2 HYPERBOLIC FUNCTIONS

Objectives

After studying this chapter you should

- understand what is meant by a hyperbolic function;
- be able to find derivatives and integrals of hyperbolic functions;
- be able to find inverse hyperbolic functions and use them in calculus applications;
- recognise logarithmic equivalents of inverse hyperbolic functions.

2.0 Introduction

This chapter will introduce you to the hyperbolic functions which you may have noticed on your calculator with the abbreviation *hyp*. You will see some connections with trigonometric functions and will be able to find various integrals which cannot be found without the help of hyperbolic functions. The first systematic consideration of hyperbolic functions was done by the Swiss mathematician *Johann Heinrich Lambert* (1728-1777).

2.1 Definitions

The **hyperbolic cosine function**, written $\cosh x$, is defined for all real values of x by the relation

$$\cosh x = \frac{1}{2}\left(e^x + e^{-x}\right)$$

Similarly the **hyperbolic sine function**, $\sinh x$, is defined by

$$\sinh x = \frac{1}{2}\left(e^x - e^{-x}\right)$$

The names of these two hyperbolic functions suggest that they have similar properties to the trigonometric functions and some of these will be investigated.

Activity 1

Show that $\cosh x + \sinh x = e^x$

and simplify $\cosh x - \sinh x$.

(a) By multiplying the expressions for $(\cosh x + \sinh x)$ and $(\cosh x - \sinh x)$ together, show that

$$\cosh^2 x - \sinh^2 x = 1$$

(b) By considering $(\cosh x + \sinh x)^2 + (\cosh x - \sinh x)^2$

show that $\cosh^2 x + \sinh^2 x = \cosh 2x$

(c) By considering $(\cosh x + \sinh x)^2 - (\cosh x - \sinh x)^2$

show that $2\sinh x \cosh x = \sinh 2x$

Activity 2

Use the definitions of $\sinh x$ and $\cosh x$ in terms of exponential functions to prove that

(a) $\cosh 2x = 2\cosh^2 x - 1$

(b) $\cosh 2x = 1 + 2\sinh^2 x$

Example

Prove that $\cosh(x-y) = \cosh x \cosh y - \sinh x \sinh y$

Solution

$$\cosh x \cosh y = \frac{1}{2}\left(e^x + e^{-x}\right) \times \frac{1}{2}\left(e^y + e^{-y}\right)$$

$$= \frac{1}{4}\left(e^{x+y} + e^{x-y} + e^{-(x-y)} + e^{-(x+y)}\right)$$

$$\sinh x \sinh y = \frac{1}{2}\left(e^x - e^{-x}\right) \times \frac{1}{2}\left(e^y - e^{-y}\right)$$

$$= \frac{1}{4}\left(e^{x+y} - e^{x-y} - e^{-(x-y)} + e^{-(x+y)}\right)$$

Subtracting gives

$$\cosh x \cosh y - \sinh x \sinh y = 2 \times \frac{1}{4}\left(e^{x-y} + e^{-(x-y)}\right)$$

$$= \frac{1}{2}\left(e^{x-y} + e^{-(x-y)}\right) = \cosh(x-y)$$

Exercise 2A

Prove the following identities.

1. (a) $\sinh(-x) = -\sinh x$ (b) $\cosh(-x) = \cosh x$

2. (a) $\sinh(x+y) = \sinh x \cosh y + \cosh x \sinh y$

   (b) $\sinh(x-y) = \sinh x \cosh y - \cosh x \sinh y$

3. $\cosh(x+y) = \cosh x \cosh y + \sinh x \sinh y$

4. $\sinh A + \sinh B = 2\sinh\left(\frac{A+B}{2}\right)\cosh\left(\frac{A-B}{2}\right)$

5. $\cosh A - \cosh B = 2\sinh\left(\frac{A+B}{2}\right)\sinh\left(\frac{A-B}{2}\right)$

2.2 Osborn's rule

You should have noticed from the previous exercise a similarity between the corresponding identities for trigonometric functions. In fact, trigonometric formulae can be converted into formulae for hyperbolic functions using **Osborn's rule**, which states that cos should be converted into cosh and sin into sinh, except when there is a product of two sines, when a sign change must be effected.

For example, $\cos 2x = 1 - 2\sin^2 x$

can be converted, remembering that $\sin^2 x = \sin x . \sin x$,

into $\cosh 2x = 1 + 2\sinh^2 x$.

But $\sin 2A = 2\sin A \cos A$

simply converts to $\sinh 2A = 2\sinh A \cosh A$ because there is no product of sines.

Activity 3

Given the following trigonometric formulae, use Osborn's rule to write down the corresponding hyperbolic function formulae.

(a) $\sin A - \sin B = 2\cos\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)$

(b) $\sin 3A = 3\sin A - 4\sin^3 A$

(c) $\cos^2\theta + \sin^2\theta = 1$

2.3 Further functions

Corresponding to the trigonometric functions $\tan x$, $\cot x$, $\sec x$ and $\operatorname{cosec} x$ we define

$$\tanh x = \frac{\sinh x}{\cosh x}, \quad \coth x = \frac{1}{\tanh x} = \frac{\cosh x}{\sinh x},$$

$$\operatorname{sech} x = \frac{1}{\cosh x} \quad \text{and} \quad \operatorname{cosech} x = \frac{1}{\sinh x}$$

By implication when using Osborn's rule, where the function $\tanh x$ occurs, it must be regarded as involving $\sinh x$.

Therefore, to convert the formula $\sec^2 x = 1 + \tan^2 x$

we must write

$$\operatorname{sech}^2 x = 1 - \tanh^2 x\ .$$

Activity 4

(a) Prove that

$$\tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}} \text{ and } \operatorname{sech} x = \frac{2}{e^x + e^{-x}}\ ,$$

and hence verify that

$$\operatorname{sech}^2 x = 1 - \tanh^2 x\ .$$

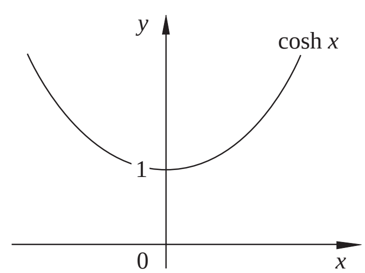
(b) Apply Osborn's rule to obtain a formula which corresponds to

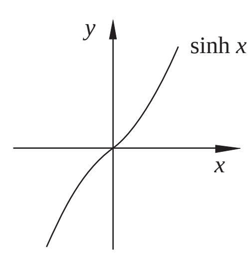
$$\operatorname{cosec}^2 y = 1 + \cot^2 y\ .$$

Prove the result by converting $\operatorname{cosech} y$ and $\coth y$ into exponential functions.

2.4 Graphs of hyperbolic functions

You could plot the graphs of $\cosh x$ and $\sinh x$ quite easily on a graphics calculator and obtain graphs as shown opposite.





The shape of the graph of $y = \cosh x$ is that of a particular chain supported at each end and hanging freely. It is often called a **catenary** (from the Latin word *catena* for chain or thread).

Activity 5

(a) Superimpose the graphs of $y = \cosh x$ and $y = \sinh x$ on the screen of a graphics calculator. Do the curves ever intersect?

(b) Use a graphics calculator to sketch the function $f : x \mapsto \tanh x$ with domain $x \in \mathbb{R}$. What is the range of the function?

(c) Try to predict what the graphs of

$$y = \text{sech}\, x,\ y = \text{cosech}\, x \text{ and } y = \coth x$$

will look like. Check your ideas by plotting the graphs on a graphics calculator.

2.5 Solving equations

Suppose $\sinh x = \frac{3}{4}$ and we wish to find the exact value of x.

Recall that $\cosh^2 x = 1 + \sinh^2 x$ and $\cosh x$ is always positive, so when $\sinh x = \frac{3}{4}$, $\cosh x = \frac{5}{4}$.

From Activity 1, we have $\sinh x + \cosh x = e^x$

so $\qquad e^x = \frac{3}{4} + \frac{5}{4} = 2$

and hence $\qquad x = \ln 2$.

Alternatively, we can write $\sinh x = \frac{1}{2}\left(e^x - e^{-x}\right)$

so $\sinh x = \frac{3}{4}$ means

$$\frac{1}{2}\left(e^x - e^{-x}\right) = \frac{3}{4}$$

$$\Rightarrow \qquad 2e^x - 3 - 2e^{-x} = 0$$

and multiplying by e^x

$$2e^{2x} - 3e^x - 2 = 0$$

$$\left(e^x - 2\right)\left(2e^x + 1\right) = 0$$

$$e^x = 2 \text{ or } e^x = -\frac{1}{2}$$

But e^x is always positive so $e^x = 2 \quad \Rightarrow \quad x = \ln 2$.

Activity 6

Find the values of x for which

$$\cosh x = \frac{13}{5}$$

expressing your answers as natural logarithms.

Example

Solve the equation

$$2\cosh 2x + 10\sinh 2x = 5$$

giving your answer in terms of a natural logarithm.

Solution

$$\cosh 2x = \frac{1}{2}\left(e^{2x} + e^{-2x}\right); \quad \sinh 2x = \frac{1}{2}\left(e^{2x} - e^{-2x}\right)$$

So $$e^{2x} + e^{-2x} + 5e^{2x} - 5e^{-2x} = 5$$

$$6e^{2x} - 5 - 4e^{-2x} = 0$$

$$6e^{4x} - 5e^{2x} - 4 = 0$$

$$\left(3e^{2x} - 4\right)\left(2e^{2x} + 1\right) = 0$$

$$e^{2x} = \frac{4}{3} \text{ or } e^{2x} = -\frac{1}{2}$$

The only real solution occurs when $e^{2x} > 0$

So $$2x = \ln\frac{4}{3} \quad \Rightarrow \quad x = \frac{1}{2}\ln\frac{4}{3}$$

Exercise 2B

1. Given that $\sinh x = \frac{5}{12}$, find the values of

   (a) $\cosh x$ (b) $\tanh x$ (c) $\operatorname{sech} x$

   (d) $\coth x$ (e) $\sinh 2x$ (f) $\cosh 2x$

   Determine the value of x as a natural logarithm.

2. Given that $\cosh x = \frac{5}{4}$, determine the values of

   (a) $\sinh x$ (b) $\cosh 2x$ (c) $\sinh 2x$

   Use the formula for $\cosh(2x + x)$ to determine the value of $\cosh 3x$.

3. In the case when $\tanh x = \frac{1}{2}$, show that $x = \frac{1}{2}\ln 3$.

4. Solve the following equations giving your answers in terms of natural logarithms.

   (a) $4\cosh x + \sinh x = 4$

   (b) $3\sinh x - \cosh x = 1$

   (c) $4\tanh x = 1 + \operatorname{sech} x$

5. Find the possible values of $\sinh x$ for which

   $$12\cosh^2 x + 7\sinh x = 24$$

   (You may find the identity $\cosh^2 x - \sinh^2 x = 1$ useful.)

   Hence find the possible values of x, leaving your answers as natural logarithms.

6. Solve the equations

   (a) $3\cosh 2x + 5\cosh x = 22$

   (b) $4\cosh 2x - 2\sinh x = 7$

7. Express $25\cosh x - 24\sinh x$ in the form $R\cosh(x-\alpha)$ giving the values of R and $\tanh\alpha$.

   Hence write down the minimum value of $25\cosh x - 24\sinh x$ and find the value of x at which this occurs, giving your answer in terms of a natural logarithm.

8. Determine a condition on A and B for which the equation

   $$A\cosh x + B\sinh x = 1$$

   has at least one real solution.

9. Given that a, b, c are all positive, show that when $a>b$ then $a\cosh x + b\sinh x$ can be written in the form $R\cosh(x+\alpha)$.

   Hence determine a further condition for which the equation

   $$a\cosh x + b\sinh x = c$$

   has real solutions.

10. Use an appropriate iterative method to find the solution of the equation

    $$\cosh x = 3x$$

    giving your answer correct to three significant figures.

2.6 Calculus of hyperbolic functions

Activity 7

(a) By writing $\cosh x = \frac{1}{2}\left(e^x + e^{-x}\right)$, prove that

$$\frac{d}{dx}(\cosh x) = \sinh x.$$

(b) Use a similar method to find $\frac{d}{dx}(\sinh x)$.

(c) Assuming the derivatives of $\sinh x$ and $\cosh x$, use the quotient rule to prove that if $y = \tanh x = \frac{\sinh x}{\cosh x}$

then $$\frac{dy}{dx} = \operatorname{sech}^2 x.$$

Note: care must be taken that Osborn's rule is **not** used to obtain corresponding results from trigonometry in calculus.

Activity 8

Use the quotient rule, or otherwise, to prove that

(a) $\dfrac{d}{dx}(\operatorname{sech} x) = -\operatorname{sech} x \ \tanh x$

(b) $\dfrac{d}{dx}(\operatorname{cosech} x) = -\operatorname{cosech} x \ \coth x$

(c) $\dfrac{d}{dx}(\coth x) = -\operatorname{cosech}^2 x$

Example

Integrate each of the following with respect to x.

(a) $\cosh 3x$ (b) $\sinh^2 x$

(c) $x \sinh x$ (d) $e^x \cosh x$

Solution

(a) $\int \cosh 3x \, dx = \frac{1}{3} \sinh 3x + \text{constant}$

(b) $\sinh^2 x \, dx$ can be found by using $\cosh 2x = 1 + 2\sinh^2 x$ giving

$$\frac{1}{2}\int (\cosh 2x - 1)\, dx$$

$$= \frac{1}{4} \sinh 2x - \frac{1}{2} x + \text{constant}$$

Alternatively, you could change to exponentials, giving

$$\sinh^2 x = \frac{1}{4}\left(e^{2x} - 2 + e^{-2x}\right)$$

$$\int \sinh^2 x \, dx = \frac{1}{8} e^{2x} - \frac{1}{2} x - \frac{1}{8} e^{-2x} + \text{constant}$$

Can you show this answer is identical to the one found earlier?

(c) Using integration by parts,

$$\int x \sinh x \, dx = x \cosh x - \int \cosh x \, dx$$

$$= x \cosh x - \sinh x + \text{constant}$$

(d) Certainly this is found most easily by converting to exponentials, giving

$$e^x \cosh x = \frac{1}{2}e^{2x} + \frac{1}{2}$$

$$\int e^x \cosh x\, dx = \frac{1}{4}e^{2x} + \frac{1}{2}x + \text{constant}$$

Exercise 2C

1. Differentiate with respect to x

   (a) $\tanh 4x$ (b) $\operatorname{sech} 2x$

   (c) $\operatorname{cosech}(5x+3)$ (d) $\sinh(e^x)$

   (e) $\cosh^3 2x$ (f) $\tanh(\sin x)$

   (g) $\cosh 5x \sinh 3x$ (h) $\sqrt{(\coth 4x)}$

2. Integrate each of the following with respect to x.

   (a) $\sinh 4x$ (b) $\cosh^2 3x$

   (c) $x^2 \cosh 2x$ (d) $\operatorname{sech}^2 7x$

   (e) $\operatorname{cosech} 2x \coth 3x$ (f) $\tanh x$

   (g) $\tanh^2 x$ (h) $e^2 \sinh 3x$

   (i) $x^2 \cosh(x^3+4)$ (j) $\sinh^4 x$

   (k) $\cosh 2x \sinh 3x$ (l) $\operatorname{sech} x$

3. Find the equation of the tangent to the curve with equation

   $y = 3\cosh 2x - \sinh x$

   at the point where $x = \ln 2$.

4. A curve has equation

   $y = \lambda \cosh x + \sinh x$

   where λ is a constant.

   (a) Sketch the curve for the cases $\lambda = 0$ and $\lambda = 1$.

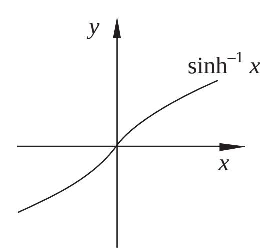
   (b) Determine the coordinates of the turning point of the curve in the case when $\lambda = \frac{4}{3}$. Is this a maximum or minimum point?

   (c) Determine the range of values of λ for which the curve has no real turning points.

5. Find the area of the region bounded by the coordinate axes, the line $x = \ln 3$ and the curve with equation $y = \cosh 2x + \operatorname{sech}^2 x$.

2.7 Inverse hyperbolic functions

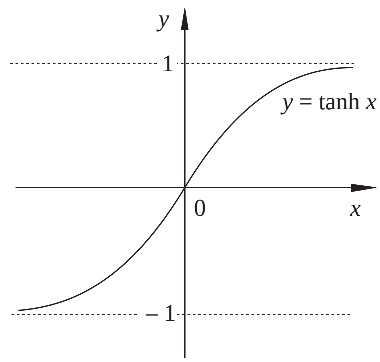
The function $f : x \mapsto \sinh x \ (x \in \mathbb{R})$ is one-one, as can be seen from the graph in Section 2.4. This means that the inverse function f^{-1} exists. The **inverse hyperbolic sine function** is denoted by $\sinh^{-1} x$. Its graph is obtained by reflecting the graph of $\sinh x$ in the line $y = x$.



Recall that $\frac{d}{dx}(\sinh x) = \cosh x$, so the gradient of the graph of $y = \sinh x$ is equal to 1 at the origin. Similarly, the graph of $\sinh^{-1} x$ has gradient 1 at the origin.

Similarly, the function $g: x \mapsto \tanh x \ (x \in \mathbb{R})$ is one-one.

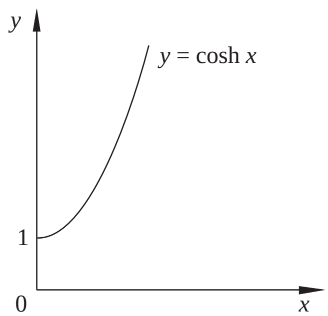
You should have obtained its graph in Activity 5. The range of g is $\{y: -1 < y < 1\}$ or the open interval $(-1, 1)$.



Activity 9

Sketch the graph of the inverse tanh function, $\tanh^{-1} x$.

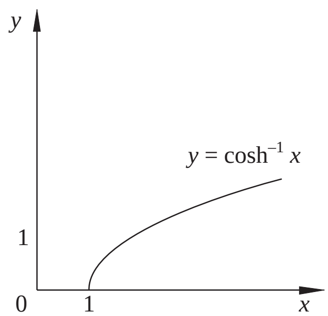
Its range is now $\mathbb{R}$. What is its domain?



The function $h: x \mapsto \cosh x \ (x \in \mathbb{R})$ is **not** a one-one function and so we cannot define an inverse function. However, if we change the domain to give the function

$$f: x \mapsto \cosh x \text{ with domain } \{x: x \in \mathbb{R}, x \geq 0\}$$

then we do have a one-one function, as illustrated.



So, provided we consider $\cosh x$ for $x \geq 0$, we can define the inverse function $f^{-1}: x \mapsto \cosh^{-1} x$ with domain $\{x: x \in \mathbb{R}, \ x \geq 1\}$.

This is called the **principal value** of $\cosh^{-1} x$.

2.8 Logarithmic equivalents

Activity 10

Let $y = \sinh^{-1} x$ so $\sinh y = x$.

Since $\cosh y$ is always positive, show that $\cosh y = \sqrt{(1 + x^2)}$

By considering $\sinh y + \cosh y$, find an expression for e^y in terms of x.

Hence show that $\sinh^{-1} x = \ln\left[x + \sqrt{(1 + x^2)}\right]$

Activity 11

Let $y = \tanh^{-1} x$ so $\tanh y = x$.

Express $\tanh y$ in the terms of e^y and hence show that

$e^{2y} = \dfrac{1+x}{1-x}$.

Deduce that $\tanh^{-1} x = \dfrac{1}{2} \ln\left(\dfrac{1+x}{1-x}\right)$.

(Do not forget that $\tanh^{-1} x$ is only defined for $|x| < 1$.)

Activity 12

Let $y = \cosh^{-1} x$, where $x \geq 1$, so $\cosh y = x$.

Use the graph in Section 2.7 to explain why y is positive and hence why $\sinh y$ is positive. Show that $\sinh y = \sqrt{(x^2 - 1)}$.

Hence show that

$$\cosh^{-1} x = \ln\left[x + \sqrt{(x^2-1)}\right].$$

The full results are summarised below.

$\sinh^{-1} x = \ln\left\{x + \sqrt{(x^2+1)}\right\}$	(all values of x)
$\tanh^{-1} x = \dfrac{1}{2}\ln\left(\dfrac{1+x}{1-x}\right)$	($\lvert x \rvert < 1$)
$\cosh^{-1} x = \ln\left\{x + \sqrt{(x^2-1)}\right\}$	($x \geq 1$)

Exercise 2D

1. Express each of the following in logarithmic form.

   (a) $\sinh^{-1}\left(\frac{3}{4}\right)$ (b) $\cosh^{-1}2$ (c) $\tanh^{-1}\left(\frac{1}{2}\right)$

2. Given that $y=\sinh^{-1}x$, find a quadratic equation satisfied by e^y. Hence obtain the logarithmic form of $\sinh^{-1}x$. Explain why you discard one of the solutions.

3. Express $\operatorname{sech}^{-1}x$ in logarithmic form for $0<x<1$.

4. Find the value of x for which

   $\sinh^{-1}x+\cosh^{-1}(x+2)=0$.

5. Solve the equation $2\tanh^{-1}\left(\frac{x-2}{x+1}\right)=\ln 2$.

2.9 Derivatives of inverse hyperbolic functions

Let $y=\sinh^{-1}x$ so that $x=\sinh y$.

$$\frac{dx}{dy}=\cosh y \quad \text{but} \quad \cosh^2 y=\sinh^2 y+1=x^2+1$$

and $\cosh y$ is always positive.

So $\frac{dx}{dy}=\sqrt{\left(x^2+1\right)}$ and therefore $\frac{dy}{dx}=\frac{1}{\sqrt{\left(x^2+1\right)}}$

In other words, $\frac{d}{dx}\left(\sinh^{-1}x\right)=\frac{1}{\sqrt{\left(x^2+1\right)}}$

Activity 13

(a) Show that $\frac{d}{dx}\left(\cosh^{-1}x\right)=\frac{1}{\sqrt{\left(x^2-1\right)}}$

(b) Show that $\frac{d}{dx}\left(\tanh^{-1}x\right)=\frac{1}{1-x^2}$

An alternative way of showing that $\frac{d}{dx}\left(\sinh^{-1}x\right)=\frac{1}{\sqrt{\left(x^2+1\right)}}$ is to use the logarithmic equivalents.

Since

$$\frac{d}{dx}\left[x+\sqrt{(x^2+1)}\right]=1+\frac{1}{2}.2x(x^2+1)^{-\frac{1}{2}}$$

$$=1+\frac{x}{\sqrt{(x^2+1)}} \quad = \frac{\sqrt{(x^2+1)}+x}{\sqrt{(x^2+1)}}$$

we can now find the derivative of $\ln\left[x+\sqrt{(x^2+1)}\right]$

$$\frac{d}{dx}\left[\ln\left\{x+\sqrt{(x^2+1)}\right\}\right]=\frac{\sqrt{(x^2+1)}+x}{\sqrt{(x^2+1)}}.\frac{1}{x+\sqrt{(x^2+1)}}$$

which cancels down to

$$\frac{1}{\sqrt{(x^2+1)}}$$

So
$$\boxed{\frac{d}{dx}\left(\sinh^{-1}x\right)=\frac{1}{\sqrt{(x^2+1)}}}$$

You can use a similar approach to find the derivatives of $\cosh^{-1}x$ and $\tanh^{-1}x$ but the algebra is a little messy.

Example

Differentiate

(a) $\cosh^{-1}(2x+1)$ (b) $\sinh^{-1}\left(\frac{1}{x}\right)$ with respect to x $(x>0)$.

Solution

(a) Use the function of a function or chain rule.

$$\frac{d}{dx}\left[\cosh^{-1}(2x+1)\right]=2.\frac{1}{\sqrt{\left\{(2x+1)^2-1\right\}}}=\frac{2}{\sqrt{(4x^2+4x)}}=\frac{1}{\sqrt{(x^2+x)}}$$

(b) $$\frac{d}{dx}\left[\sinh^{-1}\left(\frac{1}{x}\right)\right]=\frac{-1}{x^2}.\frac{1}{\sqrt{\left\{\frac{1}{x^2}+1\right\}}}=-\frac{1}{x^2}.\frac{x}{\sqrt{(1+x^2)}}$$

$$=\frac{-1}{x\sqrt{(1+x^2)}}$$

Exercise 2E

Differentiate each of the expressions in Questions 1 to 6 with respect to x.

1. $\cosh^{-1}(4+3x)$
2. $\sinh^{-1}\left(\sqrt{x}\right)$
3. $\tanh^{-1}(3x+1)$
4. $x^2\sinh^{-1}(2x)$
5. $\cosh^{-1}\left(\frac{1}{x}\right)\ (x>0)$
6. $\sinh^{-1}(\cosh 2x)$
7. Differentiate $\operatorname{sech}^{-1}x$ with respect to x, by first writing $x=\operatorname{sech}y$.
8. Find an expression for the derivative of $\operatorname{cosech}^{-1}x$ in terms of x.
9. Prove that

$$\frac{d}{dx}\left(\coth^{-1}x\right)=\frac{-1}{\left(x^2-1\right)}.$$

2.10 Use of hyperbolic functions in integration

Activity 14

Use the results from Section 2.9 to write down the values of

(a) $\displaystyle\int\frac{1}{\sqrt{\left(x^2+1\right)}}dx$ and (b) $\displaystyle\int\frac{1}{\sqrt{\left(x^2-1\right)}}dx$

Activity 15

Differentiate $\sinh^{-1}\left(\frac{x}{3}\right)$ with respect to x.

Hence find $\displaystyle\int\frac{1}{\sqrt{\left(x^2+9\right)}}dx$.

What do you think $\displaystyle\int\frac{1}{\sqrt{\left(x^2+49\right)}}dx$ is equal to?

Activity 16

Use the substitution $x=2\cosh u$ to show that

$$\int\frac{1}{\sqrt{\left(x^2-4\right)}}dx=\cosh^{-1}\left(\frac{x}{2}\right)+\text{constant}$$

Activity 17

Prove, by using suitable substitutions that, where a is a constant,

(a) $$\int \frac{1}{\sqrt{\left(x^2+a^2\right)}}\,dx = \sinh^{-1}\left(\frac{x}{a}\right) + \text{constant}$$

(b) $$\int \frac{1}{\sqrt{\left(x^2-a^2\right)}}\,dx = \cosh^{-1}\left(\frac{x}{a}\right) + \text{constant}$$

Integrals of this type are found by means of a substitution involving hyperbolic functions. They may be a little more complicated than the ones above and it is sometimes necessary to complete the square.

Activity 18

Express $4x^2 - 8x - 5$ in the form $A(x-B)^2 + C,$ where A, B and C are constants.

Example

Evaluate in terms of natural logarithms

$$\int_4^7 \frac{1}{\sqrt{\left(4x^2-8x-5\right)}}\,dx$$

Solution

From Activity 18, the integral can be written as

$$\int_4^7 \frac{1}{\sqrt{\left\{4(x-1)^2-9\right\}}}\,dx$$

You need to make use of the identity $\cosh^2 A - 1 = \sinh^2 A$ because of the appearance of the denominator.

Substitute $\quad 4(x-1)^2 = 9\cosh^2 u \quad$ in order to accomplish this.

So $\quad 2(x-1) = 3\cosh u$

and $\quad 2\dfrac{dx}{du} = 3\sinh x$

The denominator then becomes

$$\sqrt{\{9\cosh^2 u - 9\}} = \sqrt{(9\sinh^2 u)} = 3\sinh u$$

In order to deal with the limits, note that when

$x = 4, \quad \cosh u = 2 \quad \left(\text{so } u = \ln\left[2 + \sqrt{3}\right]\right)$ and when

$x = 7, \quad \cosh u = 4 \quad \left(\text{so } u = \ln\left[4 + \sqrt{15}\right]\right)$

The integral then becomes

$$\int_{\cosh^{-1} 2}^{\cosh^{-1} 4} \frac{\frac{3}{2}\sinh u\, du}{3\sinh u} = \int_{\cosh^{-1} 2}^{\cosh^{-1} 4} \frac{1}{2}\, du$$

$$= \frac{1}{2}\left[\cosh^{-1} 4 - \cosh^{-1} 2\right] = \frac{1}{2}\left\{\ln\left(2 + \sqrt{3}\right) - \ln\left(4 + \sqrt{15}\right)\right\}$$

Example

Evaluate

$$\int_{-3}^{1} \sqrt{\left(x^2 + 6x + 13\right)}\, dx$$

leaving your answer in terms of natural logarithms.

Solution

Completing the square, $x^2 + 6x + 13 = (x + 3)^2 + 4$

You will need the identity $\sinh^2 A + 1 = \cosh^2 A$ this time because of the + sign after completing the square.

Now make the substitution $x + 3 = 2\sinh\theta$

giving $\dfrac{dx}{d\theta} = 2\cosh\theta$

When $x = -3, \quad \sinh\theta = 0 \quad \Rightarrow \quad \theta = 0$

When $x = 1, \quad \sinh\theta = 2 \quad \Rightarrow \quad \theta = \sinh^{-1} 2 = \ln\left(2 + \sqrt{5}\right)$

The integral transforms to

$$\int_{0}^{\sinh^{-1} 2} \sqrt{\left(4\sinh^2\theta + 4\right)}.2\cosh\theta\, d\theta$$

$$= \int_{0}^{\sinh^{-1} 2} 4\cosh^2\theta\, d\theta$$

You can either convert this into exponentials or you can use the identity

$$\cosh 2A = 2\cosh^2 A - 1$$

giving $$\int_0^{\sinh^{-1} 2} (2 + 2\cosh 2\theta)\,d\theta$$

$$= [2\theta + \sinh 2\theta]_0^{\sinh^{-1} 2}$$

$$= 2[\theta + \sinh\theta\cosh\theta]_0^{\sinh^{-1} 2}$$

$$= 2\left(\sinh^{-1} 2 + 2\sqrt{1 + 2^2}\right)$$

$$= 2\ln\left(2 + \sqrt{5}\right) + 4\sqrt{5}$$

Activity 19

Show that

$$\frac{d}{d\phi}(\sec\phi\tan\phi) = 2\sec^3\phi - \sec\phi$$

and deduce that

$$\int \sec^3\phi\, d\phi = \frac{1}{2}\left\{\sec\phi\tan\phi + \ln(\sec\phi + \tan\phi)\right\}$$

Hence use the substitution

$$x + 3 = 2\tan\phi$$

in the integral of the previous example and verify that you obtain the same answer.

Exercise 2F

Evaluate the integrals in Questions 1 to 12.

1. $\int_0^2 \frac{1}{\sqrt{(x^2+1)}}\,dx$

2. $\int_3^4 \frac{1}{\sqrt{(x^2-4)}}\,dx$

3. $\int_0^1 \frac{1}{\sqrt{(x^2+2x+5)}}\,dx$

4. $\int_{-1}^1 \frac{1}{\sqrt{(x^2+6x+8)}}\,dx$

5. $\int_0^2 \sqrt{(4+x^2)}\,dx$

6. $\int_3^4 \sqrt{(x^2-9)}\,dx$

7. $\int_1^2 \sqrt{(x^2+2x+2)}\,dx$

8. $\int_1^2 \frac{1}{\sqrt{(x^2+2x)}}\,dx$

9. $\int_4^5 \frac{x+1}{\sqrt{(x^2-9)}}\,dx$

10. $\int_1^2 \frac{1}{\sqrt{(x^2+x)}}\,dx$

11. $\int_{-1}^0 \frac{1}{\sqrt{(2x^2+4x+7)}}\,dx$

12. $\int_4^5 \sqrt{(3x^2-12x+8)}\,dx$

13. Use the substitution $u = e^x$ to evaluate

$$\int_0^1 \operatorname{sech} x\,dx.$$

14. (a) Differentiate $\sinh^{-1} x$ with respect to x.
By writing $\sinh^{-1} x$ as $1 \times \sinh^{-1} x$, use integration by parts to find $\int_1^2 \sinh^{-1} dx$.

(b) Use integration by parts to evaluate

$$\int_1^2 \cosh^{-1} x\,dx.$$

15. Evaluate each of the following integrals.

(a) $\int_2^3 \frac{1}{\sqrt{(x^2+6x-7)}}\,dx$ (b) $\int_2^3 \sqrt{(x^2+6x-7)}\,dx$

16. Transform the integral

$$\int_1^2 \frac{1}{x\sqrt{4+x^2}}\,dx$$

by means of the substitution $x = \frac{1}{u}$. Hence, by means of a further substitution, or otherwise, evaluate the integral.

17. The point P has coordinates $(a\cosh t,\ b\sinh t)$. Show that P lies on the hyperbola with equation

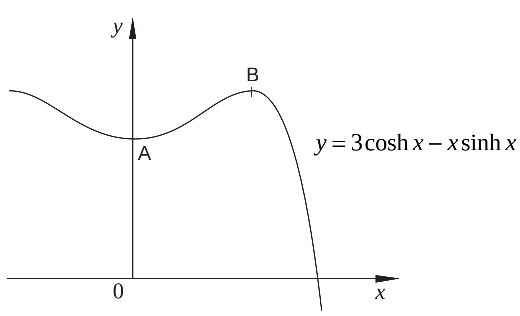
$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1.$$

Which branch does it lie on when $a>0$?

Given that O is the origin, and A is the point $(a,\ 0)$, prove that the region bounded by the lines OA and OP and the arc AP of the hyperbola has area $\frac{1}{2}abt$.

2.11 Miscellaneous Exercises

1. The sketch below shows the curve with equation $y = 3\cosh x - x\sinh x$, which cuts the y-axis at the point A. Prove that, at A, y takes a minimum value and state this value.



Given that $\frac{dy}{dx} = 0$ at B, show that the x-coordinate of B is the positive root of the equation

$$x\cosh x - 2\sinh x = 0.$$

Show that this root lies between 1.8 and 2.

Find, by integration, the area of the finite region bounded by the coordinate axes, the curve with equation $y = 3\cosh x - x\sinh x$ and the line $x = 2$, giving your answer in terms of e. (AEB)

2. Starting from the definition

$$\cosh\theta = \frac{1}{2}\left(e^{\theta}+e^{-\theta}\right) \text{ and } \sinh\theta = \frac{1}{2}\left(e^{\theta}-e^{-\theta}\right)$$

show that

$$\sinh(A+B) = \sinh A\cosh B + \cosh A\sinh B.$$

There exist real numbers r and α such that

$$5\cosh x + 13\sinh x \equiv r\sinh(x+\alpha).$$

Find r and show that $\alpha = \ln\frac{3}{2}$.

Hence, or otherwise,

(a) solve the equation $5\cosh x + 13\sinh x = 12\sinh 2$

(b) show that

$$\int_1^1 \frac{dx}{5\cosh x + 13\sinh x} = \frac{1}{12}\ln\left(\frac{15e-10}{3e+2}\right)$$

(AEB)

3. Given that $|x|<1$, prove that $\dfrac{d}{dx}\left(\tanh^{-1}x\right) = \dfrac{1}{1-x^2}$.

Show by integrating the result above that

$$\tanh^{-1}x = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$$

Use integration by parts to find $\int \tanh^{-1}x\,dx$.

4. Given that $t \equiv \tanh\frac{1}{2}x$, prove the identities

(a) $\sinh x = \dfrac{2t}{1-t^2}$ (b) $\cosh x = \dfrac{1+t^2}{1-t^2}$

Hence, or otherwise, solve the equation

$$2\sinh x - \cosh x = 2\tanh\frac{1}{2}x$$

5. Solve the equations

(a) $\cosh(\ln x) - \sinh\left(\ln\frac{1}{2}x\right) = 1\frac{3}{4}$

(b) $4\sinh x + 3e^x + 3 = 0$

6. Show that $\cosh x + \sinh x = e^x$

Deduce that $\cosh nx + \sinh nx = \sum_{k=0}^{n}\binom{n}{k}\cosh^{n-k}x\sinh^k x$

Obtain a similar expression for

$\cosh nx - \sinh nx$.

Hence prove that

$$\cosh 7x = 64\cosh^7 x - 112\cosh^5 x + 56\cosh^3 x - 7\cosh x$$

7. Define $\operatorname{cosech} x$ and $\coth x$ in terms of exponential functions and from your definitions prove that

$$\coth^2 x \equiv 1 + \operatorname{cosech}^2 x.$$

Solve the equation

$$3\coth^2 x + 4\operatorname{cosech} x = 23$$

8. Solve the equation

$$3\operatorname{sech}^2 x + 4\tanh x + 1 = 0$$ (AEB)

9. Find the area of the region R bounded by the curve with equation $y = \cosh x$, the line $x = \ln 2$ and the coordinate axes. Find also the volume obtained when R is rotated completely about the x-axis. (AEB)

10. Prove that

$$\sinh^{-1}x = \ln\left[x + \sqrt{\left(1+x^2\right)}\right]$$

and write down a similar expression for $\cosh^{-1}x$.

Given that

$$2\cosh y - 7\sinh x = 3 \text{ and } \cosh y - 3\sinh^2 x = 2,$$

find the real values of x and y in logarithmic form. (AEB)

11. Evaluate the following integrals

(a) $\displaystyle\int_1^3 \frac{1}{\sqrt{\left(x^2+6x+5\right)}}\,dx$

(b) $\displaystyle\int_1^3 \sqrt{\left(x^2+6x+5\right)}\,dx$

12. Use the definitions in terms of exponential functions to prove that

(a) $\dfrac{1-\tanh^2 x}{1+\tanh^2 x} = \operatorname{sech} 2x$

(b) $\dfrac{d}{dx}(\tanh x) = 1-\tanh^2 x$

Hence, use the substitution $t = \tanh x$ to find

$$\int \operatorname{sech} 2x\,dx$$ (AEB)

13. Sketch the graph of the curve with equation $y = \tanh x$ and state the equations of its asymptotes.

Use your sketch to show that the equation $\tanh x = 10 - 3x$ has just one root α. Show that α lies between 3 and $3\frac{1}{3}$.

Taking 3 as a first approximation for α, use the Newton-Raphson method once to obtain a second approximation, giving your answer to four decimal places.

14. Prove that $\sinh^{-1} x = \ln\left[x + \sqrt{(1+x^2)}\right]$.

(a) Given that $\exp(z) \equiv e^z$, show that $y = \exp(\sinh^{-1} x)$ satisfies the differential equation

$$(1+x^2)\frac{d^2y}{dx^2} + x\frac{dy}{dx} - y = 0$$

(b) Find the value of $\int_0^1 \sinh^{-1} x\, dx$, leaving your answer in terms of a natural logarithm.

15. Sketch the graph of $y = \tanh^{-1} x$.
Determine the value of x, in terms of e, for which $\tanh^{-1} x = \frac{1}{2}$.

The point P is on the curve $y = \tanh^{-1} x$ where $y = \frac{1}{2}$. Find the equation of the tangent to the curve at P. Determine where the tangent to the curve crosses the y-axis.

16. Evaluate the following integrals, giving your answers as multiples of π or in logarithmic form.

(a) $\displaystyle\int_0^2 \frac{dx}{\sqrt{(3x^2 - 6x + 4)}}$ (b) $\displaystyle\int_0^2 \frac{dx}{\sqrt{(1 + 6x - 3x^2)}}$

17. Find the value of x for which

$$\sinh^{-1}\frac{3}{4} + \sinh^{-1} x = \sinh^{-1}\frac{4}{3}$$

18. Starting from the definitions of hyperbolic functions in terms of exponential functions, show that

$$\cosh(x - y) = \cosh x \cosh y - \sinh x \sinh y$$

and that

$$\tanh^{-1} = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right) \text{ where } -1 < x < 1.$$

(a) Find the values of R and α such that

$$5\cosh x - 4\sinh x \equiv R\cosh(x - \alpha)$$

Hence write down the coordinates of the minimum point on the curve with equation

$$y = 5\cosh x - 4\sinh x$$

(b) Solve the equation

$$9\,\text{sech}^2 y - 3\tanh y = 7,$$

leaving your answer in terms of natural logarithms. (AEB)

19. Solve the equation $3\,\text{sech}^2 x + 4\tanh x + 1 = 0$. (AEB)

20. Define $\sinh y$ and $\cosh y$ in terms of exponential functions and show that

$$2y = \ln\left\{\frac{\cosh y + \sinh y}{\cosh y - \sinh y}\right\}$$

By putting $\tanh y = \frac{1}{3}$, deduce that

$$\tanh^{-1}\left(\frac{1}{3}\right) = \frac{1}{2}\ln 2$$ (AEB)

21. Show that the function $f(x) = (1+x)\sinh(3x-2)$ has a stationary value which occurs at the intersection of the curve $y = \tanh(3x-2)$ and the straight line $y + 3x + 3 = 0$. Show that the stationary value occurs between –1 and 0. Use Newton-Raphson's method once, using an initial value of –1 to obtain an improved estimate of the x-value of the stationary point.

22. (a) Given that $\tanh x = \frac{\sinh x}{\cosh x}$, express the value of $\tanh x$ in terms of e^x and e^{-x}.

(b) Given that $\tanh y = t$, show that $y = \frac{1}{2}\ln\left(\frac{1+t}{1-t}\right)$ for $-1 < t < 1$.

(c) Given that $y = \tanh^{-1}(\sin x)$ show that $\frac{dy}{dx} = \sec x$ and hence show that

$$\int_0^{\frac{\pi}{6}} \sec^2 x \tanh^{-1}(\sin x)\,dx = \frac{1}{\sqrt{3}}\left[\sqrt{3} - 2 + \frac{1}{2}\ln 3\right]$$

(AEB)

23. Solve the simultaneous equations

$$\cosh x - 3\sinh y = 0$$
$$2\sinh x + 6\cosh y = 5$$

giving your answers in logarithmic form. (AEB)

24. Given that $x = \cosh y$, show that the value of x is either

$$\ln\left[x + \sqrt{(x^2 - 1)}\right] \text{ or } \ln\left[x - \sqrt{(x^2 - 1)}\right]$$

Solve the equations

(a) $\sinh^2\theta - 5\cosh\theta + 7 = 0$

(b) $\cosh(z + \ln 3) = 2$

25. Sketch the curve with equation $y = \operatorname{sech} x$ and determine the coordinates of its point of inflection. The region bounded by the curve, the coordinate axes and the line $x = \ln 3$ is R.

    Calculate

    (a) the area of R

    (b) the volume generated when R is rotated through 2π radians about the x-axis.

26. Solve the equation $\tanh x + 4\operatorname{sech} x = 4$.

27. Prove the identity

    $$\cosh^2 x \cos^2 x - \sinh^2 x \sin^2 x \equiv \frac{1}{2}(1 + \cosh 2x \cos 2x)$$

28. Prove that

    $$16 \sinh^2 x \cosh^3 x \equiv \cosh 5x + \cosh 3x - 2\cosh x$$

    Hence, or otherwise, evaluate

    $$\int_0^1 16 \sinh^2 x \cosh^3 x \, dx,$$

    giving your answer in terms of e.

29. Show that the minimum value of $\sinh x + n \cosh x$ is $\sqrt{(n^2 - 1)}$ and that this occurs when

    $$x = \frac{1}{2} \ln\left(\frac{n-1}{n+1}\right)$$

    Show also, by obtaining a quadratic equation in e^x, that if $k > \sqrt{(n^2 - 1)}$ then the equation $\sinh x + n \cosh x = k$ has two real roots, giving your answers in terms of natural logarithms.

    (AEB)