Chapter 1: Mathematical reminders

1.1 PHYSICAL QUANTITIES:

As for any quantity, the value of a fundamental constant can be expressed as the product of a number and a unit. The definitions below specify the exact numerical value of each constant when its value is expressed in the corresponding SI unit. By fixing the exact numerical value, the unit becomes defined, since the product of the numerical value and the unit has to equal the value of the constant, which is postulated to be invariant.

The seven constants are chosen in such a way that any unit of the SI can be written either through a defining constant itself or through products or quotients of defining constants.

The International System of Units, the SI, is the system of units in which :

- the unperturbed ground state hyperfine transition frequency of the caesium 133 atom, $\Delta\nu_{Cs}$, is 9 192 631 770 Hz,
- the speed of light in vacuum, c, is 299 792 458 m/s,
- the Planck constant, *h*, is 6.626 070 15 × 10−34 J s,
- the elementary charge, *e*, is 1.602 176 634 × 10^{-19} C,
- the Boltzmann constant, *k*, is 1.380 649 × 10^{-23} J/K,
- the Avogadro constant, N_A, is 6.022 140 76 × 10^{23} mol^{-1},
- the luminous efficacy of monochromatic radiation of frequency 540 × 10^{12} Hz, K_{cd}, is 683 lm/W,

where the hertz, joule, coulomb, lumen, and watt, with unit symbols Hz, J, C, lm, and W, respectively, are related to the units second, meter, kilogram, ampere, kelvin, mole, and candela, with unit symbols s, m, kg, A, K, mol, and cd, respectively, According to:

$Hz = s^{-1}$,

$J = kg\ m^2\ s^{-2}$,

$C = A.\ s$,

$lm = cd.m^2.m^{-2} = cd.sr$

and $W = kg\ m^2\ s^{-3}$.

The numerical values of the seven defining constants **have no uncertainty.**

Table 1.1. The seven defining constants of the SI and the seven corresponding units they define.

Defining constant	Symbol	Numerical value	Unit
hyperfine transition frequency of Cs	$\Delta\nu_{Cs}$	9 192 631 770	Hz
Speed of light in vacuum	c	299 792 458	m s−1
Planck constant	h	$6.626\ 070\ 15 \times 10^{-34}$	J s
Elementary charge	e	$1.602\ 176\ 634 \times 10^{-19}$	C
Boltzmann constant	k	$1.380\ 649 \times 10^{-23}$	J K^{-1}
Avogadro constant	N_A	$6.022\ 140\ 76 \times 10^{23}$	mol^{-1}
Luminous efficacy	k_{cd}	683	lm W^{-1}

1.2 DIMENSIONS OF QUANTITIES:

Physical quantities can be organized in a system of dimensions, where the system used is decided by convention. Each of the seven base quantities used in the SI is regarded as having its own dimension. The symbols used for the base quantities and the symbols used to denote their dimension are shown in Table 1.2.

Table 1.2. Base quantities and dimensions used in the SI

Base quantity	Typical symbol for quantity	Symbol for dimension
Time	T	T
Length	l, x, r, etc	L
Mass	m	M
Electric current	I, i	I
Thermodynamic temperature	T	Θ
amount of substance	n	N
luminous intensity	I_V	J

All other quantities, with the exception of counts, are derived quantities, which may be written in terms of base quantities according to the equations of physics. The dimensions of the derived quantities are written as products of powers of the dimensions of the base quantities using the equations that relate the derived quantities to the base quantities. In general, the dimension of any quantity Q is written in the form of a dimensional product,

$$\textbf{\textit{dim Q}} = T^{\alpha} L^{\beta} M^{\gamma} I^{\delta} \Theta^{\varepsilon} N^{\zeta} J^{\eta}$$

where the exponents α, β, γ, δ, ε, ζ and η, which are generally small integers, which can be positive, negative, or zero, are called the dimensional exponents.

2.1 Types of Errors

Errors in length are differences between the target's true value and the measured value, or between the reference value and the measured value. They are expressed as "error = measured value - true value." In actuality, it is difficult to obtain the true value no matter how precise the measurement is, so it is unavoidable that some uncertainty will exist in the measured value. Errors can be classified into three major types according to the factor that generates the error. Such factors must be carefully considered to prevent errors.

2.1.1 Systematic errors

With this type of error, the measured value is biased due to a specific cause. Examples include measurement variations resulting from differences between individual instruments (instrumental errors), temperature, and specific ways of measuring.

2.1.2 Random errors

This type of error is caused by random circumstances during the measurement process.

2.1.3 Negligent errors

This type of error is caused by the inexperience of or the incorrect operations performed by the measuring individual.

3. Uncertainty and significant figures

3.1 ABSOLUTE UNCERTAINTY:

The absolute uncertainty in a quantity is the actual amount by which the quantity is uncertain, e.g.if $L = 6.0 \pm 0.1$ cm, the absolute uncertainty in L is 0.1 cm. Note that the absolute uncertainty of a quantity has the same units as the quantity itself.

3.2 RELATIVE UNCERTAINTY:

This is the simple ratio of uncertainty to the value reported. As a ratio of similar quantities, the relative uncertainty has no units.

3.3 CALCULATION OF ABSOLUTE UNCERTAINTY:

To calculate the absolute uncertainty of a physical quantity $f(x; y; z; ...)$, which depends on the measurable variables x; y; z…, we use the differential method by following the following steps:

- We calculate the partial derivatives $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}, ...$
- We write the differential according to these partial derivatives:

$$df = \left|\frac{\partial f}{\partial x}\right| dx + \left|\frac{\partial f}{\partial y}\right| dy + \left|\frac{\partial f}{\partial z}\right| dz + \cdots ..$$

- We approximate the differential df to its absolute uncertainty Δf: $df = \Delta f$. Likewise for the differentials of the variables $dx = \Delta x, dy = \Delta y, ...$
- We calculate the absolute uncertainty from the expression:

$$\Delta f = \left|\frac{\partial f}{\partial x}\right| \Delta x + \left|\frac{\partial f}{\partial y}\right| \Delta y + \left|\frac{\partial f}{\partial z}\right| \Delta z + \cdots ..$$

- We can then use the absolute uncertainty Δf to calculate the relative uncertainty $\frac{\Delta f}{f}$.

3.4 CALCULATION OF RELATIVE UNCERTAINTY:

To calculate the relative uncertainty of a physical quantity $f(x; y; z; ...)$ we use the logarithmic differential method by following the following steps:

- We write the logarithm lnf according to the logarithms of the variables lnx, lny, lnz ...
- We calculate the differential $d(lnf)$ in functions of the differentials $d(lnx), d(lny)$, $d(lnz)$ ...
- Knowing that for example: $d(lnf) = \frac{df}{f}$, we replace the logarithmic differentials by the differentials: $\frac{df}{f}, \frac{dx}{x}, \frac{dy}{y}, \frac{dz}{z}, \ldots$
- The relative uncertainty $\frac{\Delta f}{f}$ is obtained by replacing the differentials $\frac{df}{f}, \frac{dx}{x}, \frac{dy}{y}, \frac{dz}{z}$.., with the relative uncertainties $\frac{\Delta f}{f}, \frac{\Delta x}{x}, \frac{\Delta y}{y}, \frac{\Delta z}{z}, \ldots$
- We can then use the relative uncertainty $\frac{\Delta f}{f}$ to calculate the absolute uncertainty Δf.

3.5. Significant Figures

When we express measured values, we can only list as many digits as we measured initially with our measuring tool. For example, if we use a standard ruler to measure the length of a stick, we may measure it to be 36.7 cm. We can't express this value as 36.71 cm because our measuring tool is not precise enough to measure a hundredth of a centimeter. It should be noted that the last digit in a measured value has been estimated in some way by the person performing the measurement. For example, the person measuring the length of a stick with a ruler notices the stick length seems to be somewhere in between 36.6 cm and 36.7 cm, and he or she must estimate the value of the last digit. Using the method of significant figures**,** the rule is that the last digit written down in a measurement is the first digit with some uncertainty**.** To determine the number of significant digits in a value, start with the first measured value at the left and count the number of digits through the last digit written on the right. For example, the measured value 36.7 cm has three digits, or three significant figures.

Rules for Determining the number of significant figures

Here are some general rules for determining the number of significant figures:

- For experimental data the uncertainty in a quantity defines how many figures are significant.
- In general, the uncertainty in a measurement is equal to the estimated standard deviation for that measurement.
- The reading uncertainty is an estimate made by the experimenter
- In the case where the reading uncertainty in a measurement is larger than the estimated standard deviation, the reading uncertainty is the uncertainty in each individual measurement.
- The reading uncertainty almost by definition has one and only one significant figure.
- Thus uncertainties are specified to one or at most two digits.

Example: Express the following quantities to the correct number of significant figures:

(a) 29.625 ± 2.345
(b) 74 ± 7.136
(c) 84.26351 ± 3