Chapter I: ELECTROSTATICS

I-1 Mathematical Reminder

Vector analysis is a branch of mathematics that studies scalar and vector fields. Meaning of vector analysis arises from its intensive(severe) use in physics and engineering sciences.

Here, a vector field associates a vector with each point in space, while a scalar field associates a real number. The gradient, divergence, and curl are the three main linear first-order differential operators. This means that they only include partial derivatives of fields, however the Laplacian for second-order; they are met in electrostatics and magnetism.

The nabla operator is defined in Cartesian coordinates as follows: $\nabla=(\partial/\partial x, \partial/\partial y, \partial/\partial z)$

Written as ∇, the nabla operator has the characteristics of a vector. While it does not contain scalar values, its components are used exactly as using scalar values composing a vector.

Nabla notation provides a convenient way to express vector operators in Cartesian coordinates.

I-1-1 Gradient operator

Gradient is an operator that applies to a scalar field and describes a vector field representing the variation of the scalar field value in space. In Cartesian coordinates, the gradient is given by:

$$\overrightarrow{grad}f = \vec{\nabla}f = \frac{\partial f}{\partial x}\vec{i} + \frac{\partial f}{\partial y}\vec{j} + \frac{\partial f}{\partial z}\vec{k} \text{ where } \vec{\nabla}f \begin{pmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \\ \frac{\partial f}{\partial z} \end{pmatrix}$$

Applied to a scalar field, the nabla operator yields the gradient of the field results a vector field.

I-1-2 Divergence operator

The dot product between the nabla operator and a vector field UU (defined by its three components) gives the divergence of this vector field. The resulting divergence is a scalar field.

$$div\vec{F} = \vec{\nabla}.\vec{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z} \text{ where } \vec{F}(F_x, F_y, F_z)$$

$\vec{F}$ indicates the vector field to which the divergence operator is applied. Formally, the divergence can be seen as the dot product of the nabla operator with vector of the field to which it is applied.

I-1-3 Curl (Rotational) operator

The curl transforms a vector field into another vector field. More interesting to visualize as precisely as gradient and divergence, it states the tendency of a field to rotate around a point.

In Cartesian coordinates, the curl can be defined for $\vec{F}$ by the relation:

$$\overrightarrow{rot\vec{F}} = \vec{\nabla}x\vec{F} = \begin{bmatrix} \vec{\imath} & -\vec{\jmath} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{bmatrix} = (\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z})\vec{\imath} - (\frac{\partial F_z}{\partial x} - \frac{\partial F_x}{\partial z})\vec{\jmath} + (\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y})\vec{k} = \begin{pmatrix} \frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z} \\ \frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x} \\ \frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \end{pmatrix} \begin{matrix} \vec{\imath} \\ \vec{\jmath} \\ \vec{k} \end{matrix}$$

$\vec{F}$ designates the vector field to which the curl operator is applied. The formal analogy with the cross product justifies the notation: $\vec{\nabla}x\vec{F}$; also can be written, using a determinant: $\overrightarrow{rot\vec{F}}$.

I-1-4 Length and Volume Elements in Different Coordinate Systems

The length element is given in Cartesian coordinate systems by:

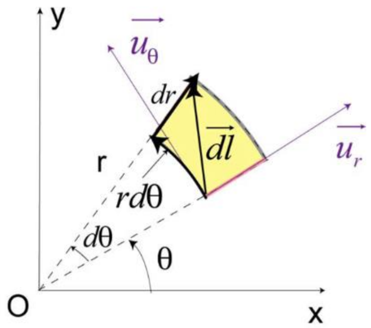
$\vec{dl} = dx.\vec{\imath} + dy.\vec{\jmath}$

In polar coordinates, the length element becomes:

$\vec{dl} = dr.\vec{u}_r + r.d\theta.\vec{u}_\theta$

Or, in the polar basis $(\vec{e}_r, \vec{e}_\theta)$ for polar coordinates

of point M (r, θ): $\vec{dl} = dr.\vec{e}_r + r.d\theta.\vec{e}_\theta$



The position vector of point M in the polar basis is: $\overrightarrow{OM} = r.\vec{u}_r$

The cylindrical coordinates of point M (ρ, θ, z) in the polar basis $(\vec{u}_\rho, \vec{u}_\theta, \vec{u}_z)$ or $(\vec{e}_\rho, \vec{e}_\theta, \vec{e}_z)$.

The position vector in cylindrical coordinates is given by:

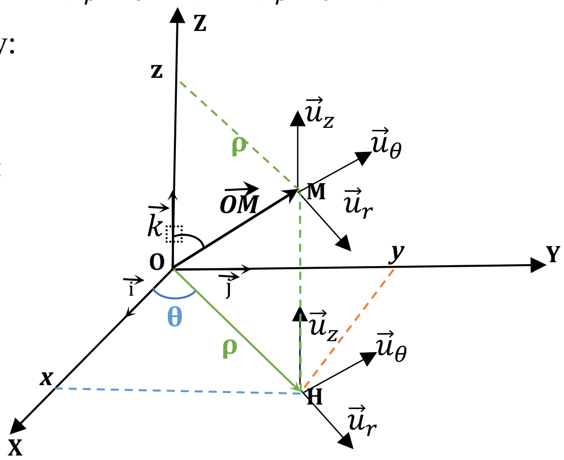
$\overrightarrow{OM} = \overrightarrow{OH} + \overrightarrow{HM} = \rho.\vec{u}_\rho + z.\vec{u}_z$

- For cylindrical coordinates, the length element becomes:

$\vec{dl} = d\rho.\vec{u}_\rho + \rho.d\theta.\vec{u}_\theta + z.\vec{u}_z$

Volume element in cylindrical coordinates can be derived:

$dV = (d\rho)(\rho.d\theta)(dz) = \rho d\rho.d\theta.dz$



The spherical coordinates of point M are (r, θ, φ),

- For the spherical basis $(\vec{u}_r, \vec{u}_\theta, \vec{u}_z)$:

The position vector of point M in the spherical basis is given by: $\overrightarrow{OM} = r.\vec{u}_r$

Infinitesimal movement in spherical coordinates: $\vec{dl} = dr.\vec{u}_r + rd\theta.\vec{u}_\theta + r sin\varphi d\varphi.\vec{u}_\varphi$

This leads to the infinitesimal volume: $dV = (dr)(r.d\theta)(r sin\varphi d\varphi) = r^2 dr.sin\varphi d\theta.d\varphi$

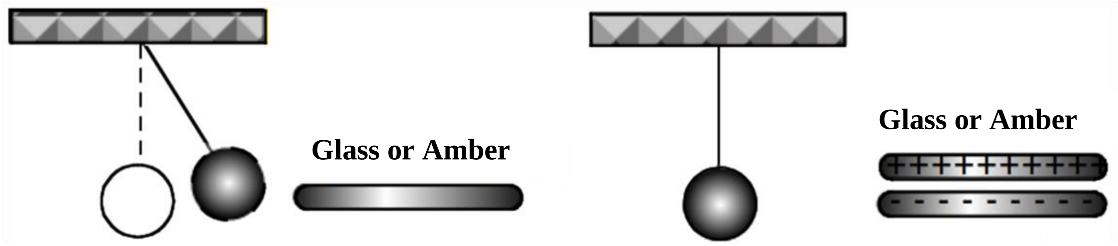
I-2 Electrostatic Field:

The static electricity consists of the attraction of small light bodies (like straw particles, as observed by Thales de Milet on 600 BCE). Electrostatic effect is perceived through an electric discharge upon contact with a foreign body. Term 'electricity' is derived from Greek word 'elektron,' meaning amber. The study of electrical phenomena intensified in the 19th century, leading to the growth of the unified theory of electrical and magnetic phenomena "electromagnetism", from which electrostatics emerges.

I-2-1 Notions of Electric Charges

Expérience 1 :

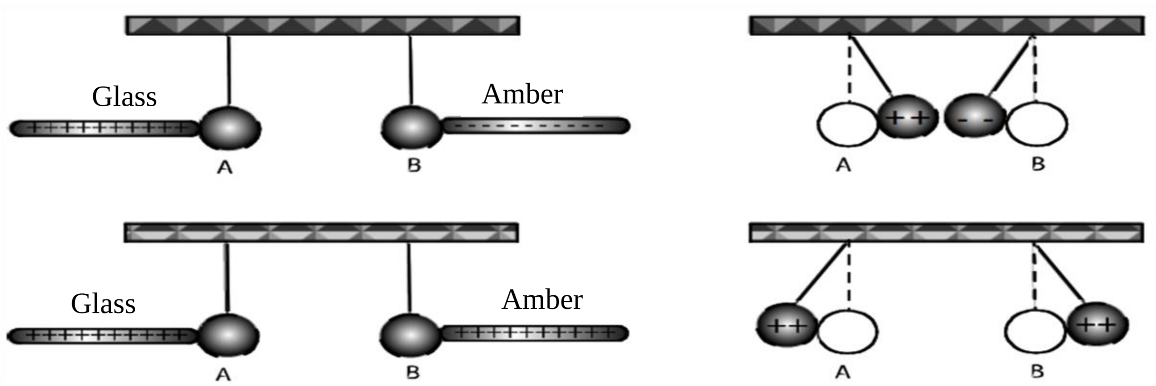
Let's take a ball of polystyrene and suspend it by a thread. Next, we bring a rod (glass or amber) after having previously rubbed it; both rods attract the ball. But, in approach of both side by side, no reaction.



Since their friction, each rod carries electricity, but this electricity can manifest itself in two opposite states. Electricity contained in glass (silk) labeled as positive, and in amber (fur) termed negative.

Expérience 2 :

Now, let's consider two balls, A and B rubbed by rod (electrified); and suspended side by side. If both into contact with the same material with a rod of, they repel each other. If not, they attract each other. Due to the fact of their attraction (contact), they lose all electrification: weight equilibrium position.



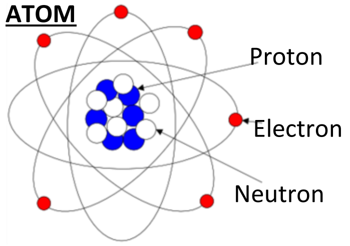
Robert A. Millikan stated that every electric charge Q is quantized in the form of electrons (multiples of elementary indivisible charges (Q=Ne). In the International System of Units, the unit of electric charge is the Coulomb (symbol C), subunits such as nanocoulombs (nC) and microcoulombs (µC). The elementary electric charge is an invariant (No destroyed or created) holds true in any reference frame. This is described by ***the concept of relativistic invariance of electric charge***.

Matter is composed of atoms, and the atom consists of a nucleus (Rutherford) with electrons orbiting around, carrying most of the mass. These electrons repel each other but remain kept around the nucleus, which has a positive electric charge that attracts them (protons). There is another type of particle (neutrons, discovered by Chadwick) with zero electric charge: (I S):

Electron : q_e = -e = -1.602 10^{-19} C ; m_e = 9.109 10^{-31} kg

Proton : q_p= +e =1.602 10^{-19} C ; m_p = 1.672 10^{-27} kg

Neutron : q_n = 0 C ; m_n =1.674 10^{-27} kg



Electron is a small and light ($m_e = 0.911\times10^{-30}$ kg) charged particle (q_e = e = -1.60×10^{-19} C) that rotates around the core. Typical atom contains many electrons that can be closer to or further from core. Atom´s core contains practically the whole mass of the atom. It consists ot protons and neutrons. The mass of proton and neutron is $m_p = m_n = 1.67\times 10^{-27}$ kg (much greater than the electron mass). Neutron is not charged. Proton charged positively, q_p=-e($1.60.10^{-19}$).

Normally atoms are not charged as the charge of their electrons balances the charge of their core. Removing electrons creates a positively charged ion, and placing additional electrons on the atom creates a negatively charged ion. Jumping of electrons from one body to the other can cause sparks.

The cohesion of matter is due to the interaction between its constituents: binding energy. Each atom possesses kinetic energy associated with its temperature, and its rigidity depends on both kinetic and binding energies. For a gas composed of neutral atoms, the interaction between two atoms is relatively weak: it occurs upon their approach, leading to electron repulsion. Each atom is free to move in space. As this gas cools, certain previously negligible electrostatic bonds can become operative, giving rise to a liquid. When this gas is heated, it provides energy to its constituents. If heating continues, it can release one or more peripheral electrons from atoms, thus producing a gas of ions (plasma). In contrast, for a solid, the bonds between each atom are stronger, and the atoms move very little (crystal: ionic bonding). The strength of this cohesion varies from one solid to another. A material consists of a large number of electric charges that are balanced (equal numbers of electrons and protons). At normal temperatures, matter is electrically neutral. Static electricity effects occur due to the movement of charges from one material to another (charging of a body). These unbalanced charges (excess/deficiency) are responsible for the electrical effects of the body. A material is considered a perfect conductor if, when electrified, the unbalanced charge carriers can move freely through the volume occupied by the material. It will be a perfect insulator (or dielectric) if the unbalanced charge carriers cannot move freely and remain localized where they were deposited. Material naturally will be between these extreme states.

I-3 Electric Field and Electrostatic Force

In physics, an electric field is defined as a field created by electrically charged particles. This concept was introduced by Michael Faraday and explains how two objects can interact at a distance without any physical connection. Newton's universal law of gravitation and Coulomb's law in electrostatics involve such remote interactions. There is no physical connection, similar between the Earth and the Sun, so far the Sun exerts its gravitational attraction from a distance. Similarly, two electric charges attract each other in a vacuum without any physical joining. Consider a charge q_1 by its presence, creates an electric field at every point in space. Interaction of this field with another charge q_2 generates an electrostatic force between these two charges. In the case of stationary charges in the reference frame under study, the electric field is called ***the electrostatic field***. If the charges are in motion, an induced electric field due to the movement of charges must be added to obtain the total field (electromagnetic field).

I-3-2 Elementary Interaction: Coulomb's Law

Charles Auguste de Coulomb (1736-1806) conducted a series of measurements (using a torsion balance) that allowed him to determine with a certain degree of precision the properties of the electrostatic force exerted by a point charge q_1 on another point charge q_2.

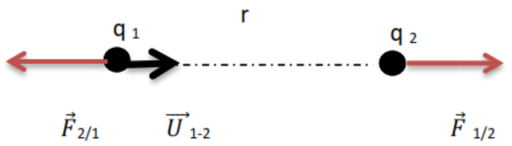
The force is radial, meaning it is directed along the line joining the two charges.

It is proportional to the product of the charges.

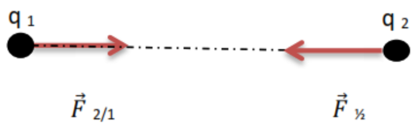
Finally, it varies inversely with the square of the distance between the two charges.

The mathematical expression for the Coulomb force exerted by charge q_1 on charge q_2, reflecting the above properties, is as follows: $\vec{F}_{1-2} = K\frac{q_1.q_2}{r^2}\vec{u}_{1-2}$

- This expression is valid only for stationary charges in a vacuum.
- This law forms the very foundation of electrostatics.
- This force adheres to the principle of action and reaction in classical mechanics.
- This force is attractive if the charges have opposite signs and repulsive if they have the same sign.



Si q $_1$q$_2$ > 0



Si q $_1$q$_2$ < 0

The Coulomb's law can be written: $\vec{F}_{1-2} = K\frac{q_1 \cdot q_2}{r^2}\vec{u}_{1-2} = K\frac{q_1 \cdot q_2}{r^2} \cdot \frac{\vec{r}}{r} = K\frac{q_1 \cdot q_2}{r^2} \cdot \frac{\vec{r}}{r^3}$

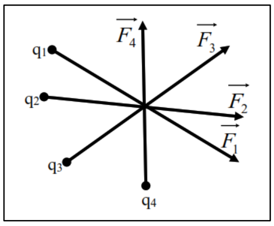
$$K = \frac{1}{4\pi\varepsilon_0} = 9.10^9 \text{ N.m}^2/\text{C}^2$$

Where K is the constant of Coulomb and ε_0 is the permittivity of vacuum, ε_0 = $\mathbf{8.85}.10^{-12}$.

In a vacuum, we have: $\mu_0\varepsilon_0 C^2 = 1$, where C is the velocity(speed) of light, $C = 3\times10^8$ m/s,

μ_0 is the permeability of free space, $\mu_0 = 4\pi\times10^{-7}$.

Principle of superposition of forces:



If a charge q_0 is surrounded by a set of electric charges q_1, q_2, q_3, q_4... q_n. The force acting on it and resulting from all the other charges is equal to the vector sum of all the forces acting.

$\vec{F}_{1/2} = \vec{F}_1 + \vec{F}_2 + \vec{F}_3 + \dots + \vec{F}_n$

I-2-3 Electrostatic Field Created by a Point Charge

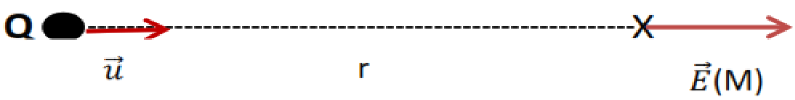
Notion of Electrostatic Field: We say that there is an electric field at a given point in space if an electrostatic force $\vec{F}_e$ acts on a point charge q_0 placed at that point.

The electric field is denoted by $\vec{E}$ and is defined as the ratio of the electrostatic force $\vec{F}_e$ to the charge q_0 experiencing this force Fe: $\vec{E} = \frac{\vec{F}_e}{q_0}$

Notion of Electrostatic Field created by a Point charge:

Consider a charge q_1 located at a point O in space, exerting an electrostatic force on another charge q_2 situated at a point M. The expression for this force is given by Coulomb's law above. However, similar to gravitational attraction, it can be expressed in a more interesting form, $\vec{F}_{1/2} = q_2.\vec{E}_1(M)$ where $\vec{E}_1(M) = \frac{1}{4\pi\varepsilon_0}\frac{q_1}{r^2}\vec{u}$

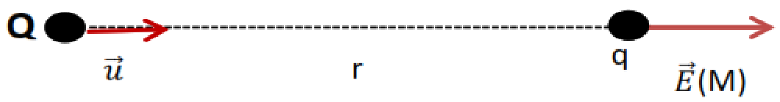
Définition: A particle of charge q located at O creates at any point M in space, distinct from O, a vector field called Electrostatic field. $\vec{E}(M) = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}\vec{u}$



The electrostatic field expresses, at every point M, the modification of the properties of space due to the presence of a source charge Q: $\vec{E}(M) = K\frac{Q}{r^2}\vec{u}$

The unit of the field in the International System (SI) is: V/m or N/C.

The electrostatic force acting on a target charge q placed at point M is: $\vec{F} = q.\vec{E}$



Principle of superposition: The electrostatic field at a point in space is equal to the sum of the electrostatic fields created by the individual charges at that point.

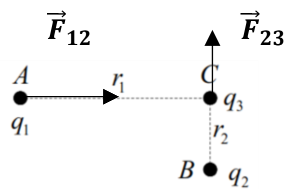
Example 1: An electron and a proton have charges of an equal magnitude but opposite sign of 1.6×10^{-19} C. If the electron and proton in a hydrogen atom are separated by a distance of 0.5×10^{-10} m, what is the electrostatic force exerted on the electron by the proton? Is it attraction or repulsion? Formulation: $q_e \equiv e = -1.6\times10^{-19}$ C, $q_p \equiv -e = 1.6\times10^{-19}$ C, $r = 5\times10^{-11}$ m, F?

Solution: Use the Coulomb´s law:

$$F_{12} = K\frac{q_1.q_2}{r^2} = -K\frac{e^2}{r^2} = -0.9\times10^{10}\left(\frac{1.6\times10^{-19}}{5\times10^{-11}}\right)^2 = -0.92\times10^{-9}N$$

The result is negative, and this means the charges are attracting each other

Example 2: Calculate the magnitude of the resultant force acting on the charge q_3: q_1=-1.5mC; q_2= 0.5 mC; q_3= 0,2 mC; r_1=AC=1.2 m and r_2= BC= 0.5 m



Solution:

Because q_1. q_3 <0, so F_{13} is attractive force; and q_2. q_3 >0, so F_{13} is repulsive force.

$$\vec{F}_{13} = -K\frac{q_1.q_3}{r^2}\vec{u}_1 \Rightarrow F_{13} = -9\times10^9\frac{1,5.10^{-3}\times0,2.10^{-3}}{(1,2)^2};\ F_{13} = 1,875.10^3\ N\ Thus\ R = \sqrt{F_{13}^2 + F_{23}^2}$$

$$\vec{F}_{23} = K\frac{q_1.q_3}{r^2}\vec{u}_1 \Rightarrow F_{13} = 9\times10^9\frac{0,5.10^{-3}\times0,2.10^{-3}}{(0,5)^2};\ F_{23} = 3,6.10^3\ N\ Thus\ R = 4,06.10^3\ N$$

Angle θ formed by Resultant $\vec{R}$ with Line $\overrightarrow{AC}$, it is equal to: $tg\alpha = \frac{F_{23}}{F_{13}} \Rightarrow tg\ \alpha = 1.92 \Rightarrow \alpha = 63°$

I-2-4 Electrostatic Field Created by a Set of Charges:

I-2-4-a Case of a Discrete Charge Distribution

The total electric field due to a group of charges is the vector sum of the individual electric fields of the charges. This is just the principle of superposition at work.

$$\vec{E}_{tot}(M) = \Sigma_{i=1}^{n} K\frac{Q_i}{r^2}\vec{u}\ i\rightarrow M$$

Similarly, for the electrostatic force created by a discrete distribution of charges (set of point charges), the resultant force is given by the principle of superposition:

$$\vec{F}_{tot}(M) = \Sigma_{i=1}^{n} K\frac{Q_iQ_M}{r^2}\vec{u}\ i\rightarrow M$$

I-2-4-b Case of a Continuous Charge Distribution

As the number of charges increases, the calculation of the total field becomes too complex. Therefore, to calculate the field at a point M in space due to a continuous charge distribution, we divide space into small elements, each containing an elemental charge dq at a distance r from point M. The sum of these elemental fields is then replaced by an integral:

$$\vec{E} = \Sigma \vec{E}(M) = \int_{0}^{distr} \frac{Kdq}{r^2} \vec{u}$$

Three cases of continuous distribution are distinguished:

A linear charge distribution

A surface charge distribution

A volumetric charge distribution

Examples of charged surfaces:

Cell membranes = surfaces charged with cell membranes

Axons of nerve cells = charged cylinders

Linear Charge Distribution

Consider a wire of length ll carrying a charge Q uniformly distributed along its length.

Let dl be an element of length carrying an elemental charge dq, then we have: $dq = \lambda . dl$

where λ represents the linear charge density (C/m).

The total charge carried by the wire can be written as: $Q = \int \lambda . dl$ and $\vec{E} = \int \overrightarrow{dE}(M)$

And the electric field E at a point M is given by: $\vec{E}(M) = \int K \frac{dq}{r^2} \vec{u}$ and $\vec{E}(M) = \int K \frac{\lambda . dl}{r^2} \vec{u}$

Surface Charge Distribution

Similarly, for a surface charge distribution, consider a surface SS carrying a charge Q uniformly distributed over its entire surface with a surface charge density σ (C/m²). Let ds be an element of surface carrying an elemental charge dq, we can write: $Q = \int \int \sigma . ds$ and $\vec{E} = \int \overrightarrow{dE}(M)$

The electric field E at a point M is then given by: $\vec{E}(M) = \int \int K \frac{dq}{r^2} \vec{u}$ and $\vec{E}(M) = \int \int K \frac{\sigma . ds}{r^2} \vec{u}$

Volumetric Charge Distribution

Similarly, for a volumetric charge distribution, consider a volume V carrying a charge Q uniformly distributed throughout its volume with a volumetric charge density ρ (C/m³). Let dv be an element of volume carrying an elemental charge dq, we can write: $Q = \int \int \int \rho . dV$

$$\vec{E}(M) = \int \int K \frac{dq}{r^2} \vec{u} \text{ and } \vec{E}(M) = \int \int K \frac{\rho . dV}{r^2} \vec{u}$$

I-3 Electrostatic Potential

The electric charge of a distribution can be described by a vector: the electric field or by a scalar quantity: the electric potential V. Therefore, the disturbance in the medium due to the presence of electric charges can be characterized by a scalar function: the potential V.

The potential is related to the work done to transport a charge from one point to another. The electrostatic field exists only if there is a variation in potential between two points.

Electric Field=Variation of Potential in SpaceElectric Field=Variation of Potential in Space

The relationship between potential and electric field is:

$$\vec{E} = -\overrightarrow{grad}\, V$$

$\vec{E} = -\overrightarrow{grad}\, V$ and $dV = -\vec{E}.\overrightarrow{dl}$ $\quad V = -\int \vec{E}.\overrightarrow{dl}$

I-3-1 Electrostatic Potential Created by a Point Charge

The potential V is therefore an algebraic scalar, expressed in volts.

The potential at a point M located at a distance r from the charge q is given by: $V = K\frac{q}{r}$

Reminder of the infinitesimal length element in polar coordinates:

$\overrightarrow{dl} = dx.\vec{i} + dy.\vec{j}$ and $\overrightarrow{dl} = dr.\vec{u}_r + r.d\theta.\vec{u}_\theta$

And so, the dot product is given in polar coordinates and similarly in spherical coordinates:

$$\vec{E}.\overrightarrow{dl} = K\frac{q_1.q_2}{r^2}\vec{u}_r.(dr.\vec{u}_r + r.d\theta.\vec{u}_\theta) = K\frac{q}{r^2}dr$$

$$\textit{Therfore, } V = -\int \vec{E}.\overrightarrow{dl} = -\int K\frac{q}{r^2}dr = K\frac{q}{r} + Cste$$

We assume that the potential at infinity is zero, V(∞)=0. Thus, 0 + Cste = 0, implying that Cste=0.

$$V = K\frac{q}{r}$$

I-3-2 Electrostatic Potential Created by a Set of Charges

I-3-2-a Case of a Discrete Charge Distribution

Potentials add up for a distribution of point charges, and we apply the principle of superposition.

$$V(M) = \Sigma_{i=1}^{n} V_i$$

I-3-2-b Case of a Continuous Charge Distribution

- Linear charge distribution: $V(M) = \int K.\frac{dq}{r} = \int K.\frac{\lambda.dl}{r}$
- Surface charge distribution : $V(M) = \int K.\frac{dq}{r} = \int\int K.\frac{\sigma.dS}{r}$
- Volumetric charge distribution: $V(M) = \int K.\frac{dq}{r} = \int\int\int K.\frac{\rho.dV}{r}$