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Exercise Serie 02

Exercise 01.

1) Let $X_1, X_2, ..., X_n, \ ...$ be an independent discrete random variables with values in $\mathbb{N}$. Show that $S_n = \sum_{i=1}^{n} X_i$ is a Markov process.

2) Let $\{X(t), \ t \geq 0\}$ be an independent increment process with discrete value space E. Show that $\{X(t), \ t \geq 0\}$ is a Markov process.

3) Let $\{X(t), \ t \geq 0\}$ be a Markov process with the transition matrix $\mathbf{P} = [p_{ij}(s,t)]_{i,j=1,..,m}, t > s$. Show that $\sum_{j=1}^{m} p_{ij}(s,t) = 1$. Demonstrate the Chapman-Kolmogorov equation

$$p_{ij}(s,t) = \sum_{k=1}^{m} p_{ik}(s,u)\, p_{kj}(u,t), \ t > u > s > 0.$$

Exercise 02.

Suppose that a point makes a random walk on the line and that it can only stop at the points of coordinates $1, 2, 3, ...m$.

In addition we assume that from state i the point can only move to state $i+1$ or to state $i-1$ with the probabilities

$$\begin{aligned} p_{i,i+1} &= P(X_{k+1} = i+1 \,| X_k = i) = p, \\ p_{i,i-1} &= P(X_{k+1} = i-1 \mid X_k = i) = q = 1-p. \end{aligned}$$

if $i \neq 1$ and $i \neq m$.

For $i = 1$ or $i = m$ we have the absorbing states

$$\begin{aligned} p_{1,1} &= P(X_{k+1} = 1 \mid X_k = 1) = 1, \\ p_{m,m} &= P(X_{k+1} = m \mid X_k = m) = 1. \end{aligned}$$

In this case determine the state space as well as the transition matrix of this Markov chain.

Exercise 03.

Let $(X(t))_{t \geq 0}$ be a birth process, is a Markov process with state space $\zeta = \{0, 1, ...,\}$, where $X(0) = 0$, and we have $\lambda_{kj} = 0$ $(j < k$ or $j > k+1)$.

1) Write the Kolmogorov's equations.

2) Find a recurrence relation between $P_n(t) = P(X(t) = n)$ et $P_{n-1}(t)$.

Exercise 4.

We assume that the lifetime X of a product is a continuous random variable with distribution function F and density f. We assume that after failure the product is repaired and the lifetime after the repair has the same distribution. We therefore observe a sequence of random variables i.i.d $X_1, X_2, ..., X_n, ...$

Denote by

$$S_n = \sum_{k=1}^{n} X_k,$$

the time of the $n-th$ failure and by $N(t)$ the number of failures in the interval $[0,t]$, F_n and f_n are respectively the distribution function and the density of S_n.

Show that

(a)

$$P\{N(l) = n\} = F_n(l) - F_{n+1}(l), \qquad (n = 0, 1, ...; F_0 \equiv 0)$$

(b)

$$H(t) = EN(t) = \sum_{n=1}^{+\infty} F_n(t)$$

(c) Let $W_t = S_{N(t)+1} - t$ denote the time between time t and the time of the first failure after t. Show that if the functions $f_n(\cdot)$ and the sum $\sum_{n=1}^{\infty} f_n(\cdot)$ are continuous on $[0, +\infty)$, then the distribution function of W_t is given by the formula

2) Show that the process $Y(t) = X(t+1) - X(t)$, for $t \geq 0$, is stationary.

2) Consider the telegraph signal process $Y(t) = Z(-1)^{X(t)}$, for $t \geq 0$. Show that $Y(\cdot)$ is a stationary process.

Exercise 6.

Let N^i, $i = 1; 2$ two independent Poisson processes with the same intensity. Show that $N^1 - N^2$ is a martingale.

Exercise 7. Let $N(\cdot)$ be a Poisson process

1. Calculate $\psi(z,t) = \sum_n z^n P(N_t = n)$.

2. Let X be a process with independent increments, with values in the set $\mathbb{N}$, such that $X_{t+s} - X_t \overset{loi}{=} X_s$ and $\psi(z,t) = \sum_n z^n P(X_t = n)$. Show that $\psi(z, t+h) = \psi(z,t)\psi(z,h)$. We assume that there exists ν such that

$$\frac{1}{h}P(X_h \geq 2) \to 0;\ \frac{1}{h}P(X_h = 1) \to \nu;\ \frac{1}{h}(1 - P(X_h = 0)) \to \nu.$$

as h goes to 0. Show that $\frac{d}{dt}\psi(z,t) = \nu(z-1)\psi(z,t)$. What is the process X?

Exercise 8.

Let T_k denote the moment of the k-th jump of the Poisson process $X(t), k = 1, 2, ...$ Let $\{\tau_k\}$ be the sequence of random variables: $\tau_k = T_k - T_{k-1}$, $T_0 = 0$.

Show that, for all finite n the random variables $\tau_1, \tau_2, ..., \tau_n$ are independent and follow the same exponential law with parameter λ.

Exercise 9.

Let $\{N_t, t \geq 0\}$ be the arrival process associated with the interarrival times X_n. That is

$$N_t = \sum_{n=1}^{\infty} n\mathbf{1}_{\{T_n \leq t < T_{n+1}\}}$$

Show that the stochastic process $\{N_t, t \geq 0\}$ is a Poisson process with parameter $\lambda > 0$.