Chapitre 7

LAPLACE TRANSFORMATION

The Laplace transformation is, along with the Fourier transformation, one of the most important integral transformations. It intervenes in many questions of mathematical physics, probability calculation, automation, etc., but it also plays a major role in classical analysis. It very legitimately bears the name of Pierre-Simon Laplace (1749-1827).

7.1 Definition, convergence abscissa

Définition 7.1.1 *Let $f : [0, +\infty[$ or $]0, +\infty[\longrightarrow \mathbb{R}$ or $\mathbb{C}$ be a piecewise continuous function on any segment. We call the Laplace transform of f the function of a real or complex variable :*

$$F(p) = \mathcal{L}(p) = \int_0^{+\infty} e^{-pt} f(t)dt.$$

Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ or $\mathbb{C}$ be a piecewise continuous function on any segment. We call the Laplace transform of f the function of a real or complex variable :

$$F(p) = (p) = \int_{-\infty}^{+\infty} e^{-pt} f(t)H(t)dt = \int_0^{+\infty} e^{-pt} f(t)dt.$$

where $H(t)$ is the Heaviside function defined by $H(t) = 0$ for $t < 0, 1$ for $t > 0$.

The function $f(t)$ is called original, object function, or causal function. The function $F(p)$ is called the image of $f(t)$. We note $f(t)]F(p)$ for this correspondence.

The following problems naturally arise :

– At what points is the function F defined ?

– What are its properties within its domain of definition ?
– What are its properties at the edge of this domain ?
– What are the algebraic properties, differential and integral, of the Laplace transformation $\mathcal{L} : f \to F$?
– Can we go back from F to f ? That is, is there an inverse Laplace transform ?

Let us denote by $D(f)$ the set of complexes $p = a + ib$ such that the function $t \longrightarrow e^{-pt}f(t)$ is integrable on $]0, +\infty[$, that is to say $\int_0^{+\infty} e^{-pt}f(t)dt$ is absolutely convergent. $D(f)$ is called the domain of absolute convergence of the Laplace transform. Like $|e^{-pt}f(t)| = e^{-at}|f(t)|, p \in D(f) \iff a = \mathcal{R}e(p) \in D(f)$. Moreover, if $p \in D(f)$, then for all $\acute{a} > a$, $e^{-\acute{a}t}f(t)$ is integrable. We deduce from this that the set $D(f)$ is of one of the following four forms :

$$\varnothing\ , \mathbb{C}, \{p; \mathcal{R}e\,(p) \in]A, +\infty[\} \text{ or } \{p; \mathcal{R}e\,(p) \in [A, +\infty[\}.$$

The real $A = a(f)$ is called the abscissa of absolute convergence of the Laplace transform. We agree that $A = +\infty$ if $D(f) = \varnothing$, $A = -\infty$ if $D(f) = \mathbb{C}$.

Exemple 7.1.1 *1) If $f(t) = \exp(t^2), D(f) = \varnothing$, because $t \longrightarrow e^{-pt}e^{t^2}$ is never integrable.*
2) If $f(t) = 0$ or if $f(t) = \exp(-t^2)$, $D(f) = \mathbb{C}$, because $t \longrightarrow e^{-pt}f(t)$ is always integrable.
3) If $f(t) = 1$ or $H(t), D(f) = \{p; \mathcal{R}e\,(p) > 0\}$ and $\mathcal{L}(1)(p) = \mathcal{L}(H)(p) = \int_0^{+\infty} e^{-pt}dt = \frac{1}{p}$.
4) If $f(t) = e^{at}$ or $e^{at}H(t), D(f) = \{p; \mathcal{R}e\,(p) > a\}$ and $\mathcal{L}(e^{at})(p) = \mathcal{L}(e^{at}H(t))(p) = \int_0^{+\infty} e^{(a-p)t}dt = \frac{1}{p-a}$.
5) If $f(t) = \frac{1}{1+t^2}, D(f) = \{p; \mathcal{R}e\,(p) \geq 0\}$.
6) If $f(t) = \frac{1}{\sqrt{t}}, D(f) = \{p; \mathcal{R}e\,(p) > 0\}$.

The following proposition gives a sufficient condition for a function f to have a Laplace transform :

Proposition 7.1.1 *Let $f :]0, +\infty[\longrightarrow \mathbb{R}$ or $\mathbb{C}$ continue piecewise on any segment. If the integral $\int_0^1 |f(t)|\, dt$ converges, and if $\exists (M, \gamma, A) \forall t \geq A |f(t)| \leq Me^{\gamma t}, D(f)$ is non-empty. The function f is said to be of exponential order if it satisfies this last condition.*

7.2 General properties

In the following, we freely use the abusive notation $F(p) = \mathcal{L}(f(t))(p)$ for $f(t)]F(p)$. The variable p is assumed to be real.

Proposition 7.2.1 (linearity) *If $D(f)$ and $D(g)$ are non-empty,, $D(\alpha f + \beta g)$ is non-empty and, on $D(f) \cap D(g)$:*

$$\mathcal{L}(\alpha f + \beta g)(p) = \alpha \mathcal{L}(f)(p) + \beta \mathcal{L}(g)(p).$$

Proposition 7.2.2 (translation) *If $D(f)$ is non-empty, for all α, $D(e^{-at} f(t))$ is non-empty and $\mathcal{L}(e^{-at} f(t))(p) = () \, (p + \alpha)$.*

Preuve. $\mathcal{L}(e^{-\alpha t} f(t))(p) = \int_0^{+\infty} e^{-pt} e^{-\alpha t} f(t) dt = \int_0^{+\infty} e^{-(p+\alpha)t} f(t) dt = (())(p + \alpha)$. ■

Proposition 7.2.3 (delay) *If $D(f)$ is non-empty, $a > 0, g(t) = f(t - a)$ for $t > a$ for $t < a$, and $\mathcal{L}(f(t - a))(p) = e^{-ap} \, () \, (p)$.*

Preuve. $\mathcal{L}(g)(p) = \int_0^{+\infty} e^{-pt} g(t) dt = \int_0^{a} e^{-pt} g\,(t)\, dt + \int_a^{+\infty} e^{-pt} g\,(t)\, dt = \int_a^{+\infty} e^{-pt} f(t - a) dt = \int_0^{+\infty} e^{-p(u+a)} f(u) du = e^{-ap} \, () \, (p)$. ■

Proposition 7.2.4 (change of scale) *Si $D(f)$ is non-empty, $D(f(at))$ is non-empty for all $a > 0$, and $\mathcal{L}(f(at))(p) = \frac{1}{a} \, () \, (\frac{p}{a})$.*

Preuve. $\mathcal{L}(f(at))(p) = \int_0^{+\infty} e^{-pt} f(at) dt = \frac{1}{a} \int_0^{+\infty} e^{\frac{pu}{a}} f(u) du = \frac{1}{a} (\frac{p}{a})$. ■

Proposition 7.2.5 (derived from the image) *If $D(f)$ is non-empty, the function $= F$ is of class $\mathcal{C}^{\infty}$ on the interval $]a(f), +\infty[$, and $\mathcal{L}(t^n f(t))(p) = (-1)^n F^{(n)}(p)$.*

Preuve. Here, the variable p is assumed to be real. Let $p > a(f)$. Let us choose b such that $a(f) < b < p$. The function $e^{-bt} f(t)$ is integrable on $]0, +\infty[$. As $t^n e^{-pt} \left| f(t) \right| = O(e^{-bt} f(t))$ at $V(+\infty)$, each of the functions $t^n e^{-pt} f(t)$ is integrable. The parameter integral differentiation theorem applies :

- Each function $t \longrightarrow t^n e^{-pt} f(t)$ is piecewise continuous and integrable ;

- Each function $p \longrightarrow t^n e^{-pt} f(t)$ is continuous ;

- For $p \geq b > a(f), t^n e^{-pt} f(t) \leq M e^{-bt} \left| f(t) \right|$, integrable upper bound. ■

Corollaire 7.2.1 *If $f(t)$ has positive real values, $F(p)$ is positive, decreasing, convex, and completely monotonic, in the sense that its n^{th} derivative has the sign of $(-1)^n$.*

Proposition 7.2.6 (Image de la dérivée) *If f is $\mathcal{C}^1$ over $\mathbb{R}_+$, then $\mathcal{L}(\acute{f})(p) = pF(p) - f(0)$.*

If f is $\mathcal{C}^2$ over $\mathbb{R}_+$, then $\mathcal{L}(\acute{\acute{f}})(p) = p^2F(p) - pf(0) - \acute{f}(0)$.

If f is $\mathcal{C}^n$ over $\mathbb{R}_+$, then $\mathcal{L}(f^{(n)})(p) = p^nF(p) - (p^{n-1}f(0) + p^{n-2}\acute{f}(0) + \ldots + pf^{(n-2)}(0) + f^{(n-1)}(0))$.

Preuve. Just integrated by parts. ■

Proposition 7.2.7 (Image of the integral) *If $D(f)$ is non-empty and if f is piecewise continuous $\mathcal{L}\left(\int_0^t f(u)\,du\right)(p) = \dfrac{F(p)}{p}$.*

Proposition 7.2.8 (Convolution) *Let f and g be two continuous functions $[0, +\infty[\longrightarrow \mathbb{C}$, exponential order, their convolution product $f * g$, defined by $\forall x \geq 0, (f*g)(x) = \int_0^x f(x-t)\,g(t)\,dt$ is continuous, of exponential order, and $\mathcal{L}(f*g)(x)(p) = \mathcal{L}(f)(p).\mathcal{L}(g)(p)$.*

Preuve. The proof scheme, based on double integrals, is as follows :

$$\begin{aligned}
\mathcal{L}(f*g)(x)(p) &= \int_0^{+\infty} e^{px}(f*g)(x)dx \\
&= \int_0^{+\infty} e^{px}\left(\int_0^x f(x-t)g(t)dt\right)dx = \iint_\Delta f(x-t)g(t)e^{-px}dtdx \\
&= \iint_\Delta f(x-t)g(t)e^{-p(x-t)}e^{-pt}dtdx = \iint_\Delta f(x-t)g(t)e^{-p(x-t)}e^{-pt}dxdt \\
&= \int_0^{+\infty}\left(\int_t^{+\infty} f(x-t)\,g(t)\,e^{-p(x-t)}e^{-pt}dx\right)dt \\
&= \int_0^{+\infty}\left(\int_t^{+\infty} f(x-t)\,e^{-p(x-t)}dx\right)g(t)\,e^{-pt}dt \\
&= \int_0^{+\infty}\left(\int_0^{+\infty} f(u)\,e^{-pu}du\right)g(t)\,e^{-pt}dt = \int_0^{+\infty} F(p)g(t)e^{-pt}dt \\
&= F(p)G(p) = \mathcal{L}(f)(p)\mathcal{L}(g)(p).
\end{aligned}$$

■

7.3 Initial value, final value

Let $f :]0, +\infty[\longrightarrow \mathbb{R}$ or $\mathbb{C}$ be a piecewise continuous function. Suppose its Laplace transform $F(p) = \int_0^{+\infty} e^{-pt}f(t)dt$ defined for $p > 0$, in other words $a(f) \leq 0$. We propose to study the

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asymptotic behavior of $F(p)$ and when $p \longrightarrow +\infty$ when $p \longrightarrow 0+$. To do this, observe that $pF(p) = p\int_0^{+\infty} e^{-pt}f(t)dt$, where $\int_0^{+\infty} pe^{-pt}dt = 1pF(p)$ is the average of the values $f(t)$ taken by f, weighted by the weights $pe^{-pt}dt$.

7.3.1 Behavior of $F(p)$ when $p \longrightarrow +\infty$

When p tends towards $+\infty$, the weights $pe^{-pt}dt$ concentrate in the vicinity of $0+$, so that $F(p)$ depends more and more on the valuesof $f(t)$ in the vicinity of $0+$ as p increases. To obtain an equivalent or an asymptotic expansion of $F(p)$ at $V(+\infty)$, it will suffice to replace, in $F(p), f(t)$ by its equivalent or its asymptotic expansion at $0+$. This is the Laplace method, or initial value property.

Théoreme 7.3.1 (Initial value theorem) *Let $f : [0, +\infty[\longrightarrow \mathbb{C}$, continue piecewise on any segment, verifying : $(L)(\exists r) f(s) = O(e^{rs})$ at $V(+\infty)$.$F(p)$ is defined for $p > r$, and $\lim\limits_{p \longrightarrow +\infty} pF(p) = \lim\limits_{t \longrightarrow 0+} f(t)$.*

We will find in exercises applications and generalizations of this important result.

7.3.2 Behavior of $F(p)$ when $p \longrightarrow 0+$

When 0 is inside $D(f)$, i.e. $a(f) < 0, F(p)$ is expandable as an integer series at 0 and there is no problem. If 0 is on the edge of $D(f)$, i.e. $a(f) = 0$, the weights $pe^{-pt}dt$ are distributed more and more homogeneously as $p \longrightarrow 0+$, so that $F(p)$ depends more and more of the values taken by $f(t)$ in $+\infty$, or, let's say, of its average general behavior on $\mathbb{R}_+^*$. This is the final value property.

Théoreme 7.3.2 (Final Value Theorem) *1) If f is integrable over $\mathbb{R}_+^*$, then $F = \mathcal{L}(f)$ is defined for $p \geq 0$, and continues to 0.*

2) If f is integrable over $]0, 1]$ and has a limit ω in $+\infty$, $F(p)$ is defined for $p > 0$ and $\lim\limits_{p \longrightarrow 0+} pF(p) = \lim\limits_{t \longrightarrow +\infty} f(t) = \omega$.

Preuve. left in exercise. ■

7.4 Table of usual Laplace transforms

Just as there are tables of usual primitives, tables of usual limited expansions, there exist tables of Fourier transforms and tables of Laplace transforms of usual functions. In the table below, it is strictly necessary to indicate the convergence abscissa. From this table and the calculation rules above, we deduce that the Laplace transformation induces an isomorphism of the vector space of exponential-polynomials, that is to say the linear combinations of the functions $t^n e^{at}$ (a réel ou complexe), on the vector space of rational fractions of degree < 0. (See appendices).

7.5 Inverse Laplace transform

If $f(t)$ has Laplace transform $F(p), F =$, we symbolically write $f = \mathcal{L}^{-1}F$ and we say that f is a Laplace transform inverse of F.

Warning : the Laplace transformation is not injective !

- On the one hand, only the values taken by $f(t)$ on $t > 0$. come into play. The functions 1 and $H(t)$ even have a Laplace transform.
- On the other hand, two functions which differ on $\mathbb{R}_+^*$ can have the same Laplace image. A zero function almost everywhere has a zero Laplace transform.

The functions $f(t) = e^{-2t}$ and $g(t) = 0$ for $t = 5$, e^{-2t} for $t \neq 5$, even have a Laplace transform : $()(p) = ()(p) = \dfrac{1}{p+2}$.

However, the Laplace transformation is injective if we restrict it to certain classes of functions : exponential-polynomials, Lerch's theorem...

7.6 Introduction to symbolic calculus

Symbolic calculus, or operational calculus, was invented by Heaviside to solve in particular linear differential equations and systems, but also certain integral equations. It bridges the gap between analysis and algebra. We will develop it using a few examples.

Exemple 7.6.1 (Solve the differential equation) $\acute{\acute{y}} + 3\acute{y} + 2y = t, y(0) = \acute{y}(0) = 0$. *It is a linear differential equation with constant coefficients. Let us denote $F(p) = (Lf)(p)$ as the Laplace transform of $y(t)$.*

$$L(\acute{\acute{y}} + 3\acute{y} + 2y)(p) = \mathcal{L}(t)(p)$$

$$p(pF(p) - y(0)) - \acute{y}(0) + 3p(F(p) - y(0)) + 2F(p) = \frac{1}{p^2}$$

$$(p^2 + 3p + 2)F(p) - 4py(0) - \acute{y}(0) = \frac{1}{p^2}$$

$$F(p) = \frac{1}{p^2\left(p^2+3p+2\right)} = \frac{1}{p^2\left(p+1\right)\left(p+2\right)} = \frac{1}{2}\frac{1}{p^2} - \frac{3}{4}\frac{1}{p} + \frac{1}{p+1} - \frac{1}{4}\frac{1}{p+2}.$$

The decomposition into simple elements of the fraction allows us to go back to the causal function. $F(p)$ is Laplace transform of :

$$y(t) = \frac{1}{2}t - \frac{3}{4} + e^{-t} - \frac{1}{4}e^{-2t}.$$

This method provides the correct result, but it poses problems of rigor.

1st problem : does the solution $y(t)$ have a Laplace transform ? It would be necessary to show that the solutions of linear differential equations with constant coefficients and with an exponential-polynomial second member are all dominated by $O(e^{Mt})$ for a suitable M. This is indeed the case.

2nd problem : a uniqueness argument is missing to go back from $F(p)$ to the source $y(t)$. It would be necessary to demonstrate that the Laplace transformation $y(t) \longrightarrow F(p)$ is injective on a sufficiently large class of functions (exponential-polynomials in particular).

Exemple 7.6.2 (Find the continuous function f from $\mathbb{R}$ in $\mathbb{R}$) *checking :*

$$\forall x \in \mathbb{R} \qquad f(x) = x^2 + \int_0^x \sin(x-t)f(t)dt. \tag{7.1}$$

It is a functional convolution equation, which is written : : $f(x) = x^2 + (\sin * f)(x)$. Let us denote $F(p) = ()(p)$ as the Laplace transform of $f(x)$. It comes $F(p) = \frac{2}{p^3} + \frac{F(p)}{p^2+1}$, so $F(p) = \frac{2}{p^3} + \frac{2}{p^5}$.$F(p)$ is the Laplace transform of $f(x) = x^2 + \frac{1}{12}x^4$. The converse is easy.

NB : We could give a more rigorous and more basic direct solution. Indeed, (7.1) is written : $\forall x \in \mathbb{R} \qquad f(x) = x^2 + \sin(x)\int_0^x \cos(t) f(t)dt - \cos(x)\int_0^x \sin(t) f(t)dt$. We deduce that f is $\mathcal{C}^1$ and, step by step, $\mathcal{C}^{+\infty}$. If we differentiate it twice, we come across a differential equation...

Chapitre 8

FOURIER TRANSFORMATION

8.1 Definitions

Let $f : \mathbb{R}^d \longrightarrow \mathbb{C}$ be a piecewise continuous function (or more generally locally integrable in the Riemann sense). We will say that f belongs to the space $L^1(\mathbb{R}^d)$, if :

$$\int_{\mathbb{R}^d} |f(x)|\, dx < \infty,$$

that is to say if the integral above is convergent. Likewise we will say that f belongs to the space $L^2(\mathbb{R}^d)$ if :

$$\int_{\mathbb{R}^d} |f(x)|^2\, dx < \infty,$$

We notice

$$\|f\|_1 := \int_{\mathbb{R}^d} |f(x)|\, dx, \qquad \text{for } f \in L^1(\mathbb{R}^d),$$
$$\|f\|_2 := \left(\int_{\mathbb{R}^d} |f(x)|^2\, dx\right)^{\frac{1}{2}}, \qquad \text{for } f \in L^2(\mathbb{R}^d)$$

The quantities $\|f\|_1$ and $\|f\|_1$ are norms, that is to say that : $\|f+g\|_i \leq \|f\|_i + \|g\|_i$, $\|\lambda f\|_i = |\lambda|\, \|f\|_i$ and $\|f\|_i = 0 \implies f = 0$. For $f \in L^1(\mathbb{R}^d)$, we set :

$$\hat{f}(k) := \int_{\mathbb{R}^d} e^{-ikx} f(x) dx, \qquad k \in \mathbb{R}^d.$$

where $k.x = \sum_{i=1}^{d} k_i x_i$. The function $\hat{f}$ is called the **Fourier transform** of the function f. We

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also write :

$$\hat{f} = \mathcal{F}f \qquad \text{or} \qquad \mathcal{F}(f),$$

where the transformation :

$$\mathcal{F} : f \mapsto \hat{f}$$

is called the Fourier transform. It is therefore an operator which transforms functions of the variable x into functions of the variable k. If the variable x represents a position (its dimension is therefore m), the variable k represents an impulse, its dimension is m^{-1}. In signal processing, we have $d = 1$, the variable x is denoted t and has the dimension of a time (s), the variable k is denoted and has the dimension of a frequency (s^{-1}).

8.2 Properties

We use the following notations : the symbol ∂_{x_j}, designates the derivation operator with respect to x_j :

$$\partial_{x_j} f(x) := \frac{\partial}{\partial_{x_j}} f(x).$$

The symbol x_j denotes the multiplication operator by x_j :

$$x_j f(x) := x_j f(x).$$

We use the same conventions for the symbols ∂_{k_j} and k_j, which act on functions of the variable k.

Proposition 8.2.1 *(1) if $f \in L^1(\mathbb{R}^n)$, $\mathcal{F}f$ is a continuous and bounded function on $\mathbb{R}^d$,*
(2) si $f \in L^1(\mathbb{R}^d)$ and $x_j f \in L^1(\mathbb{R}^d)$, $\mathcal{F}f$ is a function of class $\mathcal{C}^1$ and :

$$\partial_{k_j} \mathcal{F}f(k) = -iF(x_j f)(k);$$

(3) if $f \in L^1(\mathbb{R}^d)$ and $\partial_{x_j} f \in L^1(\mathbb{R}^d)$, then $k_j \mathcal{F}f$ is bounded and :

$$k_j \mathcal{F}(f)(k) = -i\mathcal{F}(\partial_{x_j} f)(k).$$

The thing to remember is that the Fourier transform $\mathcal{F}$ transforms the momentum operator $D_j = i^{-1}\partial_{x_j}$; in the multiplication operator k_j :

$$\mathcal{F}(D_j f) = k_j \mathcal{F}(f).$$

Proposition 8.2.2 (Link to convolution) *Let $f, g \in L^1(\mathbb{R}^d)$. We set :*

$$h(x) := f * g(x) = \int_{\mathbb{R}^d} f(x-y)g(y)dy.$$

The function $f * g$ is called the convolution product of f and g. We have :

(1) $f * g = g * f$,

(2) $f * g \in L^1(\mathbb{R}^d)$ and $\|f * g\|_1 \leq \|f\|_1 \|g\|_1$;

(3) $\mathcal{F}(f * g) = \mathcal{F}(f)\mathcal{F}(g)$.

In other words, the Fourier transform transforms the convolution product into the ordinary product of functions.

8.3 Table of usual Fourier transforms

We now give some Fourier transforms of usual functions :

1. We start with the case $d = 1$.

$$f(x) = \mathbb{I}_{[-a,a](x)}, \qquad \mathcal{F}f(k) = \begin{cases} 2\dfrac{\sin(ak)}{a} & k \neq 0 \\ 0 & k = 0 \end{cases}.$$

   We recall that $\mathbb{I}_I(x)$ designates the indicator function of the set I, equal to 1 if $x \in I$ and to 0 otherwise.

$$f(x) = e^{-a|x|}, a > 0, \qquad \mathcal{F}f(k) = 2\frac{a}{a^2 + k^2}.$$

$$f(x) = e^{\frac{-ax^2}{2}}, a > 0, \qquad \mathcal{F}f(k) = (\frac{2\pi}{a})^{\frac{1}{2}} e^{-a^{-1}k^2/2}. \tag{8.1}$$

   (The Fourier transform of a Gaussian is a Gaussian).

2. In any dimension d, the last formula generalizes :

$$f(x) = e^{-\sum_1^d a_i x_i^2/2}, \qquad \mathcal{F}f(k) = \prod_1^d (\frac{2\pi}{a_i})^{\frac{1}{2}} e^{-\sum_1^d a_i^{-1} k_i^2/2}, \qquad \text{for} \qquad a_i > 0.$$

3. For more details see the annexes.

8.4 Inverse Fourier Transform

Proposition 8.4.1 *Let $f \in L^1(\mathbb{R}^d)$ be a function such that $\hat{f} \in L^1(\mathbb{R}^d)$. So we have :*

$$f(x) = (2\pi)^{-d} \int_{\mathbb{R}^d} e^{ik.x} \hat{f}(k) dk.$$

We can rewrite this result as :

$$\mathcal{F}^{-1} g(x) = (2\pi)^{-d} \int_{\mathbb{R}^d} e^{ik.x} g(k) dk.$$

where $\mathcal{F}^{-1}$ denotes the inverse Fourier transform, which transforms functions of the variable k into functions of the variable x.

8.5 Fourier transform on $L^2(\mathbb{R}^d)$

Proposition 8.5.1 (Plancherel's formula) *Let $f \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$. Then $\hat{f} \in L^2(\mathbb{R}^d)$ and we have :*

$$\int_{\mathbb{R}^d} |f(x)|^2 dx = (2\pi)^{-d} \int_{\mathbb{R}^d} \left|\hat{f}\right|^2 (k) dk.$$

This proposal makes it possible to extend the Fourier transformation from the space $L^1(\mathbb{R}^d)$ to the space $L^2(\mathbb{R}^d)$.

Proposition 8.5.2 (Fourier transformation on $L^2(\mathbb{R}^d)$) *Let $f \in L^2(\mathbb{R}^d)$. We set :*

$$\hat{f}_\varepsilon(k) = \int_{\mathbb{R}^d} e^{-ik.x} e^{-\varepsilon x^2} f(x) dx.$$

So

$$\hat{f}(k) := \mathcal{F}f(k) := \lim_{\varepsilon \longrightarrow 0^+} \hat{f}_\varepsilon(k)$$

exists and is called the Fourier transform of f.

The limit is to be understood in the sense L^2, that is to say that :

$$\int \left| \hat{f}(k) - \hat{f}_\varepsilon(k) \right|^2 dk \longrightarrow 0 \qquad when \qquad \varepsilon \longrightarrow 0.$$

The Fourier transformation extended to functions in the space $L^2(\mathbb{R}^d)$ still has the same properties.

8.6 Application of Fourier transform to solve differential equations

A powerful application of Fourier methods is in the solution of differential equations. This is because of the following identity for the FT of a derivative :

$$FT\left[f^{(p)}(x)\right] = FT\left[\frac{d^p f}{dx^p}\right] = (ik)^p \tilde{f}(k)$$

Thus applying a FT to terms involving derivatives replaces the differential equation with an algebraic equation for $\tilde{f}$, which may be easier to solve. Let's remind ourselves of the origin of this fundamental result. The simplest approach is to write a function $f(x)$ as a Fourier integral : $f(x) = \int \tilde{f}(k) exp(ikx) dk/2\pi$. Differentiation with respect to x can be taken inside the integral, so that $df/dx = \int \tilde{f}(k) exp(ikx) dk/2\pi$. From this we can immediately recognise ik $\tilde{f}(k)$ as the FT of df/dx. The same argument can be made with a Fourier series.

Fourier Transforms can also be applied to the solution of differential equations. To introduce this idea, we will run through an Ordinary Differential Equation (ODE) and look at how we can use the Fourier Transform to solve a differential equation.

Consider the ODE in Equation :

$$\frac{d^2 y(t)}{dt^2} - y(t) = -g(t) \tag{8.2}$$

We are looking for the function $y(t)$ that satisfies Equation 8.2 above. We know that we can take the Fourier Transform of a function, so why not take the fourier transform of an equation ? It turns out there is no reason we can't. And since the Fourier Transform is a linear operation, the time domain will produce an equation where each term corresponds to the a term in the frequency domain. Taking the Fourier Transform of Equation 8.2 , we get Equation 8.3 :

$$F\left(\frac{d^2y(t)}{dt^2}\right) - F\left(y(t)\right) = F\left(-g(t)\right) \Longleftrightarrow F\left(\frac{d^2y(t)}{dt^2}\right) - Y(f) = -G(f) \tag{8.3}$$

Hence, Equation 8.3 becomes :

$$(2\pi if)^2\, Y(f) - Y(f) = -G(f) \tag{8.4}$$

Equation 8.4 is a simple algebraic equation for $Y(f)$! This can be easily solved. This is the utility of Fourier Transforms applied to Differential Equations : They can convert differential equations into algebraic equations. Equation 8.4 can be easiliy solved for $Y(f)$:

$$Y(f) = \frac{-G(f)}{(2\pi if)^2 - 1} = \frac{G(f)}{1 + 4\pi^2 f^2} \tag{8.5}$$

In general, the solution is the inverse Fourier Transform of the result in Equation 8.5. For this case though, we can take the solution farther. Recall that the multiplication of two functions in the time domain produces a convolution in the Fourier domain, and correspondingly, the multiplication of two functions in the Fourier (frequency) domain will give the convolution in the time domain. Hence, Equation 8.5 becomes :

$$\begin{aligned} y(t) &= F^{-1}\left(Y(f)\right) = F^{-1}\left(\frac{1}{1+4\pi^2 f^2} . G(f)\right) \\ &= F^{-1}\left(\frac{1}{1+4\pi^2 f^2}\right) * F^{-1}\left(G(f)\right) \end{aligned} \tag{8.6}$$

Equation 8.6 might not look helpful, but note that we already know the inverse Fourier Transform for the left-most inverse Fourier transform in the second line of 8.6 : it's one half of the two-sided decaying exponential function. Hence, we can start to simplify equation 8.6 :

$$\begin{aligned} y(t) &= F^{-1}\left(\frac{1}{1+4\pi^2 f^2}\right) * F^{-1}\left(G(f)\right) = \frac{e^{-|t|}}{2} * g(t) \\ &= \frac{1}{2}\int_{-\infty}^{+\infty} e^{-|t-\tau|} g(\tau) d\tau \end{aligned}$$

Now for the fine print. When we went from Step 1 to Step 2, we assumed the Fourier Transform for $y(t)$ existed. This is a non-trivial assumption. You may recall from your differential equations class that the solution should also contain the so-called homogeneous solution, when $g(t) = 0$:

$$\frac{d^2 y_h(t)}{dt^2} - y_h(t) = 0 \Longrightarrow y_h(t) = c_1 e^t + c_2 e^{-t} \tag{8.7}$$

The "total" solution is the sum of the solution we obtained in equation 8.6 and the homogeneous solution y_h of equation 8.7. So why does the homogeneous solution not come out of our method ?

The answer is simple : the non-decaying exponentials of equation 8.7 do not have Fourier Transforms. That is, if you try to take the Fourier Transform of $exp(t)$ or $exp(-t)$, you will find the integral diverges, and hence there is no Fourier Transform. This is a very important caveat to keep in mind.