sqrt{x}-1)(\sqrt{x-1})}{(\sqrt{x})^2-1^2}$$

$$= \frac{(\cancel{\sqrt{x}-1})(\sqrt{x-1})}{(\cancel{\sqrt{x}-1})(\sqrt{x}+1)} = \frac{\sqrt{x-1}}{\sqrt{x}+1}$$

$$\boxed{\lim_{x\to 1}\frac{\sqrt{x-1}}{\sqrt{x}+1} = 0.}$$

$$\lim_{x\to 1}\frac{\sqrt{x}-1}{\sqrt[4]{x}-1} = \lim_{x\to 1}\frac{x^{1/2}-1}{x^{1/4}-1}$$

1st way: $\lim_{x\to 1}\frac{x^{1/2}-1}{x^{1/4}-1} = \lim_{x\to 1}\frac{(x^{1/4})^2-1^2}{x^{1/4}-1} = \lim_{x\to 1} x^{1/4}+1 = 2$

2nd way: we pose $y^4 = x \Rightarrow (x\to 1) \Rightarrow (y \to 1)$

$4 = \text{LCM}(2,4)$

$$\lim_{x\to 1}\frac{x^{1/2}-1}{x^{1/4}-1} = \lim_{y\to 1}\frac{y^2-1}{y-1} = \lim_{y\to 1} y+1 = 2$$

- $$\lim_{x\to 1}\frac{\sqrt[3]{x}-1}{\sqrt[4]{x}-1} = \lim_{x\to 1}\frac{x^{1/3}-1}{x^{1/4}-1}$$

we pose $y^{16} = x$ $(16 = \text{LCM}(3,4))$

$x \to 1 \Rightarrow y \to 1.$

$$\Rightarrow \lim_{x\to 1}\frac{\sqrt[3]{x}-1}{\sqrt[4]{x}-1} = \lim_{y\to 1}\frac{y^4-1}{y^3-1} = \lim_{y\to 1}\frac{(\cancel{y-1})(y+1)(y^2+1)}{(\cancel{y-1})(y^2+y+1)} = \boxed{4/3}$$

$$\boxed{\begin{cases} y^4-1 = (y^2-1)(y^2+1) = (y-1)(y+1)(y^2+1) \\ y^3-1 = (y-1)(y^2+y+1) \end{cases}}$$

- $\lim_{x\to 0}(1+ax)^{1/x} = ?$

we have $$\lim_{x\to 0}\ln(1+ax)^{1/x} = \lim_{x\to 0} a \cdot \frac{\ln(1+ax)}{ax}$$
$$= a \lim_{x\to 0} \frac{\ln(1+ax)}{ax} = \boxed{a}$$

So $$\lim_{x\to 0}(1+ax)^{1/x} = \lim_{x\to 0} e^{\ln(1+ax)^{1/x}} = e^{\lim_{x\to 0}\ln(1+ax)^{1/x}} = \boxed{e^a}$$

- $$\lim_{x\to\infty}\left(\frac{x^2+x+2-2}{n^2+n+2}\right)^{n^2+n} = \lim_{x\to\infty}\left(1+\frac{-2}{n^2+n+2}\right)^{n^2+n}$$

$$= \lim_{x\to\infty}\left(1+\frac{-2}{n^2+n+2}\right)^{n^2+n+2} * \left(1+\frac{-2}{n^2+n+2}\right)^{-2}$$

$$= \lim_{x\to\infty}\left(1-\frac{2}{x^2+x+2}\right)^{x^2+x+2} * \lim_{x\to\infty}\left(1+\frac{-2}{x^2+x+2}\right)^{-2}.$$

$$\lim_{x\to+\infty}\left(1-\frac{2}{x^2+x+2}\right)^{-2}=1.$$

$$\lim_{x\to+\infty}\left(1-\frac{2}{x^2+x+2}\right)^{x^2+x+2}=\ ? \longrightarrow (*)$$

we pose $y=\frac{1}{x^2+x+2}$. $\Rightarrow (x\to+\infty \Rightarrow y\to 0)$

$(*) \Leftrightarrow \lim_{y\to 0}(1+ay)^{1/y}=e^a$ with $a=-2$.

$$\Rightarrow \lim_{x\to+\infty}\left(\frac{x^2+1}{x^2+x+2}\right)^{x^2+x}=e^{-2}.$$

with same steps we obtain $\lim_{x\to-\infty}\left(\frac{x^2+1}{x^2+x+2}\right)^{x^2+x}=e^{-2}$

- $\lim_{x\to\pm\infty} f_n(x)e^{-x} = \lim_{x\to\pm\infty} e^{-x} = \begin{cases} 0 & \text{when } x\to+\infty \\ +\infty & \text{when } x\to-\infty.\end{cases}$

- $\lim_{x\to\pm\infty}\frac{\ln(f_n(x))}{x}=\lim_{x\to\pm\infty}\frac{1}{x}=0$

$\boxed{\text{As } f_n(x)=o(e^x) \text{ and } \ln(f_n(x))=o(x) \text{ when } x\to\infty}$

Corrigé ▼

Dans toute la suite, on va poser $f(x)=\ln(x)-ax$, définie pour $x>0$. Chercher une solution de (E_a), c'est chercher un zéro de f. Remarquons d'abord que la fonction f est continue sur $]0,+\infty[$ et que $\lim_{x\to0^+}f(x)=-\infty$. De plus, f est dérivable sur $]0,+\infty[$ et pour tout $x>0$, on a

$$f'(x)=\frac{1}{x}-a.$$

On a $f'(x)=0 \iff x=1/a$. Si $a\leq 0$, la fonction f est strictement croissante sur $]0,+\infty[$. Si $a>0$, la fonction f est strictement croissante sur $]0,1/a[$ et strictement décroissante sur $]1/a,+\infty[$.

1. Si $a\leq 0$, alors la fonction f est strictement croissante sur $]0,+\infty[$ et de plus $\lim_{x\to+\infty}f(x)=+\infty$ (ce n'est pas une forme indéterminée). La fonction f réalise donc une bijection de $]0,+\infty[$ sur $\mathbb{R}$, et l'équation $f(x)=0$ admet une unique solution dans $]0,+\infty[$. On peut même préciser l'emplacement de ce zéro. En effet, $f(1)=-a\geq 0$, et donc $0\in]\lim_{x\to0^+}f(x),f(1)]$. On en déduit que la solution à (E_a) est dans l'intervalle $]0,1]$.
2. Si $a>0$, on a $\lim_{x\to+\infty}f(x)=-\infty$ par croissance comparée de la fonction logarithme et des fonctions puissance. On a donc le tableau de variations suivant pour la fonction f :

x	0		$1/a$		$+\infty$
$f'(x)$	‖	$-$	0	$+$	
f	‖ $-\infty$	↗	$\ln(1/a)-1$	↘	$-\infty$

$a\in]0,1/e[$, alors $f'(x)>0$ sur $]0,1/a[$ et $f'(x)<0$ sur $]1/a,0[$. f réalise donc une bijection de $]0,1/a]$ sur $]-\infty,f(1/a)=\ln(1/a)-1]$ et de $[1/a,+\infty[$ sur $[\ln(1/a)-1,\lim_{x\to+\infty}f(x)[=[\ln(a)-1,-\infty[$ (la limite en $+\infty$ est ici une conséquence de la croissance comparée de la fonction logarithme et des fonctions puissance). Puisque $a<1/e$, $\ln(1/a)>1$ et on trouve bien deux solutions à l'équation $f(x)=0$: l'une dans l'intervalle $]0,1/a[$ et l'autre dans l'intervalle $]1/a,+\infty[$.
3. Si $a=1/e$, alors f admet un maximum en e qui vaut 0. La fonction étant strictement croissante sur $]0,1/e[$ et strictement décroissante sur $]1/e,+\infty[$, l'équation (E_a) admet pour unique solution $1/e$.
4. Si $a>1/e$, alors f admet un maximum en $1/a$ et $f(1/a)=-\ln(a)-1<0$. Ainsi, l'équation (E_a) n'admet pas de solutions.

EXO5

Recall that f is continous at x_0 mean:

$$\lim_{x \overset{<}{\to} x_0} f(x) = \lim_{x \overset{>}{\to} x_0} f(x) = f(x_0).$$

- $f(x) = \begin{cases} x^2 + 2x & \text{if } x \geq 1 \\ -x + c & \text{if } x < 1. \end{cases}$ $f(x)$ is polynomial

From the expression of $f \Rightarrow f$ is continous on $\mathbb{R} \setminus \{1\}$

So f is continous on $\mathbb{R}$ iff f is continous at $x_0 = 1$.

$$\left.\begin{aligned} \lim_{x \overset{<}{\to} 1} f(x) &= \lim_{x \overset{<}{\to} 1} -x + c = c - 1 \\ \lim_{x \overset{>}{\to} 1} f(x) &= \lim_{x \to 1} x^2 + 2x = 3 \\ f(1) &= 3 \end{aligned}\right\} \Rightarrow c - 1 = 3 \Rightarrow \boxed{c = 4}$$

We conclude that f is continous on $\mathbb{R}$ iff $\boxed{c = 4}$.

- $g(x) = \begin{cases} x^2 & \text{if } x \leq 0 \\ ae^x + b & \text{if } 0 < x < \pi \\ 1 - \cos(x) & \text{if } x \geq \pi \end{cases} \Rightarrow g(x)$ is continous on $\mathbb{R} / \{0, \pi\}$.

1st case: $x_0 = 0$:

$$\begin{cases} f(0) = 0 \\ \lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} ae^x + b = a + b \Rightarrow \boxed{a + b = 0} \quad (*) \\ \lim_{x \to 0^+} f(x) = 0 \end{cases}$$

1

Case 2 $x_0 = \pi$

$$\left.\begin{aligned} g(\pi) &= 2. \\ \lim_{x \overset{<}{\to} \pi} f(x) &= \lim_{x \overset{<}{\to} \pi} ae^x + b = ae^\pi + b \\ \lim_{x \overset{>}{\to} \pi} f(x) &= \lim_{x \overset{>}{\to} \pi} 1 - \cos(x) = 2. \end{aligned}\right\} \Rightarrow ae^\pi + b = 2. \quad (**)$$

Thus, in order that the function g be continous on $\mathbb{R}$ a and b must statisfie the following:

$$\begin{cases} a + b = 0 \\ ae^\pi + b = 2 \end{cases} \Rightarrow \begin{cases} a = \dfrac{2}{e^\pi - 1} \\ b = \dfrac{-2}{e^\pi - 1}. \end{cases}$$

- $h(x) = \begin{cases} 1 & \text{if } x \leq 0 \\ ae^{-x} + be^x + cx(e^x - e^{-x}) & 0 < x < 1 \\ e^{-x} & x \geq 1. \end{cases}$

Proceeding with same maner as (2) (i.e. with function g) we conclude that h is continous on $\mathbb{R}$ iff:

$$\begin{cases} \lim_{x \to 0^+} ae^{-x} + be^x + cx(e^x - e^{-x}) = a + b = 1. \\ \lim_{x \overset{<}{\to} 1} ae^{-x} + be^x + cx(e^x - e^{-x}) = ae^{-1} + be^1 + c(e^1 - e^{-1}) = e^{-1} \end{cases}$$

$$\Rightarrow (a, b, c) = (c, 1 - c, c) = (0, 1, 0) + c(1, -1, 1)$$

i.e: h is continous on $\mathbb{R}$ for all $(a, b, c) \in \{(0,1,0) + c(1,-1,1) / c \in \mathbb{R}\}$

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EXO. IV

$$f(x) = E(x) \Rightarrow f(x) = \begin{cases} k-1 & \text{if } k-1 < x < k \\ k & \text{if } x = k \\ k & \text{if } k < x < k+1 \end{cases} \quad \forall k \in \mathbb{Z}.$$

~~f(k)~~ = from the expression of f it clear that f is continous on all intervals of form $]k, k+1[$ with $k \in \mathbb{Z}$. —— $(*)$.

Let verifie the continuity of f at $x_0 = k$.

$f(k) = k$.

$\lim\limits_{x \to k^-} f(x) = k-1$ $\Rightarrow$ f is left-continous at $x_0 = k$ but not right continous at $x_0 = k$.

$\lim\limits_{x \to k^+} f(x) = k$.

we conclude that $E(x)$ is continous on $\mathbb{R}/\mathbb{Z}$.

Conclusion:

$\boxed{\text{not all functions are continous on their domain.}}$

EXO6:

$D_{f_1} = \{x \in \mathbb{R} / x^2 \neq 0\} =]-\infty, 0[\cup]0, +\infty[$ $(\mathbb{R}^*)$

$D_{f_2} = \{x \in \mathbb{R} / x \neq 0\} =]-\infty, 0[\cup]0, +\infty[$ $(\mathbb{R}^*)$.

$D_{f_3} = \{x \in \mathbb{R} / 1 + x^3 \neq 0\} =]-\infty, -1[\cup]-1, +\infty[$ $(\mathbb{R}/\{-1\})$

$D_{f_4} = \{x \in \mathbb{R} / |x+1| \neq 0\} =]-\infty, -1[\cup]-1, +\infty[$ $(\mathbb{R}/\{-1\})$

$D_{f_5} = \mathbb{R}/_{\{2\pi k / k \in \mathbb{Z}\}}$ (see exo 2).

$D_{f_6} = \{x \in \mathbb{R} / x \neq 0\} =]-\infty, 0[\cup]0, +\infty[$ $(\mathbb{R}^*)$.

f_1: f_1 is not continous at $x_0 = 0$, we can remouve this discontinuty if: $\lim\limits_{x \to 0^-} f(x) = \lim\limits_{x \to 0^+} f(x) = \ell$ (exist)

$$\left\{ \begin{array}{l} \lim\limits_{x \to 0^-} f_1(x) = \lim\limits_{x \to 0^-} e^{-1/x^2} = 0 \\ \lim\limits_{x \to 0^+} f(x) = \lim\limits_{x \to 0^+} e^{-1/x^2} = 0 \end{array} \right. \Rightarrow F_1(x) = \begin{cases} f_1(x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$$

$D_{F_1} = \mathbb{R}$

f_2: f_2 is not continous at $x_0 = 0$.

$\lim\limits_{x \to 0^-} f_2(x) = \lim\limits_{x \to 0^-} e^{-1/x} = +\infty \neq \lim\limits_{x \to 0^+} f_2(x) = 0$

f_2 have not a removable discontinuity.

f_3: is not continous at $x_0 = -1$.

$$\lim_{\substack{x \to -1 \\ <}} f_3(x) = \lim_{\substack{x \to -1 \\ <}} \frac{(1+x)}{1+x^3} = \lim_{\substack{x \to -1 \\ <}} \frac{\cancel{(1+x)}}{\cancel{(1+x)}(x^2-x+1)} = 1$$

$$\lim_{\substack{x \to -1 \\ >}} f_3(x) = 1$$

So, the discontinuity of f_3 at $x_0 = -1$ can be removed:

$$F_3(x) = \begin{cases} f_3(x) & \text{if } x \neq -1 \\ 1 & \text{if } x = -1. \end{cases}$$

$$D_{F_3} = \mathbb{R}.$$

---

f_4: is not continuous at $x_0 = -1$. ~~because~~

$$\lim_{\substack{x \to -1 \\ <}} f_4(x) = \lim_{\substack{x \to -1 \\ <}} \sin(x+1)\ln(-x-1).$$

$$= \lim_{x \to -1} \left(\frac{\sin(x+1)}{x+1}\right)\left(-\frac{\ln(-x-1)}{-x-1}\right)$$

$$= \underbrace{\lim_{\substack{x \to -1 \\ <}} \left(\frac{\sin(x+1)}{x+1}\right)}_{1} * \underbrace{\left[-\lim_{x \to -1} \frac{\ln(-x-1)}{-x-1}\right]}_{0}$$

As $\begin{cases} \lim\limits_{\substack{x \to -1 \\ <}} f_4(x) = 0 \\ \lim\limits_{\substack{x \to -1 \\ >}} f_4(x) = 0 \end{cases}$ $\Rightarrow$ we can remove the discontinuity of f_4 at $x_0 = -1$. where

$$F_4(x) = \begin{cases} f_4(x) & \text{if } x \neq -1 \\ 0 & \text{if } x = -1. \end{cases}$$

$$\boxed{D_{F_4} = \mathbb{R}}$$

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f_5: is discontinuous at $x_0 = 2\pi k \,/\, k \in \mathbb{Z}$.

$\lim\limits_{x \to 2\pi k} f_5(x) = (\sqrt{2})^{2n} = 2^n$, So we can remove (from Exo2)

all the discontinuity points. where the new function is given by:

$$F_5(x) = \begin{cases} f_5(x) & \text{if } x \neq 2\pi k \,/\, k \in \mathbb{Z}. \\ 2^n & \text{if } x = 2\pi k \,/\, k \in \mathbb{Z}. \end{cases}$$

$$D_{F_5} = \mathbb{R}.$$

f_6 is discontinuous at $x_0 = 0$.

~~let $x_n = \frac{1}{n}$ so $f_6(x_n) \neq f_6(x_0)$~~

$$\lim_{x \to 0} f_6(x) = \lim_{x \to 0} \underbrace{\cos(x)}_{1} \underbrace{\cos\left(\frac{1}{x}\right)}_{\nexists} = \nexists$$

So f_6 is not continuous at $x_0 = 0$

f_6 is continuous on $\mathbb{R}/\{0\}$.

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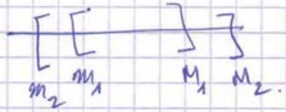
Exo7:

let $f(I) = [m_1; M_1]$ and $g(I) = [m_2, M_2]$.

1st case: $f(I) \subset g(I)$

$f(I) \subset g(I) \Rightarrow m_2 \leq m_1 \leq M_1 \leq M_2$.

let define $h(x) = f(x) - g(x)$ -



$h(I) = [m_1 - M_2 ; M_1 - m_2]$.

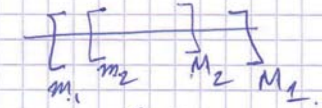
it clear that $\begin{cases} m_1 - M_2 \leq 0 \\ \text{and} \\ M_1 - m_2 \geq 0 \end{cases} \Rightarrow h$ changes the sig in I.

the according the intermidiar value therem we conclude that $\exists c \in I$ such as $h(c) = 0$

$\Rightarrow f(c) - g(c) = 0 \Rightarrow f(c) = g(c)$ —— (1)

2° Case: $g(I) \subset f(I) \Rightarrow m_1 \leq m_2 \leq M_2 \leq M_1$.

$h(I) = [m_1 - M_2 ; M_1 - m_2]$.



$\begin{cases} m_1 - M_2 \leq 0 \\ M_1 - m_2 \geq 0 \end{cases} \Rightarrow \exists c \in I$ such as $h(c) = 0$

$\Rightarrow f(c) = g(c)$ — (2)

from (1) and (2) we conclude that the propostion is true.

Remark $h(I)$ is obtained as follow: $\underset{x \in I}{m_1 \leq f(x) \leq M_1}$ — (*)

$m_2 \leq g(x) \leq M_2 \Rightarrow -M_2 \leq -g(x) \leq -m_2$ — (**) | (*) + (**) $\Rightarrow h(I)$ ✓.

Exo7: II

let defined the real function f on $]-\infty, 2[$ as follow $f(x) = \sin(x) - \frac{2x+1}{x-2}$.

we have $\lim_{x \to 2} f(x) = \lim_{x \to 2} \sin(x) - \frac{2x+1}{x-2} = +\infty$.

on the other hand, $f(x)$ is negative close to $-\infty$ (*), then according to the intermediare value theorem and as f changes the sign on $]-\infty, 2[\Rightarrow \exists c \in]-\infty, 2[$ such as $f(c) = 0$

ie $\sin(c) - \frac{2c+1}{c-2} = 0 \Rightarrow \sin(c) = \frac{2c+1}{c-2}$.

Remark: the limit of f when $x \to -\infty$ do not exist but the values of f are negative.

for a smallest value of x we have $-1 \leq \sin(x) \leq 1$

$\Rightarrow \sin(x) - \frac{2x+1}{x-2} \leq 0$

$\Rightarrow 1 - \frac{2x+1}{x-2} \leq 0$

$\Rightarrow -\frac{2x+1}{x-2} \leq -1 \Rightarrow \frac{2x+1}{x-2} \geq 1$.

$\Rightarrow 2x+1 \leq x-2$.

$\Rightarrow \boxed{x \leq -3}$

ie for all value $x \leq -3$, $f(x)$ is negative.

Exo8:

$f'(c)$ exists that mean $\lim\limits_{x \to x_0} \frac{f(x)-f(x_0)}{x-x_0} = \ell$ exist

$\lim\limits_{h \to 0} \frac{f(x+h)-f(x)}{h} = \ell$. (we can use this latest formula also).

$$\lim_{x \to c} f(x) = \lim_{x \to c} f(x) - f(c) + f(c)$$
$$= \lim_{x \to c} \left[\frac{f(x) - f(c) + f(c)}{(x-c)}\right] * (x-c)$$
$$= \lim_{x \to c} \left[\left(\frac{f(x)-f(c)}{x-c}\right)(x-c) + f(c)\right]$$
$$= \underbrace{\lim_{x \to c} \left[\frac{f(x)-f(c)}{x-c}\right]}_{f'(c)} * \underbrace{\lim_{x \to c} (x-c)}_{0} + f(c).$$

$0'$.

$\lim\limits_{x \to c} f(x) = f(c) \Rightarrow f$ is continous at c.

- $(e^x)' = e^x$?

$$(e^x)' = \lim_{h \to 0} \frac{e^{x+h} - e^x}{h} = \lim_{h \to 0} \frac{e^x(e^h - 1)}{h}$$
$$= e^x \lim_{h \to 0} \frac{e^h - 1}{h} = ?$$

Let $y = e^h - 1 \Rightarrow y+1 = e^h \Rightarrow h = \ln(1+y)$

$$\Rightarrow \lim_{h \to 0} \frac{e^h - 1}{h} = \lim_{y \to 0} \frac{y}{\ln(1+y)} = \lim_{y \to 0} \frac{1}{\frac{\ln(1+y)}{y}} = 1$$

$$\Rightarrow (e^x)' = \lim_{h \to 0} e^x \frac{e^h - 1}{h} = e^x$$

- $$\left(\frac{f(x)}{g(x)}\right)' = \lim_{h \to 0} \frac{\frac{f(x+h)}{g(x+h)} - \frac{f(x)}{g(x)}}{h}$$
$$= \lim_{h \to 0} \frac{g(x) f(x+h) - f(x) g(x+h)}{h} * \frac{1}{g(x+h) g(x)}$$
$$= \lim_{h \to 0} \frac{g(x) f(x+h) - g(x) f(x) - f(x) g(x+h) + g(x) f(x)}{h} * \frac{1}{g(x+h) g(x)}$$
$$= \lim_{h \to 0} \frac{g(x)\left[\frac{f(x+h) - f(x)}{h}\right] - f(x)\left[\frac{g(x+h) - g(x)}{h}\right]}{g(x+h) g(x)}$$
$$= \frac{g(x) \underbrace{\lim_{h \to 0} \left(\frac{f(x+h) - f(x)}{h}\right)}_{f'} - f(x) \underbrace{\lim_{h \to 0} \left(\frac{g(x+h) - g(x)}{h}\right)}_{g'}}{\underbrace{\lim_{h \to 0} (g(x+h) g(x))}_{g^2}}$$
$$= \frac{g(x) f'(x) - f(x) g'(x)}{g^2(x)}$$

- Recall that $f^{-1} \circ f(x) = x$.

$\arcsin(\sin(x)) = x \Rightarrow (\arcsin(\sin(x)))' = 1$.

$\Rightarrow \cos(x) \arcsin(\sin(x))' = 1$

$\Rightarrow \arcsin(\sin(x))' = \frac{1}{\cos(x)}$.

~~we~~ pose $y = \sin(x)$. $\Rightarrow$ $\cos(x) = \sqrt{1-\sin^2(x)}$

$\sin^2 + \cos^2 = 1$.

$$\Rightarrow \arcsin(\sin(x))' = \frac{1}{\sqrt{1-\sin^2(x)}}$$

$$\Rightarrow \arcsin(y)' = \frac{1}{\sqrt{1-y^2}}$$

ie $\boxed{\arcsin(x)' = \frac{1}{\sqrt{1-x^2}}}$

- $\arctan(\tan(x)) = x \Rightarrow [\arctan(\tan(x))]' = 1$

$$\tan(x)' \arctan'(\tan(x)) = 1.$$

$$\Rightarrow \arctan(\tan(x))' = \frac{1}{\tan(x)'} \quad \text{—— } (*)$$

$$\tan(x)' = \left(\frac{\sin(x)}{\cos(x)}\right)' = \frac{\sin^2(x) + \cos^2(x)}{\cos^2(x)} = 1 + (\tan(x))^2$$

$$(*) \Rightarrow \arctan(\tan(x))' = \frac{1}{1+\tan(x)^2}$$

~~tan~~ we pose $y = \tan(x) \Rightarrow$

$$\arctan'(y) = \frac{1}{1+y^2}$$

so, $\boxed{\arctan(x)' = \frac{1}{1+x^2}}$

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- $(f^{-1}(x))' = \frac{1}{f' \circ f^{-1}(x)}$?

we have $f \circ (f^{-1}(x)) = x \Rightarrow [f(f^{-1}(x))]' = 1$.

$$\Rightarrow (f^{-1}(x))' \cdot f'(f^{-1}(x)) = 1$$

$$\Rightarrow (f^{-1}(x))' = \frac{1}{f' \circ f^{-1}(x)}.$$

- $g \circ f(x) = ?$

$$g \circ f(x)' = \lim_{h \to 0} \frac{g \circ f(x+h) - g \circ f(x)}{h} = \lim_{h \to 0} \frac{g(f(x+h)) - g(f(x))}{h}$$

$$= \lim_{h \to 0} \frac{g(f(x+h)) - g(f(x))}{h} * \frac{\frac{f(x+h) - f(x)}{f(x+h) - f(x)}}{}$$

$$= \lim_{h \to 0} \frac{g(f(x+h)) - g(f(x))}{f(x+h) - f(x)} * \frac{f(x+h) - f(x)}{h}.$$

$$= \lim_{h \to 0} \frac{g(f(x+h)) - g(f(x))}{f(x+h) - f(x)} * \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{g(f(x+h)) - g(f(x))}{f(x+h) - f(x)} * f'(x).$$

To compute the above limit let take.

$y = f(x+h) - f(x) \Rightarrow y \to 0$ when $h \to 0$

and $f(x+h) = f(x) + y$.

So, $\lim_{h \to 0} \frac{g(f(x+h)) - g(f(x))}{f(x+h) - f(x)} = \lim_{y \to 0} \frac{g(f(x)+y) - g(f(x))}{y}$

by definition this last is nothing that $g'(f(x))$

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Exo: f is differentiable at $x_0 \Rightarrow \begin{cases} \text{f is continuous at } x_0 \\ \lim_{x\to x_0} f(x) = \ell \text{ exists.} \end{cases}$

sol: for f we must check $c=4$.

for g " " " $a=-b=\frac{2}{e^{\pi}-1}$

for h " " " $(a,b,c)\in\{(0,1,0)+c(1,-2,1)\}$

① $f(x)=\begin{cases} x^2+2x, & \text{if } x\geq 1 \\ -x+4 & \text{if } x<1. \end{cases}$

$\lim\limits_{x\to 1^+}\frac{f(x)-f(1)}{x-1}=\lim\limits_{x\to 1^+}\frac{x^2+2x-3}{x-1}=\lim\limits_{x\to 1}(x+3)=\boxed{4}$

$\lim\limits_{x\to 1^-}\frac{f(x)-f(1)}{x-1}=\lim\limits_{x\to 1^-}\frac{-x+4-3}{x-1}=\boxed{-1}$

$\lim\limits_{x\to 1^+}\frac{f(x)-f(1)}{x-1}\neq\lim\limits_{x\to 1^-}\frac{f(x)-f(1)}{x-1}\Rightarrow$ f is not derivative at $x_0=1$.

• f is differentiable on $\mathbb{R}\setminus\{1\}$.

② $g(x)=\begin{cases} x^2 & \text{if } x\leq 0 \\ \frac{2}{e^{\pi}-1}(e^x-1) & 0<x\leq\pi \\ 1-\cos(x) & x\geq\pi \end{cases}$ (13)

$\lim\limits_{x\to 0^-}\frac{g(x)-g(0)}{x-0}=0\ ;\ \lim\limits_{x\to 0^+}\frac{g(x)-g(0)}{x}=2/(e^{\pi}-1)$ — ①

$\lim\limits_{x\to \pi^-}\frac{g(x)-g(\pi)}{x-\pi}=\frac{2e^{\pi}}{e^{\pi}-1}\ ;\ \lim\limits_{x\to \pi^+}\frac{g(x)-g(\pi)}{x-\pi}=0$ — ②

from ① and ② $\Rightarrow$ g is differentiable on $\mathbb{R}\setminus\{0,\pi\}$.

$h(x)=\begin{cases} 1 & \text{if } x\leq 0 \\ c(x-1)(e^x-e^{-x}) & \text{if } 0<x<1 \\ e^{2-x} & \text{if } x\geq 1 \end{cases}$

• $\lim\limits_{x\to 0^-}\frac{h(x)-h(0)}{x-0}=0\ ;\ \lim\limits_{x\to 0^+}\frac{h(x)-h(0)}{x-0}=\cancel{1}\,2c$ — ①

• $\lim\limits_{x\to 1^-}\frac{h(x)-h(1)}{x-1}=e^1+c(e^1-e^{-1});\ \lim\limits_{x\to 1^+}\frac{h(x)-h(0)}{x-0}=-e^1$ ②.

from ① and ② $\Rightarrow$ h is differentiable on $\mathbb{R}\setminus\{0,1\}$.

• $f(x)=\begin{cases} \sqrt{x} & \text{if } 0\leq x\leq 1 \\ ax^2+bx+c & \text{if } x>1. \end{cases}$

* 1° the continuity:

$\lim\limits_{x\to 1^-}f(x)=\lim\limits_{x\to 1}\sqrt{x}=1.$

$\lim\limits_{x\to 1^+}f(x)=\lim\limits_{x\to 1}ax^2+bx+c=a+b+c$ $\Rightarrow a+b+c=1$ ①

$f(1)=1.$

* $\lim\limits_{x\to 1^-}\frac{f(x)-f(1)}{x-1}=\lim\limits_{x\to 1}\frac{\sqrt{x}-1}{x-1}=1/2$ — ②

$\lim\limits_{x\to 1^+}\frac{f(x)-f(1)}{x-1}=\lim\limits_{x\to 1}\frac{ax^2+bx+c-1}{x-1}$ (14)

from ① $\Rightarrow c=1-a-b\Rightarrow ax^2+bx+c-1=ax^2+bx-a-b$

$=a(x^2-1)+b(x-1)=[a(x+1)+b](x-1).$

Thus

$$\lim_{x\to 1}\frac{f(x)-f(1)}{x-1}=\lim_{x\to 1}\frac{[a(x+1)+b]\cancel{(x-1)}}{\cancel{(x-1)}}=2a+b \quad — (3)$$

from (2) and (3) $\Rightarrow 2a+b=\frac{1}{2}$ ———— (4)

In conclusion, we have:

$$\begin{cases} a+b=1-c \\ 2a+b=\frac{1}{2}\end{cases} \Rightarrow \begin{cases} a=c-\frac{1}{2} \\ b=\frac{3}{2}-2c \\ c=c\end{cases}$$

we conclude that if $(a,b,c)\in\{(c-\frac{1}{2};\frac{3}{2}-2c,c)\}_{c\in\mathbb{R}}$

then f is differentiable on $\mathbb{R}$.

- $f_1(x)=\begin{cases} x^2\cos(\frac{1}{x}) & \text{if } x\neq 0 \\ 0 & \text{else.}\end{cases}$

$$\lim_{x\to 0}\frac{f(x)-f(0)}{x-0}=\lim_{x\to 0}\frac{x^2\cos(\frac{1}{x})-0}{x-0}$$

$$=\lim_{x\to 0}x\cos(\frac{1}{x})=0$$

Proof: $-1\leq\cos(\frac{1}{x})\leq 1$

(*) for $x\to 0^+$ $\quad -x\leq x\cos(\frac{1}{x})\leq x$

$$\lim_{x\to 0^+}-x\leq\lim_{x\to 0^+}x\cos(\tfrac{1}{x})\leq\lim_{x\to 0^+}x$$

$$0\leq\lim_{x\to 0}x\cos(\tfrac{1}{x})\leq 0$$

$$\Rightarrow\lim_{x\to 0^+}x\cos(\tfrac{1}{x})=0$$

(*) for $x\to 0^-$

$$-x\geq x\cos(\tfrac{1}{x})\geq x$$

$$\lim_{x\to 0^-}x\cos(\tfrac{1}{x})=0$$

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$\Rightarrow f_1$ is differentiable on $\mathbb{R}$.

- $f_2(x)=\begin{cases}\sin(x)\sin(\frac{1}{x}) & \text{if } x\neq 0 \\ 0 & \text{else.}\end{cases}$

$$\lim_{x\to 0}\frac{f_2(x)-f_2(0)}{x-0}=\lim_{x\to 0}\frac{\sin(x)\sin(\frac{1}{x})-0}{x}$$

$$=\lim_{x\to 0}\frac{\sin(x)}{x}*\lim\sin(\tfrac{1}{x})$$

$\frac{\sin(x)}{x}\to 1$, $\quad \lim\sin(\frac{1}{x})\ \nexists$

$=\nexists$

the function f_2 is differentiable on $\mathbb{R}/\{0\}$.

- $f_3(x)=\begin{cases}\frac{|x|\sqrt{x^2-2x+1}}{x-1} & \text{if } x\neq +1 \\ 1 & \text{if } x=+1\end{cases}$

f_3 can be rewriten as follow:

$$f_3(x)=\begin{cases}\frac{|x|\,|x-1|}{x-1} & \text{if } x\neq 1 \\ 1 & \text{else}\end{cases}=$$

$$=\begin{cases}+x & \text{if } x\leq 0 \\ -x & \text{if } 0<x<1 \\ x & \text{if } x>1 \\ 1 & \text{if } x=1\end{cases}$$

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After calculation of the limits $\lim_{x\to x_0} \frac{f(x)-f(x_0)}{x-x_0}$

$x_0 \in \{0,1\}$ we conclude that

the function f_3 is differentiable only on $\mathbb{R} / \{0,1\}$

Remark: f_3 is not continuous at $x_0 = 1$ because.

$$\boxed{\lim_{x \to 1^<} f(x) = -1 \neq \lim_{x \to 1^>} f(x) = 1.}$$

- f_1 differentiable at $x_0 = 0$?

$$\lim_{x\to 0} \frac{f_1(x) - f_1(0)}{x - 0} = \lim_{x\to 0} \frac{\sqrt{x}}{x} = \lim_{x \to 0} \frac{1}{\sqrt{x}} = +\infty$$

$\Rightarrow f_1$ is not differentiable at $x_0 = 0$.

- $f_2(x) = (1-x)\sqrt{1-x^2}$, $x_0 = -1$ (Remark $\left\{ \begin{array}{l} \sqrt{1-x^2} \text{ is not diff at } -1 \\ 1-x \text{ is diff at } -1 \end{array} \right.$

$$\lim_{x\to -1} \frac{(1-x)\sqrt{1-x^2}}{x+1} = \lim_{x \to -1} \frac{(1-x)\sqrt{1-x}\sqrt{1+x}}{1+x}$$

$$= \lim_{x\to -1} \frac{(1-x)\sqrt{1-x}}{\sqrt{1+x}} = \frac{2\sqrt{2}}{0^+} = +\infty$$

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$\Rightarrow f_2$ is not diff at $x_0 = -1$

- $f_3(x) = (1-x)\sqrt{1-x^2}$, $x_0 = +1$ $\left\{ \begin{array}{l} \sqrt{1-x^2} \text{ is not diff at } x_0 = 1 \\ 1-x \text{ is diff at } x_0 = 1 \end{array} \right.$

$$\lim_{x\to 1} \frac{f_3(x) - f_3(1)}{x-1} = \lim_{x\to 1} \frac{\cancel{(1-x)}\sqrt{1-x}\sqrt{1+x}}{-\cancel{(1-x)}} = 0 \Rightarrow f_3 \text{ is diff at } +1$$

we see that if f or g does not diff at x_0 the we can conclude the diff of $f*g$ at x_0.

EXO 10:

- $\left(e^{\sin(x^3)}\right)' = \sin(x^3)' \, e^{\sin(x^3)}$

$$= (x^3)' \sin'(x^3)\, e^{\sin(x^3)}$$

$$= 3x^2 \cos(x^3)\, e^{\sin(x^3)}$$

- $\ln(x^2 + e^{-x^2})' = (x^2 + e^{-x^2})' / (x^2 + e^{-x^2})$

$$= (2x - 2x e^{-x^2}) / (x^2 + e^{-x^2})$$

- $\ln\left(\frac{x+1}{x-1}\right) = \left(\frac{x+1}{x-1}\right)' * \left(\frac{x-1}{x+1}\right)$

$$= \frac{x-1 - x-1}{(x-1)^2} * \left(\frac{x-1}{x+1}\right) = \frac{-2}{(x-1)^{\cancel{2}}} * \left(\frac{\cancel{x-1}}{x+1}\right)$$

$$\boxed{\ln\left(\frac{x+1}{x-1}\right)' = \frac{-2}{(x^2-1)}}$$

- $\sin(2x^2 + \cos(x)) = (2x^2 + \cos(x))' \sin'(2x^2 + \cos(x))$

$$= (4x - \sin(x)) \cos(2x^2 + \cos(x)).$$

- $\arcsin(x^2+x)' = (x^2+x)' \arcsin'(x^2+x)$

$$= \frac{(2x+1)}{\sqrt{1-(x^2+x)^2}}$$

- $\arctan(x^2+x)' = \frac{2x+1}{1+(x^2+x)^2}$

- $\left(\sqrt[3]{x^2+x}\right)' = (x^2+x)^{\frac{1}{3}} = \frac{1}{3}(x^2+x)'(x^2+x)^{\frac{1}{3}-1}$

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$$\left(\sqrt[3]{x^2+x}\right)' = \frac{\frac{1}{3}(2x+1)}{(x^2+x)^{2/3}}.$$

$$\left(a^{\left(\frac{x-1}{x+1}\right)}\right)' = \left(e^{\ln(a)\left(\frac{x-1}{x+1}\right)}\right)'$$

$$= \left(\ln(a)\left(\frac{x-1}{x+1}\right)\right)' e^{\ln(a)\left(\frac{x-1}{x+1}\right)}$$

$$= \frac{2\ln(a)}{(x+1)^2}\, a^{\left(\frac{x-1}{x+1}\right)}$$

- $$\left(e^{e^{(x^2+1/x)}}\right)' = \left(e^{(x^2+1/x)}\right)' e^{e^{(x^2+\frac{1}{x})}}$$

$$= (x^2+1/x)'\, e^{x^2+1/x}\, e^{e^{(x^2+\frac{1}{x})}}$$

$$= \left(2x-\frac{1}{x^2}\right) e^{x^2+1/x}\, e^{e^{(x^2+\frac{1}{x})}}$$

- $$\left(\log_a(\arcsin(x))\right)' = (\arcsin(x))' \log_a'(\arcsin(x))$$

$$= (\arcsin(x))' \frac{\ln'(\arcsin(x))}{\ln(a)}$$

$$= \frac{1}{\sqrt{1-x^2}} * \frac{1}{\ln(a)\arcsin(x)}.$$

- $$\left(\sqrt{|x^2+4x+3|}\right)' = \left(\sqrt{|(x-1)(x+3)|}\right)' = ?$$

x		-3		1	
$x+3$	$-$	0	$+$		$+$
$x-1$	$-$		$-$	0	$+$
	$+$	0	$-$	0	$+$

so
$$\begin{cases} \left(\sqrt{x^2+4x+3}\right)' = \dfrac{2x-4}{2\sqrt{x^2-4x+3}} & \text{if } x \notin]-3,1[\\ \left(\sqrt{-x^2+4x-3}\right)' = \dfrac{-2x+4}{2\sqrt{-x^2+4x-3}} & \text{else.} \end{cases}$$

$$\left(\frac{1-\tan^2(x)}{(1+\tan(x))^2}\right)' = \frac{\left(1-\tan^2(x)\right)'\left(1+\tan^2(x)\right) - \left((1+\tan(x))^2\right)'\left(1-\tan^2(x)\right)}{(1+\tan(x))^4}$$

$$= \frac{\left(\frac{-2}{\cos^2(x)}\tan(x)\right)\left(1+\tan^2(x)\right) - 2\frac{(1+\tan(x))}{\cos^2(x)}\left(1-\tan^2(x)\right)}{(1+\tan(x))^4}$$

$$= \frac{2\left(\tan^2(x)-2\tan(x)-1\right)}{\cos^2(x)\left(1+\tan(x)\right)^2}.$$

<u>Remark</u>: In general way for any differentiable f we have the following:

- $\left(a^{f(x)}\right)' = \ln(a) f'(x)\, a^{f(x)}$
- $\log_a(f(x))' = \dfrac{f'(x)}{\ln(a) f(x)}$
- $\arcsin(f(x))' = \dfrac{f'(x)}{\sqrt{1-f^2(x)}}$
- $\arctan(f(x))' = \dfrac{f'(x)}{1+f^2(x)}.$

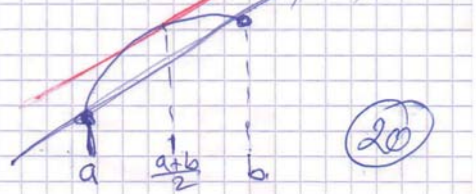
<u>EXO11</u>: f is a $P_2(x)$, so f is continous on $\mathbb{R}$.

$$\frac{f(b)-f(a)}{b-a} = f'(c) \Rightarrow \frac{(\alpha b^2+\beta b+\gamma)-(\alpha a^2+\beta a+\gamma)}{b-a} = 2\alpha c+\beta$$

$$\Rightarrow \frac{\alpha(b^2-a^2)+\beta(b-a)}{b-a} = 2c+\beta$$

$$\Rightarrow \frac{\alpha\cancel{(b-a)}(b+a)+\beta\cancel{(b-a)}}{\cancel{(b-a)}} = 2c+\beta \Rightarrow \cancel{\alpha}(b+a)+\cancel{\beta} = 2\cancel{\alpha}c+\cancel{\beta}$$

$$\Rightarrow \boxed{c = \frac{b+a}{2}.}$$



If we have two points $(a, f(a))$ and $(b, f(b))$, then the ligne passing trough these points is // to the tangente of f at the point $m_0 = (a+b)/2$.

② we have from the mean ~~value~~ theorem's value

$$\exists c \in]x, y[\text{ such as } \frac{\ln(y) - \ln(x)}{y - x} = f'(c).$$

$$\Rightarrow \frac{\ln(y) - \ln(x)}{y - x} = \frac{1}{c} \Rightarrow c = \frac{y - x}{\ln(y) - \ln(x)}.$$

As $c \in]x, y[$ ie $x < c < y$ then

$$x < \frac{y - x}{\ln(y) - \ln(x)} < y$$

Exo 12

① let the restriction $x \in]a, x_0[$.

~~so~~ Using the mean value theorem on the interval $]a, x[$ we have $\exists c \in]a, x[$ such as $\frac{f(x) - f(a)}{x - a} = f'(c)$.

As for any $c \in]a, b[$ $f'(c) \neq 0 \Rightarrow \forall c \in]a, x[$ we have $f'(c) \neq 0$

$$\Rightarrow \frac{f(x) - f(a)}{x - a} \neq 0 \Rightarrow f(x) - f(a) \neq 0$$
$$\Rightarrow f(x) \neq f(a).$$

② let compute $h(a)$ and $h(b)$

$$\begin{cases} h(a) = f(a) - \alpha g(a) \\ h(b) = f(b) - \alpha g(b) \end{cases} \Rightarrow \begin{array}{l} h(a) = f(a)g(b) - f(b)g(a) \quad (*) \\ h(b) = f(a)g(b) - f(b)g(a) \end{array}$$

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from the question ① we know that $f(b) - f(a) \neq 0$

f and g are continuous on $[a, b]$ —— (**) (**)

So, from (*), (**) and (***) $\Rightarrow$ h is a continuous function on $[a,b]$ and ~~$h(a) = h(b)$~~ is a differentielle function on $]a, b[$

in addition $h(a) = h(b)$

$$\Rightarrow \exists c \in]a, b[\Rightarrow \frac{h(b) - h(a)}{b - a} = h'(c) = 0$$

$$h'(c) = f'(c) - \alpha g'(c) = 0.$$

$$\Rightarrow \alpha = \frac{f'(c)}{g'(c)} \quad \text{ie} \quad \frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)}. \quad / g'(c) \neq 0$$

③ $$\lim_{x \to b} \frac{f(x) - f(b)}{g(x) - g(b)} = \lim_{x \to b^-} \frac{\frac{f(x) - f(b)}{x - b}}{\frac{g(x) - g(b)}{x - b}} = \lim_{x \to b^-} \frac{f'(x)}{g'(x)} = \ell.$$

More details: let consider $c(x) \in]x, b[$.
then, when $x \to b \Rightarrow c(x) \to b$. —— (*)

~~From~~ Using the result of question ② and (*)

we have:
$$\lim_{x \to b} \frac{f(x) - f(b)}{g(x) - g(b)} = \lim_{x \to b} \frac{f(b) - f(x)}{g(b) - g(x)} = \lim_{x \to b} \frac{f'(c(x))}{g'(c(x))} = \ell.$$

④ $$\lim_{x \to 1} \frac{\arccos(x)}{\sqrt{1 - x^2}} = \lim_{x \to 1} \frac{\arccos(x)'}{(\sqrt{1 - x^2})'} = \lim_{x \to 1} \frac{\frac{-1}{\sqrt{1 - x^2}}}{\frac{-2x}{2\sqrt{1 - x^2}}}$$
$$= \lim_{x \to 1} \frac{1}{x} = 1.$$

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EXO 13:

① $\lim\limits_{x\to 0} \frac{e^{3x-2} - e^{-2}}{x-0} = e^{3x-2}\Big|'_{x=0} = 3e^{3x-2}\Big|_{x=0} = 3e^{-2}$

② $\lim\limits_{x\to 1} \frac{\ln(2-x) - \ln(1)}{x-1} = \ln(2-x)\Big|'_{x=1} = \frac{-1}{2-x}\Big|_{x=1} = -1.$

③ $\lim\limits_{x\to \pi} \frac{\sin(x)}{x^2-\pi^2} = ?$ let put $y = x^2 \Rightarrow \cancel{x} \to \pi \Rightarrow y \to \pi^2.$

$\Rightarrow x = \begin{cases} +\sqrt{y} \\ -\sqrt{y} \end{cases}$ reject the fact $\underset{x\to\pi}{x} > 0.$

$\Rightarrow \cancel{\lim \frac{\sin}{y - \pi^2}}$

$\Rightarrow \lim\limits_{x\to\pi} \frac{\sin(x)}{x^2-\pi^2} = \lim\limits_{y\to\pi^2} \frac{\sin(\sqrt{y}) - \sin(\pi)}{x - \pi^2} = \cancel{\sin(\sqrt{y})'\Big|_{x=\pi^2} = \cos(\pi^2)}$

$= \sin(\sqrt{y})'\Big|_{y=\pi^2} = \frac{1}{2\sqrt{y}}\cos(\sqrt{y})\Big|_{\pi^2} = \frac{-1}{2\pi}.$

④ $\lim\limits_{x\to\pi/2} \frac{e^{\cos(x)} - e^{\cos(\pi/2)}}{(x-\pi/2)} = \left(e^{\cos(x)}\right)'\Big|_{x=\pi/2} = -\sin(x)\,e^{\cos(x)}\Big|_{x=\pi/2}$

$= -1.$

⑤ $\lim\limits_{x\to 0} \frac{\ln(1-\sin(x)) - \ln(1-\sin(0))}{x-0} = \ln(1-\sin(x))\Big|'_{x=0}$

$= \frac{-\cos(x)}{1-\sin(x)}\Big|_{x=0} = -1.$

⑥ $\lim\limits_{x\to+\infty} \left(\ln(x+1) - \ln(x)\right) = ?$

from EXO 11 we have

$x < \frac{y-x}{\ln(y)-\ln(x)} < y.$

$\Rightarrow \frac{1}{y} < \frac{\ln(y)-\ln(x)}{y-x} < \frac{1}{x}$

Remark

% the aim of the EXO is to use the ≠ derivate notion else

$\ln(x+1) - \ln(x) = \ln(1+\frac{1}{x})$

$y = \frac{1}{x} \Rightarrow \ln(1+y)$

$\lim\limits_{y\to 0} \ln(1+y) = 0.$

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$\Rightarrow$ if $y = x+1 \Rightarrow \quad \frac{1}{x+1} < \ln(x+1) - \ln(x) < \frac{1}{x}$

$\Rightarrow \lim\limits_{x\to\infty}\left(\frac{1}{1+x}\right) < \lim\limits_{x\to+\infty}\left(\ln(x+1) - \ln(x)\right) < \lim\limits_{x\to+\infty} \frac{1}{x}.$

$0 < \lim\limits_{x\to+\infty}\left(\ln(x+1) - \ln(x)\right) < 0$

$\Rightarrow \lim\limits_{x\to\infty}\left(\ln(x+1) - \ln(x)\right) = 0.$

EXO 14:

$f^{(1)}(x) = 2kx^{k-1}$

$f^{(2)}(x) = 2k(k-1)x^{k-2}$

$f^{(3)}(x) = 2k(k-1)(k-2)x^{k-3}$

$f^{(n)}(x) = \begin{cases} 2k(k-1)\times\cdots\times(k-n+1)x^{k-n} & \text{if } n <= k \\ 0 & \text{else.} \end{cases}$ — (*)

Proof: let prove by induction that the formula (*) is true for $n <= k$.

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- $f^{(1)}(x) = 2kx^{k-1}$ ✓ — ①
- suppose tha $f^{(n)}(x)$ is true — ②
- $P(n+1)$? $\quad f^{(n+1)}(x) = \left(f^{(n)}(x)\right)' = \left(2\times k(k-1)\cdots(k-n+1)x^{k-n}\right)'$

$= 2k(k-1)\cdots(k-n+1)(k-n)x^{k-n-1}.$

from (*) $\Rightarrow f^{(n+1)}(x) = 2k(k-1) - (k-n+1)(k-n)x^{k-n-1}$ } P_{n+1} is true.

- we have $f^{(k)} = 2k! = $ constante $\Rightarrow f^{(n)}(x) = 0$ for all $n > k$.

- $f(x) = \frac{1}{x}$

$f'(x) = \frac{-1}{x^2}$

$f^{(2)}(x) = \frac{2}{x^3}$

$f^{(3)}(x) = \frac{-2\times 3}{x^4}$

$\vdots$

$f^{(n)}(x) = \frac{(-1)^n n!}{x^{n+1}}$?

$\forall n \in \mathbb{N}^*$

- $n=0 \Rightarrow f^{(1)}(x) = \frac{(-1)^1 1!}{x^2} = \frac{-1}{x^2}$ ✓
- Suppose that $f^{(n)}(x)$ is true
- $f^{(n+1)}(x) = \frac{(-1)^{n+1}(n+1)!}{x^{n+2}}$ from the formula.

$f^{(n+1)}(x) = \left(f^{(n)}(x)\right)' = \left(\frac{(-1)^n n!}{x^{n+1}}\right)'$

$= \frac{(-1)^n n! \, (-(n+1))}{x^{n+2}}$

$= \frac{(-1)^{n+1}(n+1)!}{x^{n+2}}$ =

- $f(x) = \frac{1}{x^2}$

~~$f^{(1)}(x) =$ the example prude~~

from the above example ($f(x) = \frac{1}{x}$) we can deduce that when $f(x) = \frac{1}{x^2}$ then

$f^{(n)}(x) = \frac{(-1)^{n+1}(n+1)!}{x^{n+1}}$, $\forall n \in \mathbb{N}^*$.

- $f(x) = \sin(2x)$

$f^{(1)}(x) = 2\cos(2x) = 2\sin\left(2x + \frac{\pi}{2}\right)$

$f^{(2)}(x) = \cancel{-2^2\sin(2x)} = 2^2\cos\left(2x + \frac{\pi}{2}\right) = 2^2\sin\left(2x + 2\times\frac{\pi}{2}\right)$

$f^{(3)}(x) = \cancel{-2^3\cos(2x)}\ 2^3\cos\left(2x + 2\frac{\pi}{2}\right) = 2^3\sin\left(2x + 3\frac{\pi}{2}\right)$

$\boxed{f^{(n)}(x) = 2^n \sin\left(2x + n\frac{\pi}{2}\right)}$ $n \in \mathbb{N}^*$.

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$f(x) = \sin(x)\cos(x)$

$= \frac{1}{2}\left(2\sin(x)\cos(x)\right)$

$f(x) = \frac{1}{2}\left(\sin(2x)\right)$.

$\Rightarrow \boxed{f^{(n)}(x) = 2^{n-1}\sin\left(2x + \frac{n\pi}{2}\right) \quad \forall n \in \mathbb{N}^*}$

Note that

$f(x) = \frac{1}{1-x^2} = \left(\frac{1}{1-x} + \frac{1}{1+x}\right) * \frac{1}{2}$

let $g(x) = \frac{1}{1-x}$ and $h(x) = \frac{1}{1+x}$.

thus $f^{(n)}(x) = \frac{1}{2}\left(g^{(n)}(x) + h^{(n)}(x)\right)$

$g'(x) = \frac{-1}{(1+x)^2}$

$g''(x) = \frac{+1\times 2}{(1+x)^3}$

$g'''(x) = \frac{-1\times 2\times 3}{(1+x)^4}$

$\vdots$

$g^{(n)}(x) = \frac{(-1)^n (n!)}{(1+x)^{n+1}}$

$h'(x) = \frac{1}{(1-x)^2}$

$h''(x) = \frac{1\times 2}{(1-x)^3}$

$h'''(x) = \frac{1\times 2\times 3}{(1-x)^4}$

$\vdots$

$h^{(n)}(x) = \frac{n!}{(1-x)^{n+1}}$

$\Rightarrow f^{(n)}(x) = \frac{(-1)^n (n!)}{2(1+x)^{n+1}} + \frac{n!}{2(1-x)^{n+1}}$

- $f(x) = x^2 e^x \Rightarrow f^{(n)}(x) = \left[(n*(n-1)) + 2nx + x^2\right]e^{x}$

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