Chapter 1

Integral

1.0.1 Primitive of a function

Definition 1 *Let f a function defined on an interval I. We call primitive function of f on I any differentiable function F satisfying:*

$$F'(x) = f(x),\ x \in I.$$

Proposition 2 *If F is a primitive of f on an interval I, then every primitive of f on I is of the form $F + c$*

$$F(x) = \int f(x)dx + c,$$

where c is a real constant

Proposition 3 *Any continuous function on an interval admits primitives on this interval.*

Remark 4 *-Find a primitive of a function is the inverse operation of calculating a derivative.*

-A function does not a single primitive.

Example 5 *Evaluate the following integral:* $\int (x^2 + 2x)dx$

$$\int (x^2 + 2x)dx = \frac{1}{3}x^3 + x + c.$$

Linearity: Let f and g be two continuous functions on I and a, b be two real numbers of I.

1. For $\lambda \in \mathbb{R}$, $\int \lambda f(x)dx = \lambda \int f(x)dx$.

2. $\int (f(x) + g(x))dx = \int f(x)dx + \int_a^b g(x)dx$.

1.0.2 Integral of a continuous function

Definition 6 *Let f be a continuous and positive real function taking its values in $I = [a, b]$, then the integral of f over I, denoted*

$$\int_a^b f(x)dx.$$

is the area of a surface delimited by the graphic representation of f and by the three straight lines of equation $x = a, x = b$, $0 \leq y \leq f(x)$.

Definition 7 *Let f be a continuous function on un interval $I = [a, b]$. We call integral of f on $I = [a, b]$ the numbre $F(b) - F(a)$ where F is any primitive of f on I. We also write:*

$$\int_a^b f(x)dx = F(b) - F(a)$$

Theorem 8 *Let f is a continuous and positive function on an interval $I = [a, b]$. The function F defined on $[a, b]$ by $F(x) = \int_a^x f(t)dt$ is a primitive of f or $F(x)$ is deferentiable on I and its derevative is the function f; $F'(x) = f(x)$.*

Remark 9 *The variable x can be replaced by any letter:*

$$\int_a^b f(x)dx, \int_a^b f(t)dt \text{ or } \int_a^b f(u)du$$

1.0.3 Properties of integrals

Relationship of Chasles: Let f be a continuous function on I and a, b and c three real numbers of I:

1. $\int_a^a f(x)dx = 0$.
2. $\int_a^b f(x)dx = F(b) - F(a) = -\int_b^a f(x)dx$
3. $\int_a^b f(x)dx + \int_b^c f(x)dx = \int_a^c f(x)dx$

Example 10 *Evaluate the following integral:* $\int_1^3 (x^2 + 2x + 1)dx$

Primitive of $(x^2 + 2x + 1)$ is $\frac{1}{3}x^3 + x^2 + x + C$ then we we substitute the bounded $a = 1$ and $b = 3$ or also We can divide this interval like $a = 1$, $b = 2$ and $c = 3$. We obtain the same result

$$\begin{aligned}
\int_1^3 (x^2 + 2x + 1)dx &= \left[\frac{1}{3}x^3 + x^2 + x + C\right]_1^3 \\
&= (\frac{1}{3}27 + 9 + 3 + C) - (\frac{1}{3} + 1 + 1 + C) \\
&= \frac{56}{3}
\end{aligned}$$

and

$$\begin{aligned}
& \int_1^2 (x^2+2x+1)dx + \int_2^3 (x^2+2x+1)dx \\
= & \left[\frac{1}{3}x^3+x^2+x+C\right]_1^2 + \left[\frac{1}{3}x^3+x^2+x+C\right]_1^2 \\
= & \frac{19}{3}+\frac{37}{3} \\
= & \frac{56}{3}
\end{aligned}$$

Linearity: Let f and g be two continuous functions on I and a, b be two real numbers of I.

1. For $\lambda \in \mathbb{R}$, $\int_a^b \lambda f(x)dx = \lambda \int_a^b f(x)dx$.

2. $\int_a^b (f(x)+g(x))dx = \int_a^b f(x)dx + \int_a^b g(x)dx$.

Example 11 *We take the same example* $\int_1^3 (x^2+2x+1)dx$

$$\begin{aligned}
\int_1^3 (x^2+2x+1)dx &= \int_1^3 x^2dx + 2\int_1^3 xdx + \int_1^3 dx \\
& \left[\frac{1}{3}x^3\right]_1^3 + \left[x^2\right]_1^3 + [x]_1^3 \\
& \frac{56}{3}
\end{aligned}$$

Inequality: Let f and g be two continuous functions on I and a, b be two real numbers of I.

If $f(x) \leq g(x)$ then $\int_a^b f(x)dx \leq \int_a^b g(x)dx$.

Example 12 *We know that* $\sin(x) \leq 1$ *and* $x\sin(x) \leq x$ $(I = \left[0, \frac{\pi}{2}\right])$. *We calculate the integral* $\int_0^{\frac{\pi}{2}} x\sin(x) = 1$ *and* $\int_0^{\frac{\pi}{2}} x = \frac{1}{8}\pi^2$.

it's clear that $\int_0^{\frac{\pi}{2}} x\sin(x) \leq \frac{1}{8}\pi^2$

Function	Primitive	Function	Primitive
x^n	$\frac{1}{n+1}x^{n+1}+c$, $\mathbb{R}$	$\frac{1}{\sqrt{x-a}}$	$2\sqrt{x-a}+c$, $(]a,+\infty[)$
$x^{\alpha+1}, \alpha \in \mathbb{R}-\{-1\}$	$\frac{1}{\alpha+1}x^{\alpha+1}+c$, $\mathbb{R}$	$ln(x)$	$x\ln(x)-x$, $\mathbb{R}^+$
$(x-a)^n$	$\frac{1}{n+1}(x-a)^{n+1}+c$, $\mathbb{R}$	$(x-a)^\alpha, \alpha \in \mathbb{R}-\{-1\}$	$\frac{1}{\alpha+1}(x-a)^{\alpha+1}+c$
$\frac{1}{x-a}, a \in \mathbb{R}$	$\ln(\|x-a\|)+c$, $\mathbb{R}-\{a\}$	$\frac{1}{x^2+1}$	$arctan(x), \mathbb{R}$
$\frac{1}{(x-a)^n}, a \in \mathbb{R}, n \geq 2$	$\frac{-1}{(n-1)(x-a)^{n-1}}+c$, $\mathbb{R}-\{a\}$	$u'u^\alpha, \alpha \neq 1$	$\frac{1}{\alpha+1}u^\alpha+c$
$\frac{1}{x}$	$\ln(\|x\|)+c$, $\mathbb{R}^*$	$\frac{u'}{u}$	$\ln(\|u\|)+c$
$\cos(ax)$, $a \in \mathbb{R}^*$	$\frac{1}{a}\sin(ax)+c$, $\mathbb{R}$	$\frac{u'}{\sqrt{u}}$	$2\sqrt{u}+c$
$\sin(ax)$, $a \in \mathbb{R}^*$	$-\frac{1}{a}\cos(ax)+c$, $\mathbb{R}$	$u'\exp(u)$	$exp(u)+c$
$\exp(ax)$, $a \in \mathbb{R}^*$	$\frac{1}{a}\exp(ax)+c$, $\mathbb{R}$	$u'\sin(u)$	$-\cos(u)+c$
a^x, $a \in \mathbb{R}^*$	$\frac{1}{\ln(a)}a^x+c$, $\mathbb{R}$	$u'\cos(u)$	$sin(u)+c$
$\sqrt{x-a}$	$\frac{2}{3}(x-a)^{\frac{3}{2}}+c$, $[a,+\infty[$		

Table 1.1: Table of primitives of usual functions.

1.0.4 Primitive functions of elementary functions

In some cases it is not easy to determine a primitive of a function and therefore to calculate the integral. Techniques are used to solve this problem such as integral by parts, integral by change of variable and integration by decomposition. We cite these two techniques in the next section.

1.0.5 Integration by part

Integration by parts is a technique for solving integrals; given a single function to integrate, the latter consists of separating this single function into a product of two functions $u(x)v(x)$ such that the residual integral of the integration formula by parts is easier to evaluate that single function. The following formula illustrate the integration by parts

$$\int u'(x)v(x)dx = u(x)v(x) - \int u(x)v'(x)dx$$

On the right-hand side, u is differentiated and v is integrated; consequently it is useful to choose u as a function that simplifies when differentiated, or to choose v as a function that simplifies when integrated.

Remark 13 *For calculating integration by parts on a closed interval* $[a, b]$*, we get*

$$\int_a^b u'(x)v(x)dx = [u(x)v(x)]_a^b - \int_a^b u(x)v'(x)dx$$

where $[u(x)v(x)]_a^b = u(b)v(b) - u(a)v(a)$.

Examples

Polynomials and trignometric functions or exponontial functions: In order to calculate: $\int x\cos(x)dx$

Let:

$$\begin{aligned} u &= x \Rightarrow u' = 1 \\ v' &= \cos(x) \Rightarrow v = \sin(x) \end{aligned}$$

then

$$\begin{aligned} \int x\cos(x)dx &= \int uv' \\ &= uv - \int u'v \\ &= x\sin(x) - \int \sin(x)dx \\ &= x\sin(x) + \cos(x) + c \end{aligned}$$

also $\int xe^x dx$. Let

$$\begin{aligned} u &= x \Rightarrow u' = 1 \\ v' &= e^x \Rightarrow v = e^x \end{aligned}$$

then

$$\begin{aligned} \int xe^x dx &= \int uv' \\ &= uv - \int u'v \\ &= xe^x - \int e^x dx \\ &= xe^x - e^x + c \\ &= e^x(x-1) + c. \end{aligned}$$

Exponentials and trignometric functions: We can also used integration by parts when the integral is product of Exponential function and trignometric function such as: $\int e^x \cos(x)dx$, $\int e^x \sin(x)dx$. In that case integration by parts is performed twice. we take the following example: $\int e^x \cos(x)dx$

First let

$$\begin{aligned} u &= \cos(x) \Rightarrow du = -\sin(x)dx \\ dv &= e^x dx \Rightarrow v = \int e^x dx = e^x \end{aligned}$$

then

$$\int e^x \cos(x)dx = e^x \cos(x) + \int e^x \sin(x)dx$$

and by integration by parts second time of $\int e^x \sin(x)dx$

$$\begin{aligned} u &= \sin(x) \Rightarrow du = \cos(x)dx \\ dv &= e^x dx \Rightarrow v = \int e^x dx = e^x \end{aligned}$$

then

$$\begin{aligned} \int e^x \cos(x)dx &= e^x \cos(x) + \int e^x \sin(x)dx \\ &= e^x \cos(x) + e^x \sin(x) - \int e^x \cos(x)dx \end{aligned}$$

The same integral shows up on the both sides of the equation. by adding the two sides we get

$$2 \int e^x \cos(x)dx = e^x \cos(x) + e^x \sin(x) + c$$

so

$$\int e^x \cos(x)dx = \frac{1}{2}(e^x \cos(x) + e^x \sin(x)) + c',$$

where $c' = c/2$.

Functions multiplied by unity: Integration by parts is applied to a function expressed as a product of 1 and itself. it's used when the derivative of the function is known, and the integral of this derivative times x is also known.

An example: $\int \ln(x)dx$. We write this as: $\int 1. \ln(x)dx$

Let

$$\begin{aligned} u &= \ln(x) \Rightarrow du = \frac{1}{x}dx \\ dv &= 1dx \Rightarrow v = x \end{aligned}$$

then

$$\begin{aligned}\int 1.\ln(x)dx &= x\ln x - \int \frac{x}{x}dx \\ &= x\ln(x) - \int dx \\ &= x\ln(x) - x + c.\end{aligned}$$

so

$$\int \ln(x)dx = x\ln(x) - x + c.$$

1.0.6 Integration by change of variable (Integration by Substitution)

Suppose that $g(x)$ is a differentiable function and f is continuous on the range of g. Integration by substitution is given by the following formulas:

$$\int f(g(x))g'(x)dx = \int f(u)du,$$

where $u = g(x)$.

and to integrate f on a closed interval $[a, b]$, integration by substitution is given by

$$\int_a^b f(g(x))g'(x)dx = \int_{g(a)}^{g(b)} f(u)du.$$

The goal of using integral by change of variable is making the integral easier to compute. we summarize this technique in the following steps:

1. Choose u to be the function that is "inside" the function;

2. Differentiate $u = g(x)$ to conclude $du =g(x)'dx$. If we have boundes in the integral, we must change them. For $x = a$ implies $u = g(a)$ and for $x = b$ implies $u = g(b)$.

3. Rewrite the integral by replacing all instances of $x$with the new variable and compute the integral.

4. Write final answer back in terms of the original variables.

Example 14 *Evaluate* $\int \frac{x}{\sqrt{2x-1}}dx$

by putting $u = 2x - 1.$ *Then* $du = 2dx,$ *or* $dx = \frac{1}{2}du$

Substituting into the integral:

$$\begin{aligned}
& \int \frac{u+1}{2\sqrt{u}}\frac{1}{2}du \\
= & \frac{1}{4}\int \frac{u+1}{\sqrt{u}}du \\
= & \frac{1}{4}\int (u+1)u^{-\frac{1}{2}}du \\
= & \frac{1}{4}\int (u^{\frac{1}{2}} + u^{-\frac{1}{2}})du \\
= & \frac{1}{4}(\frac{2}{3}u^{\frac{3}{2}} + 2u^{\frac{1}{2}}) + c \\
= & \frac{1}{6}u^{\frac{3}{2}} + \frac{1}{2}u^{\frac{1}{2}} \\
& \frac{1}{3}u^{\frac{1}{2}}(\frac{1}{2}u + \frac{3}{2}) \\
= & \frac{1}{3}\sqrt{(2x-1)}(\frac{1}{2}(2x-1) + \frac{3}{2}) + c \\
= & \frac{1}{3}\sqrt{(2x-1)}(x+1) + c
\end{aligned}$$

Example 15 *Calculate:* $\int_2^3 \frac{x}{\sqrt{x-1}}dx$

by putting: $u = x - 1$ *then* $du = dx$ *and for*

$$\begin{aligned}
x &= 2 \Rightarrow u = 1 \\
x &= 3 \Rightarrow u = 2
\end{aligned}$$

we get

$$\begin{aligned}
\int_2^3 \frac{x}{\sqrt{x-1}}dx &= \int_1^2 \frac{(u+1)}{\sqrt{u}}du \\
&= \int_1^2 (\frac{u}{\sqrt{u}} + \frac{1}{\sqrt{u}})du \\
&= \int_1^2 (\sqrt{u} + \frac{1}{\sqrt{u}})du \\
&= \int_1^2 u^{\frac{1}{2}}du + \int_1^2 \frac{1}{\sqrt{u}}du \\
&= \left[\frac{2}{3}u^{\frac{3}{2}} + 2\sqrt{u}\right]_1^2 \\
&= (\frac{2}{3}2^{\frac{3}{2}} + 2\sqrt{2}) - (\frac{2}{3}1^{\frac{3}{2}} + 2\sqrt{1}) \\
&= \frac{10}{3}\sqrt{2} - \frac{8}{3}.
\end{aligned}$$

1.0.7 Primitive functions of rational functions

$$R(x) = \frac{P(x)}{Q(x)}$$

$$R(x) = E(x) + \sum pe + \sum se$$

where $E(x)$ is a polynom function, $\sum pe$ is sum of simple element of first kind and $\sum se$ is sum of simple element of second kind.

The degree of $E(x)$ depend of the degrees of $P(x)$ and $Q(x)$ according the following relations:

1. If $Q^0 > P^0$ then $E(x) = 0$.

2. If $P^0 = Q^0$ then $E(x) = k$ (k is constant).

3. If $P^0 > Q^0$ then $E^0 = P^0 - Q^0$.

We take examples:

1. Let $R(x) = \frac{1}{x^2-x-2}$, we observe that $P^0 = 0$ and $Q^0 = 2$ then $E(x) = 0$
2. Let $R(x) = \frac{x^2}{x^2-x-2}$, we observe that $P^0 = Q^0 = 2$ then $E(x) = k$.
3. Let $R(x) = \frac{x^3}{x^2-x-2}$, we observe that $P^0 = 3$ and $Q^0 = 2$ then $E^0 = 3 - 2 = 1$ so $E(x) = ax + b$ and in that case we must determine a and b.

Simple element of first kind: The general form of simple of first kind is given by:

$$pe = \frac{k}{(x-a)^n},$$

where k and a are real constant. It is very easy to integrate pe because its primitive function is knowen,

- for $n = 1$ and $k = 1$:

$$\int \frac{1}{(x-a)} dx = \ln|x-a|$$

- for $n > 1$ and $k = 1$

$$\begin{aligned} \int \frac{1}{(x-a)^n} dx &= \int (x-a)^{-n} \, dx \\ &= \frac{1}{-n+1} (x-a)^{-n+1} + c \\ &= \frac{1}{1-n} \frac{1}{(x-a)^{n-1}} + c \end{aligned}$$

- and for $n > 1$

$$\int \frac{k}{(x-a)^n} dx = \frac{1}{1-n} \frac{k}{(x-a)^{n-1}} + c$$

Simple element ofsecond kind: The general form of simple of first kind is given by:

Example 16 $\int \frac{1}{x^2-x-2}dx$, *we have* $P(x) = 1$ *and* $P^0 = 0$, $Q(x) = x^2 - x - 2$ *so* $Q^0 = 2$. *We observe that* $P^0 < Q^0$ *then* $E(x) = 0$

we calculate the determinant of $x^2 - x - 2 : \Delta = (-1)^2 - 4(1)(-2) = 9 > 0$ *so we have two solution*

$$x_1 = \frac{1-3}{2} = -1, \; x_2 = \frac{1+3}{2} = 2$$

so we can write $\frac{1}{x^2-x-2}$ *by* $\frac{1}{(x+1)(x-2)} = \frac{a}{x+1} + \frac{b}{x-2}$ *and by identification we find the values of* a *and* b

$$\begin{aligned} \frac{a}{x+1} + \frac{b}{x-2} &= \frac{ax - 2a + bx + b}{(x+1)(x-2)} \\ &= \frac{(a+b)x - 2a + b}{(x+1)(x-2)} \end{aligned}$$

we obtain a system of two equations and two inconnus

$$\begin{cases} a + b = 0 \\ -2a + b = 1 \end{cases}$$

$$a = -\frac{1}{3}, \; b = \frac{1}{3}$$

then we calculate the integral $\int (\frac{-1}{3(x+1)} + \frac{1}{3(x-2)})dx$

$$\begin{aligned} \int \frac{1}{x^2 - x - 2}dx &= -\frac{1}{3}\int \frac{1}{x+1}dx + \frac{1}{3}\int \frac{1}{x-2}dx \\ &= \frac{-1}{3}\ln|x+1| + \frac{1}{3}\ln|x-2| + c \\ &= \frac{1}{3}\ln\left|\frac{x-2}{x+1}\right| + c \end{aligned}$$

Example 17 $\int \frac{x}{x^2-1}dx$

$\int \frac{x^3}{x^2-x-2}dx = x + \frac{1}{3}\ln(x+1) + \frac{8}{3}\ln(x-2) + \frac{1}{2}x^2$

$\int \frac{(x-1)(x+1)+1}{(x+1)(x-2)}dx = \int dx + \int \frac{1}{(x-2)}dx + \int \frac{1}{(x+1)(x-2)}dx = x - \frac{1}{3}\ln(x+1) + \frac{4}{3}\ln(x-2)$

$\int \frac{1}{(x-2)}dx + \int \frac{x}{(x+1)(x-2)}dx = \frac{1}{3}\ln(x+1) + \frac{5}{3}\ln(x-2)$

$= \int_3^4 \frac{1}{x+1}dx + \int_3^4 \frac{1}{(x+1)(x-2)}dx = \frac{2}{3}\ln 5 - \ln 2 = 0.379\,81$

$\int_4^3 \frac{1}{x+1}dx = -0.223\,14$

$\int_3^4 \frac{1}{(x+1)(x-2)}dx = \ln 2 - \frac{1}{3}\ln 5$

$\int \frac{x^2}{1-x}dx = -x - \ln(x-1) - \frac{1}{2}x^2$

$\int \frac{x^2}{x-1}dx = x + \ln(x-1) + \frac{1}{2}x^2$

$(x-1)(x-2) = x^2 - 3x + 2$

$\int \frac{x^2}{(x+4)(x-3)}dx = x + \frac{9}{7}\ln(x-3) - \frac{16}{7}\ln(x+4)$

$\int \frac{1}{(x+4)(x-3)}dx = \frac{1}{7}\ln(x-3) - \frac{1}{7}\ln(x+4)$

$\frac{d\left(x - \frac{1}{3}\ln(x+1) + \frac{4}{3}\ln(x-2)\right)}{dx} = -\frac{x^2}{-x^2+x+2}$

$\frac{d\left(\frac{1}{3}\ln(x+1) + \frac{5}{3}\ln(x-2)\right)}{dx} = -\frac{2x+1}{-x^2+x+2}$