Mohamed Khider University of Biskra

Faculty of FSESNV
Department of SM
University Year 2023/2024

Module: Series and Diff. Eq
Level: 2^{nd} Year LMD
Specialty: Physics

Dirigated Work N°5

(Fourier series, Fourier Transform, Laplace Transform)

Exercise 1 *(Explicit calculations of Laplace transforms) Calculate the Laplace transforms of the following functions:*

$$f(t) = 1, \qquad f(t) = t^n, \qquad f(t) = e^{-at},$$
$$f(t) = \sin(\omega t), \qquad f(t) = \cos(\omega t), \qquad f(t) = t\sin(\omega t),$$

Exercise 2 *(Explicit calculations of inverse Laplace transforms) For each of the following functions, find a function $f(t)$ such that $\mathcal{L}[f(t)] = F(p)$:*

$$F(p) = \frac{1}{(p+2)(p-1)}, \qquad F(p) = \frac{p}{(p+1)(p^2+1)},$$

Exercise 3 *(Application of Laplace transforms in the resolution of Diff Eqs)*
We consider the differential equation:

$$\begin{cases} \ddot{y} + 2\dot{y} + y = e^{-t} \\ y(0) = 0 \\ \dot{y}(0) = 2 \end{cases}, \qquad t \geq 0.$$

Exercise 4 *(Trigonometric form of Fourier series)*
Calculate the Fourier series, in trigonometric form, of the 2π-periodic function $f : \mathbb{R} \longrightarrow \mathbb{R}$ such that: $f(x) = x^2$ on $[0, 2\pi[$. Does the series converge to f?

Exercise 5 *(Complex form of Fourier series)*
Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be the 2π-periodic function such that $f(x) = e^x$ for all $x \in]-\pi, \pi]$.

1. *Calculate the complex Fourier coefficients of the function f.*
2. *Study the (simple, uniform) convergence of the Fourier series of f.*

Exercise 6 *(Fourier transform)*
Let $a > 0$ and f be the function defined on $\mathbb{R}$ by $f(x) = e^{-a|x|}$.

1. *We consider a function $g : \mathbb{R} \longrightarrow \mathbb{R}$ which is integrable and even. Show that:*
$$\hat{g}(\omega) = 2\int_0^{\infty} g(x)\cos(\omega x)dx.$$
2. *Use the previous question to calculate the Fourier transform of f.*
3. *With a particular value of a, deduce the value of the integral:* $\int_0^{+\infty} \frac{\cos(\omega x)}{1+\omega^2} d\omega$.

Charged of courses

Dr. OUAAR, F

Fonction	Transformée de Laplace et inverse
$\delta(t)$	1
1	$\frac{1}{p}$
t	$\frac{1}{p^2}$
t^n	$\frac{n!}{p^{n+1}}$
$\sqrt{t}$	$\frac{1}{2}\sqrt{\frac{\pi}{p^3}}$
$\frac{1}{\sqrt{t}}$	$\sqrt{\frac{\pi}{p}}$
$e^{-c.t}$	$\frac{1}{p+c}$
$t.e^{-c.t}$	$\frac{1}{(p+c)^2}$
$t^2.e^{-c.t}$	$\frac{2}{(p+c)^3}$
$t^n.e^{-c.t}$	$\frac{n!}{(p+c)^{n+1}}$
a^t	$\frac{1}{p-\ln a}$
$\sin(a.t)$	$\frac{a}{p^2+a^2}$
$t.\sin(a.t)$	$\frac{2a.p}{(p^2+a^2)^2}$
$t^2.\sin(a.t)$	$\frac{2a(3p^2-a^2)}{(p^2+a^2)^3}$
$\cos(a.t)$	$\frac{p}{p^2+a^2}$
$t.\cos(a.t)$	$\frac{p^2-a^2}{(p^2+a^2)^2}$
$t^2.\cos(a.t)$	$\frac{2p(p^2-3a^2)}{(p^2+a^2)^3}$
$\sin(a.t+b)$	$\frac{a\cos b+p\sin b}{p^2+a^2}$
$\cos(a.t+b)$	$\frac{p.\cos b-a\sin b}{p^2+a^2}$
$\sinh(a.t)$	$\frac{a}{p^2-a^2}$
$t.\sinh(a.t)$	$\frac{2a.p}{(p^2-a^2)^2}$
$\cosh(a.t)$	$\frac{p}{p^2-a^2}$
$t.\cosh(a.t)$	$\frac{p^2+a^2}{(p^2-a^2)^2}$

Transformée de Laplace	Fonction
1	$\delta(t)$
$\frac{1}{p}$	1
$\frac{1}{p^2}$	t
$\frac{1}{p^n}$	$\frac{t^{n-1}}{(n-1)!}$
$\frac{1}{\sqrt{p}}$	$\frac{1}{\sqrt{\pi t}}$
$\frac{1}{\sqrt{p^3}}$	$2\sqrt{\frac{t}{\pi}}$
$\frac{1}{p+a}$	$e^{-a.t}$
$\frac{1}{p(p+a)}$	$\frac{1}{a}\left(1-e^{-a.t}\right)$
$\frac{1}{p^2(p+a)}$	$\frac{e^{-a.t}}{a^2}+\frac{t}{a}-\frac{1}{a^2}$
$\frac{1}{p(p+a)^2}$	$\frac{1}{a^2}\left(1-e^{-a.t}-a.t.e^{-a.t}\right)$
$\frac{1}{(p+a)(p+b)}$	$\frac{e^{-b.t}-e^{-a.t}}{a-b}$
$\frac{p}{(p+a)(p+b)}$	$\frac{ae^{-a.t}-be^{-b.t}}{a-b}$
$\frac{1}{(p+a)(p+b)(p+c)}$	$\frac{e^{-a.t}}{(b-a)(c-a)}+\frac{e^{-b.t}}{(a-b)(c-b)}+\frac{e^{-c.t}}{(a-c)(b-c)}$
$\frac{1}{(p+a)^2}$	$t.e^{-a.t}$
$\frac{p}{(p+a)^2}$	$e^{-a.t}(1-a.t)$
$\frac{1}{(p+a)(p+b)^2}$	$\frac{e^{-a.t}-[1+(b-a)t]\,e^{-b.t}}{(b-a)^2}$
$\frac{1}{p(p+a)(p+b)}$	$\frac{1}{a.b}\left(1+\frac{b.e^{-a.t}-a.e^{-b.t}}{a-b}\right)$
$\frac{p+c}{p(p+a)(p+b)}$	$\frac{c}{a.b}+\frac{c-a}{a(a-b)}.e^{-a.t}+\frac{c-b}{b(b-a)}.e^{-b.t}$
$\frac{p^2+c.p+d}{p(p+a)(p+b)}$	$\frac{d}{a.b}+\frac{a^2-a.c+d}{a(a-b)}.e^{-a.t}+\frac{b^2-b.c+d}{b(b-a)}.e^{-b.t}$
$\frac{1}{(p+a)^3}$	$\frac{t^2.e^{-a.t}}{2}$
$\ln\left(\frac{p+a}{p+b}\right)$	$\frac{e^{-b.t}-e^{-a.t}}{t}$
$\frac{1}{p^2+a^2}$	$\frac{1}{a}\sin(a.t)$

$e^{-c.t}.\sin(a.t)$	$\frac{a}{(p+c)^2+a^2}$
$e^{-c.t}.\cos(a.t)$	$\frac{p+c}{(p+c)^2+a^2}$
$e^{-c.t}.\sin(a.t+b)$	$\frac{a\cos b+(p+c)\sin b}{(p+c)^2+a^2}$
$e^{-c.t}.\cos(a.t+b)$	$\frac{(p+c)\cos b+a\sin b}{(p+c)^2+a^2}$
$e^{-c.t}.\sinh(a.t)$	$\frac{a}{(p+c)^2-a^2}$
$e^{-c.t}.\cosh(a.t)$	$\frac{p+c}{(p+c)^2-a^2}$
$\sin^2(a.t)$	$\frac{2a^2}{p(p^2+4a^2)}$
$\sin^3(a.t)$	$\frac{6a^3}{(p^2+a^2)(p^2+9a^2)}$
$\cos^2(a.t)$	$\frac{p^2+2a^2}{p(p^2+4a^2)}$
$\cos^3(a.t)$	$\frac{p(p^2+7a^2)}{(p^2+a^2)(p^2+9a^2)}$
$\sinh^2 t$	$\frac{2}{p(p^2-4)}$
$\cosh^2 t$	$\frac{p^2-2}{p(p^2-4)}$
$\sin(a.t).\sin(b.t)$	$\frac{2a.b.p}{[(p^2+(a-b)^2].[(p^2+(a+b)^2]}$
$\cos(a.t).\cos(b.t)$	$\frac{p^2(p^2+a^2+b^2)}{[(p^2+(a-b)^2].[(p^2+(a+b)^2]}$
$\sin(a.t).\cos(b.t)$	$\frac{a(p^2+a^2-b^2)}{[(p^2+(a-b)^2].[(p^2+(a+b)^2]}$

$\frac{1}{p(p^2+a^2)}$	$\frac{1}{a^2}(1-\cos(a.t))$
$\frac{p}{p^2+a^2}$	$\cos(a.t)$
$\frac{p+a}{p(p^2+b^2)}$	$\frac{a}{b^2}-\frac{\sqrt{a^2+b^2}}{b^2}\cos\left(b.t+\arctan\frac{b}{a}\right)$
$\frac{p^2+c.p+d}{p(p^2+b^2)}$	$\frac{d}{b^2}-\frac{\sqrt{(d-b^2)^2+c^2b^2}}{b^2}\cos\left(b.t+\arctan\frac{bc}{d-b^2}\right)$
$\frac{1}{p^2-a^2}$	$\frac{1}{a}\sinh(a.t)$
$\frac{p}{p^2-a^2}$	$\cosh(a.t)$
$\frac{1}{(p+b)^2+a^2}$	$\frac{1}{a}e^{-b.t}.\sin(a.t)$
$\frac{p+b}{(p+b)^2+a^2}$	$e^{-bt}.\cos(a.t)$
$\frac{1}{(p^2+a^2)^2}$	$\frac{\sin(a.t)}{2a^3}-\frac{t.\cos(a.t)}{2a^2}$
$\frac{p}{(p^2+a^2)^2}$	$\frac{t}{2a}\sin(a.t)$
$\frac{p^2}{(p^2+a^2)^2}$	$\frac{1}{2a}(\sin(a.t)+a.t.\cos(a.t))$
$\frac{1}{p^3+a^3}$	$\frac{1}{3a^2}\left[e^{-at}-e^{\frac{at}{2}}\left(\cos\left(\frac{\sqrt{3}}{2}at\right)-\sqrt{3}\sin\left(\frac{\sqrt{3}}{2}at\right)\right)\right]$
$\frac{p}{p^3+a^3}$	$\frac{1}{3a}\left[-e^{-at}-e^{\frac{at}{2}}\left(\cos\left(\frac{\sqrt{3}}{2}at\right)+\sqrt{3}\sin\left(\frac{\sqrt{3}}{2}at\right)\right)\right]$
$\frac{p^2}{p^3+a^3}$	$\frac{1}{3}\left[e^{-at}+2e^{\frac{at}{2}}\cos\left(\frac{\sqrt{3}}{2}at\right)\right]$
$\frac{1}{(\tau1p+1)(\tau2p+1)p^2}$	$t-(\tau1+\tau2)+\frac{1}{(\tau1-\tau2)}.(\tau1^2.e^{-t/\tau1}-\tau2^2.e^{-t/\tau2})$