Exo1: let's $C \in \mathbb{R}$.

① $\int 2\,dx = 2x + C$

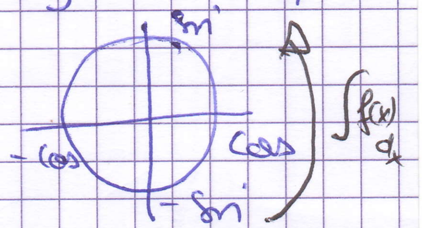
② $\int x^2 + 2x - 1\,dx = \frac{1}{3}x^3 + x^2 - x + C$

③ $\int \frac{\theta^2 + 3}{\sqrt{\theta}}\,d\theta = \int \frac{\theta^2}{\sqrt{\theta}} + \frac{3}{\sqrt{\theta}}\,d\theta = \int \theta^{3/2} + 3\theta^{-1/2}\,d\theta = \frac{2}{5}\theta^{5/2} + 6\sqrt{\theta} + C$

④ $\int \underline{\underline{x}}\,y^2 + y^2\sqrt[3]{y}\,\underline{\underline{dy}} = \underline{\underline{x}}\int y^2\,dy + \int y^2 \times y^{1/3}\,dy = \frac{x}{3}y^3 + \frac{3}{10}y^{10/3} + C$

⑤ $\int x^2 + \sqrt{x} + \frac{1}{x^4\sqrt{x}}\,dx = \int x^2 + x^{1/2} + x^{-9/2}\,dx = \frac{1}{3}x^3 + \frac{2}{3}x^{3/2} - x^{-7/2} \times \frac{2}{7} + C$

⑥ $\int \sin(\theta) - \cos(\theta)\,d\theta = -\cos(\theta) - \sin(\theta) + C$



Exo2:

① we have $f'(x) = 3x^2 + e^{-x} \Rightarrow f(x) = \int 3x^2 + e^{-x}\,dx = \boxed{x^3 - e^{-x} + C}$

as $f(0) = 1$ then $0^3 - e^{-0} + C = 1 \Rightarrow \boxed{C = 2}$

so $\boxed{f(x) = x^3 - e^{-x} + 2}$

② $f'(x) = \sqrt[3]{x^2} - \frac{1}{x^2} \Rightarrow f(x) = \int x^{2/3} - x^{-2}\,dx = \frac{3}{5}x^{5/3} + x^{-1} + C$

$f(1) = 3 \Rightarrow \frac{3}{5} + 1 + C = 3 \Rightarrow C = \frac{7}{5} \Rightarrow \boxed{f(x) = \frac{3}{5}x^{5/3} + \frac{1}{x} + \frac{7}{5}}$

③ $f''(x) = \sin(x) - e^{-2x} \Rightarrow f'(x) = \int \sin(x) - e^{-2x}\,dx = \boxed{-\cos(x) + \frac{1}{2}e^{-2x} + C_1}$

$f'(0) = \frac{5}{2} \Rightarrow -1 + \frac{1}{2} + C_1 = \frac{5}{2} \Rightarrow \boxed{C_1 = 3}$

$f'(x) = -\cos(x) + \frac{e^{-2x}}{2} + 3 \Rightarrow f(x) = -\sin(x) - \frac{e^{-2x}}{4} + 3x + C_2$

$f(0) = 0 \Rightarrow -\frac{1}{4} + C_2 = 0 \Rightarrow \boxed{C_2 = \frac{1}{4}} \Rightarrow \boxed{f(x) = -\sin(x) - \frac{e^{-2x}}{4} + 3x + \frac{1}{4}}$

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④ $f''(x) = \sin(x) - \cos(x) \Rightarrow f'(x) = -\cos(x) - \sin(x) + C_1$

$f'(0) = 0 \Rightarrow -1 + C_1 = 0 \Rightarrow \boxed{C_1 = 1} \Rightarrow \boxed{f'(x) = 1 - \cos(x) - \sin(x)}$

$\Rightarrow f(x) = x - \sin(x) + \cos(x) + C_2$; $f(\pi/2) = 0 \Rightarrow \frac{\pi}{2} - 1 + C_2 = 0$

$\Rightarrow C_2 = 1 - \pi/2 \Rightarrow \boxed{f(x) = x - \sin(x) + \cos(x) + 1 - \frac{\pi}{2}}$

Exo2:

- $\int \tan(x)\,dx = \int \frac{\sin(x)}{\cos(x)}\,dx = \int -\frac{1}{t}\,dt = -\ln(t) + C \quad / t = \cos(x)$.
- $\int \sin^3(x)\cos^2(x)\,dx = \int \cos^2(x)\sin^2(x)\,d(\cos(x)) = -\int t^2(1-t^2)\,dt$
  $= -\int t^2 - t^4\,dt = -\frac{1}{3}t^3 - \frac{1}{5}t^5 + C$

  $\boxed{\text{we put } t = \cos(x) \Rightarrow \cancel{dt = \cos(x)\,dx}\ dt = -\sin(x)\,dx}$

- $I = \int \frac{\sin(x)\cos(x)}{(1+\cos(2x))^2}\,dx = \int \frac{\frac{1}{2}\sin(2x)}{(1+\cos(2x))^2}\,dx = ?$

  we put: $t = 1 + \cos(2x) \Rightarrow dt = -2\sin(2x)\,dx$.

  $\Rightarrow I = -\frac{1}{4}\int \frac{dt}{t^2} = -\frac{1}{4}\left(-\frac{1}{t}\right) + C = \frac{1}{4t} + C$

- $\int \arcsin(x) \times \frac{1}{\sqrt{1-x^2}}\,dx = ?$ we put $t = \arcsin(x) \Rightarrow dt = \frac{dx}{\sqrt{1-x^2}}$

  $\Rightarrow \int \frac{\arcsin(x)}{\sqrt{1-x^2}}\,dx = \int t\,dt = \frac{1}{2}t^2 + C = \frac{1}{2}\arcsin(x)^2 + C$.

- $\int \frac{x}{x^2+a^2}\,dx = ?$ we put $t = x^2 \Rightarrow dt = 2t\,dt$

  $\Rightarrow \int \frac{x}{x^4+a^2}\,dx = \frac{1}{2}\int \frac{1}{t^2+a^2}\,dt = \frac{1}{a}\arctan\left(\frac{t}{a}\right) + C$.

- $\int \frac{1}{x\ln(x)}dx = ?$ we put $t = \ln(x) \Rightarrow dt = \frac{1}{x}dx$.

So $\int \frac{1}{x\ln(x)}dx = \int \frac{1}{t}dt = \ln(|t|) + c = \ln(|\ln(x)|) + c$.

- $\int x a^{x^2}dx = \int x e^{x^2\ln(a)}dx$; we put $t = x^2\ln(a)$

$\Rightarrow dt = 2x\ln(a) \Rightarrow \int x a^{x^2}dx = \frac{1}{2\ln(a)}\int e^t dt$

$= \frac{1}{2\ln(a)}e^t + c = \boxed{\frac{1}{2\ln(a)} a^{x^2} + c}$.

- $\int \frac{a^x - b^x}{a^x b^x}dx = \int \frac{\cancel{a^x}}{\cancel{a^x} b^x} - \frac{\cancel{b^x}}{a^x \cancel{b^x}}dx = \underbrace{\int b^{-x}dx}_{I_1} - \underbrace{\int a^{-x}dx}_{I_2}$.

$\begin{cases} \text{in } I_1 \text{ we put } t = -x\ln(a) \\ \text{in } I_2 \text{ we put } t = -x\ln(b). \end{cases} \Rightarrow \begin{cases} dt = -\ln(a)dx \\ dt = -\ln(b)dx. \end{cases}$

$I_1 = \frac{-1}{\ln(a)}\int e^t dt = \frac{-e^t}{\ln(a)} + c_1$; $I_2 = \frac{-1}{\ln(b)}\int e^t dt = \frac{-e^t}{\ln(b)} + c_2$

So $I = \frac{e^{-x\ln(b)}}{\ln(b)} - \frac{e^{-x\ln(a)}}{\ln(a)} + c = \boxed{\frac{b^{-x}}{\ln(b)} - \frac{a^{-x}}{\ln(a)} + c}$

Exo2 (Students).

① $I_1 = \int \sin^{2p+1}(x)\cos^q(x)\,dx = -\int \sin^{2p}(x)\cos^q(x)\overbrace{-\sin(x)}^{u'}dx$

we put $t = \cos(x) \Rightarrow dt = -\sin(x)dx$.

$\Rightarrow I_1 = -\int (1-t^2)^p t^q dt = \dots$

$I_2 = \int \sin^p(x)\cos^{2q+1}(x)dx \Rightarrow$ we put $t = \sin(x)$

② we put $t = \ln(x) \Rightarrow dt = \frac{1}{x}dx \Rightarrow I = \int \frac{1}{t^n}dt = \begin{cases} \nearrow n = 1 \\ \searrow n \neq 1 \end{cases}$

EXO3 :

- $I_1 = \int \arctan(x)\,dx = \int 1\ \arctan(x)\,dx$

we put $\begin{cases} u' = 1 \\ v = \arctan(x) \end{cases} \Rightarrow \begin{cases} u = x \\ v' = \frac{1}{1+x^2} \end{cases}$

$\Rightarrow I_1 = x\arctan(x) - \int \frac{x}{1+x^2}\,dx$

$= x\arctan(x) - \frac{1}{2}\ln(1+x^2) + c.$

- $I_2 = \int \arcsin(x)\,dx = \int 1 * \arcsin(x)\,dx$

$\begin{cases} u' = 1 \\ v = \arcsin(x) \end{cases} \Rightarrow \begin{cases} u = x \\ v' = \frac{1}{\sqrt{1-x^2}} \end{cases}$

$I_2 = x\arcsin(x) + \int \frac{\cancel{-}x}{\sqrt{1-x^2}}\,dx = x\arcsin(x) + \sqrt{1-x^2} + c$

this stay the same for $\int \arccos(x)\,dx$, $\int \ln(x)\,dx$ ...

- $I_3 = \int \frac{x\arctan(x)}{(1+x^2)^2}\,dx = \int \frac{x}{(1+x^2)^2} * \frac{\arctan(x)}{\cancel{1+x^2}}\,dx.$

we put : $u = \arctan(x)$, $v' = \frac{x}{(1+x^2)^2}$ $\Rightarrow \begin{cases} u' = \frac{1}{1+x^2} \\ v = \end{cases}$

so $I_3 = \frac{-\arctan(x)}{2(1+x^2)} + \frac{1}{2}\underbrace{\int \frac{1}{(1+x^2)^2}\,dx}_{I_{3}.\text{bis}}$

to compute I_3.bis see exercice N° 4.

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• $I_4 = \int \frac{\arcsin(\sqrt{x})}{\sqrt{x}} dx$, we put $t = \sqrt{x} \Rightarrow dt = \frac{1}{2\sqrt{x}}$.

$\Rightarrow I_4 = 2 \int \arcsin(t)\, dt = 2 I_2$. (see example 2).

2. $I_{2.1} = \int \ln(x) dx = \int 1 * \ln(x)\, dx$

$$u' = 1, \quad v = \ln(x) \Rightarrow \begin{cases} u = x \\ v' = \frac{1}{x}. \end{cases}$$

$\Rightarrow I_{21} = x \ln(x) - \int x * \frac{1}{x} dx = x \ln(x) - x + C$.

$I_{22} = \int x \ln(x)\, dx \quad \begin{cases} u' = x \\ v = \ln(x) \end{cases} \Rightarrow \begin{cases} u = \frac{1}{2} x^2 \\ v' = \frac{1}{x} \end{cases}$

$\Rightarrow I_{22} = \frac{1}{2} x^2 \ln(x) - \frac{1}{2} \int x^2 * \frac{1}{x} dx$

$I_{22} = \frac{1}{2} x^2 \ln(x) - \frac{1}{4} x^2 + C$.

$I_{23} = \int \ln(x)^2 dx = \int 1 * \ln(x)^2 dx$

$$\begin{cases} u' = 1 \\ v = \ln(x)^2 \end{cases} \Rightarrow \begin{cases} u = x \\ v' = \frac{2 \ln(x)}{x} \end{cases}$$

$\Rightarrow I_{2.3} = x \ln(x)^2 - 2\int \ln(x)\, dx$.

$= x \ln(x)^2 - 2(x \ln(x) - x) + C$. (see $I_{2.1}$)

$I_{2.4} = \int x^n \ln(x)\, dx$.

$$\begin{cases} u' = x^n \\ v = \ln(x) \end{cases} \Rightarrow \begin{cases} u = \frac{1}{n+1} x^{n+1} \\ v' = \frac{1}{x}. \end{cases}$$

$I_{2.4} = \frac{x^{n+1} \ln(x)}{n+1} - \int \frac{1}{n+1} x^n dx$

$= \frac{1}{n+1} x^{n+1} \ln(x) - \frac{1}{(n+1)^2} x^{n+1} + C$.

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EX03:

$I_{31} = \int x^2 e^{-x}\,dx = ?$ $\begin{cases} u' = e^{-x} \\ v = x^2 \end{cases} \Rightarrow \begin{cases} u = -e^{-x} \\ v' = 2x \end{cases}$

$\Rightarrow I_{3.1} = -x^2 e^{-x} + \underbrace{\int 2x e^{-x}\,dx}_{I}$, $\begin{cases} u' = e^{-x} \\ v = 2x \end{cases} \Rightarrow \begin{cases} u = -e^{-x} \\ v' = 2 \end{cases}$

$\Rightarrow I_{3.1} = -x^2 e^{-x} - 2x e^{-x} + 2\int e^{-x}\,dx$

$$\boxed{I_{3.1} = -e^{-x}(x^2 + 2x + 2) + C}$$

$I_n = \int x^n e^{-x}\,dx = ?$ $\begin{cases} u' = e^{-x} \\ v = x^n \end{cases} \Rightarrow \begin{cases} u = -e^{-x} \\ v' = n x^{n-1} \end{cases}$

$I_n = -x^n e^{-x} + n\int x^{n-1} e^{-x}\,dx$

$$\boxed{I_n = -x^n e^{-x} + n I_{n-1}}$$

$\Rightarrow I_n = -x^n e^{-x} + n\left(-x^{n-1} + (n-1) I_{n-2}\right)$

$= -e^{-x}\left(x^n + n x^{n-1}\right) + n(n-1) I_{n-2}$

$= -e^{-x}\left(x^n + n x^{n-1} + n(n-1) x^{n-2}\right) + n(n-1)(n-2) I_{n-3}$

$= -e^{-x}\left(x^n + n x^{n-1} + \cdots + n(n-1)\cdots(n-k) x^{n-k-1} + \cdots\right.$

$\left. + n(n-1)\cdots 2 x^1\right) + n!\, I_0$

$$I_n = -e^{-x}\left(x^n + \cdots + n(n-1)\cdots(n-k) x^{n-k-1} + \cdots + n!\right) + C$$

$I_4 = \int \sin(x) e^x\,dx = ? \Rightarrow \begin{cases} u' = \sin(x) \\ v = e^x \end{cases} \Rightarrow \begin{cases} u = -\cos(x) \\ v = e^x \end{cases}$

$\Rightarrow I_4 = -\cos(x) e^x + \underbrace{\int \cos(x) e^x}_{I_{4.1}}$ —(*)

$\begin{cases} u' = \cos(x) \\ v = e^x \end{cases} \Rightarrow \begin{cases} u = \sin(x) \\ v' = e^x \end{cases}$

$I_{41} = \sin(x) e^x - \int \sin(x) e^x\,dx$

$I_{41} = \sin(x) e^x - I_4$ —(**)

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from (*) and (**) we get:

$$I_4 = -\cos(x)e^x + \sin(x)e^x - I_4$$

$$\Rightarrow 2I_4 = \left(\sin(x) - \cos(x)\right)e^x$$

$$\Rightarrow \boxed{I_4 = \frac{1}{2}\left(\sin(x) - \cos(x)\right)e^x + C}$$

$$I_4 = \int \cos(\beta x)\, e^{\alpha x}\, dx = ?$$

$$\begin{cases} u' = \cos(\beta x) \\ v = e^{\alpha x} \end{cases} \Rightarrow \begin{cases} u = \frac{+1}{\beta}\sin(\beta x) \\ v' = \alpha e^{\alpha x} \end{cases}$$

$$\Rightarrow I_4 = \frac{1}{\beta}\sin(\beta x)\, e^{\alpha x} - \frac{\alpha}{\beta}\underbrace{\int \sin(\beta x)\, e^{\alpha x}\, dx}_{I_{4_1}} \quad \text{—— } (*)$$

$$I_{4_1} = ? \quad \begin{cases} u' = \sin(\beta x) \\ v = e^{\alpha x} \end{cases} \Rightarrow \begin{cases} u = -\frac{1}{\beta}\cos(\beta x) \\ v' = \alpha e^{x} \end{cases}$$

$$\Rightarrow I_{4_1} = -\frac{\cos(\beta x)}{\beta} e^{\alpha x} + \frac{\alpha}{\beta}\int \cos(\beta x)\, e^{x}\, dx$$

$$I_{4_1} = -\frac{\cos(\beta x)}{\beta} e^{\alpha x} + \frac{\alpha}{\beta} I_4 \quad \text{—— } (**)$$

from (*) and (**) we get:

$$I_4 = \frac{1}{\beta}\sin(\beta x)\, e^{\alpha x} - \frac{\alpha}{\beta}\left(-\frac{1}{\beta}\cos(\beta x)\, e^{\alpha x} + \frac{\alpha}{\beta} I_4\right)$$

$$= \frac{1}{\beta}\sin(\beta x)\, e^{\alpha x} + \frac{\alpha}{\beta^2}\cos(\beta x)\, e^{\alpha x} - \left(\frac{\alpha}{\beta}\right)^2 I_4$$

$$\Rightarrow \left(1 + \left(\frac{\alpha}{\beta}\right)^2\right) I_4 = \left(\frac{1}{\beta}\sin(\beta x) + \frac{\alpha}{\beta^2}\cos(\beta x)\right) e^{\alpha x}$$

$$\Rightarrow I_4 = \frac{1}{\beta\left(1 + \left(\frac{\alpha}{\beta}\right)^2\right)} e^{\alpha x}\left(\sin(\beta x) + \frac{\alpha}{\beta}\cos(\beta x)\right)$$

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with the same sketch we obtain the expression of $\int \sin(\beta x)\, e^{\alpha x}\, dx$.

$$I_5 = \int \sqrt{a^2-x^2}\,dx = \int \frac{a^2-x^2}{\sqrt{a^2-x^2}}\,dx.$$

$$= \underbrace{\int \frac{a^2}{\sqrt{a^2-x^2}}\,dx}_{I_{51}} - \underbrace{\int \frac{x^2}{\sqrt{a^2-x^2}}\,dx}_{I_{52}}, \quad \text{——} (*)$$

$$(**) \text{——} \; I_{51} = a\int \frac{1}{\sqrt{1-\left(\frac{x}{a}\right)^2}}\,dx = a^2 \arcsin\left(\frac{x}{a}\right) + C_1 \;\Big/\; \text{After substitution } t = \frac{x}{a}$$

$$I_{52} = \int \frac{x^2}{\sqrt{a^2-x^2}}\,dx = \int x * \frac{x}{\sqrt{a^2-x^2}}\,dx$$

$$\begin{cases} u' = \dfrac{x}{\sqrt{a^2-x^2}} \\ v = x \end{cases} \Rightarrow \begin{cases} u = -\sqrt{a^2-x^2} \\ v' = 1 \end{cases}$$

$$\Rightarrow I_{52} = -x\sqrt{a^2-x^2} + \int \sqrt{a^2-x^2}\,dx.$$

$$(***) \text{——} \; I_{52} = -x\sqrt{a^2-x^2} + I_5.$$

from $(*)$, $(**)$ and $(***)$ $\Rightarrow$

$$I_5 = a^2 \arcsin\left(\frac{x}{a}\right) + x\sqrt{a^2-x^2} - I_5 + C.$$

$$\Rightarrow \boxed{I_5 = \frac{a^2}{2}\arcsin\left(\frac{x}{a}\right) + \frac{x}{2}\sqrt{a^2-x^2} + C}$$

• $I_6 = \int \cos^2(x)\,dx \Rightarrow \begin{cases} u' = 1 \\ v = \cos^2(x) \end{cases} \Rightarrow \begin{cases} u = x \\ v' = -2\sin(x)\cos(x) \end{cases}$

$$\Rightarrow I_6 = x\cos^2(x) + 2\int x\sin(x)\cos(x)\,dx$$

this choice coplicate the calculation of I_6.

So, before calculation of I_6 let consider the substitution

$$t = \sin(x) \Rightarrow dt = \cos(t)\,dt$$

$$I_6 = \int \cos^2(x)\,dx = \int \cos(x)\cos(x)\,dx = \int \sqrt{1-\sin^2(x)}\,\cos(x)\,dx$$
$$= \int \sqrt{1-t^2}\,dt = I_5 \text{ with } a = 1.$$

Ex04

$$I_1 = \int \frac{x+1}{x^2+x+2}dx = \frac{1}{2}\int \frac{2x+1+1}{x^2+x+2}dx = \frac{1}{2}\int \frac{2x+1}{x^2+x+2}dx + \frac{1}{2}\int \frac{1}{x^2+x+2}dx$$

$$= \frac{1}{2}\ln(x^2+x+2) + \frac{1}{2}\underbrace{\int \frac{1}{x^2+x+2}dx}_{J}$$

we note that $\Delta = 1^2 - 4\times 2 < 0 \Rightarrow$ we must write $x^2+x+2 = (x+a)^2 + b^2$.

$$x^2+x+2 = x^2 + 2\times\frac{1}{2}x + \frac{1}{4} - \frac{1}{4} + 2$$

$$= \left(x+\frac{1}{2}\right)^2 + \frac{7}{4}$$

$$J = \int \frac{1}{(x+\frac{1}{2})^2 + (\frac{\sqrt{7}}{2})^2}dx = \frac{2}{\sqrt{7}}\arctan\left(\left(x+\frac{1}{2}\right)\frac{2}{\sqrt{7}}\right)$$

thus $I = \frac{1}{2}\ln(x^2+x+2) + \frac{\sqrt{7}}{7}\arctan\left(\left(x+\frac{1}{2}\right)\times\frac{2\sqrt{7}}{7}\right) + C.$

$I_2 = \int \frac{x+1}{x^2+x-2}dx$, we note that $\Delta = 1+4\times 2 = 9 > 0$.

$$\Rightarrow x_1 = \frac{-1-3}{2} = -2$$

$$x_2 = \frac{-1+3}{2} = 1$$

So $x^2+x-2 = (x+2)(x-1) \Rightarrow \frac{x+1}{x^2+x-2} = \frac{x+1}{(x+2)(x-1)}$

$$\Rightarrow \frac{x+1}{(x+2)(x-1)} = \frac{a}{x+2} + \frac{b}{x-1} \Rightarrow \left\{ \frac{x+1}{x-1} = a + \frac{b(x+2)}{x-1} \right.$$

if we take $x = -2 \Rightarrow a = \frac{1}{3}$

$\Rightarrow \frac{x+1}{x+2} = \frac{a(x-1)}{x+2} + b \Rightarrow$ if we put $x = 1$ then

$$b = \frac{2}{3}.$$

So $I_2 = \int \frac{x+1}{x^2+x-2}dx = \int \frac{1/3}{(x+2)}dx + \int \frac{2/3}{x-1}dx$

$$I_2 = \frac{1}{3}\ln(|x+2|) + \frac{2}{3}\ln(|x-1|) + C.$$

Faculté : FSESNV
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Module : Programmation Linéaire

Interrogation N°01

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$$I_1 = \int \frac{x+1}{x^2+x+2}\,dx = \frac{1}{2}\int \frac{2x+1+1}{x^2+x+2}\,dx$$

$$= \frac{1}{2}\int \frac{(x^2+x+2)'}{x^2+x+2}\,dx + \frac{1}{2}\int \frac{1}{x^2+x+2}\,dx$$

$$= \frac{1}{2}\ln(|x^2+x+2|) + \frac{1}{2} I_{11}. \quad \text{———} (*)$$

$I_{11} = ?$ we have $\Delta = 1-8 < 0 \Rightarrow x^2+x+2 = (x+a)^2 + k^2$.

$$x^2+x+2 = x^2 + 2\times\frac{1}{2}x + \frac{1}{4} - \frac{1}{4} + 2.$$

$$= \left(x+\frac{1}{2}\right)^2 + \left(\sqrt{\frac{7}{4}}\right)^2$$

$$= \left(x+\frac{1}{2}\right)^2 + \left(\frac{\sqrt{7}}{2}\right)^2$$

thus:

$$I_{11} = \int \frac{1}{\left(x+\frac{1}{2}\right)^2 + \left(\frac{\sqrt{7}}{2}\right)^2}\,dx = \frac{1}{\frac{\sqrt{7}}{2}} \arctan\left(\frac{x+\frac{1}{2}}{\frac{\sqrt{7}}{2}}\right) \quad \text{———} (**)$$

Recall that: $\int \frac{1}{x^2+a^2}\,dx = \frac{1}{a}\arctan(x/a) + c.$

from ① and ② we deduce that:

$$I = \frac{1}{2}\ln(x^2+x+2) + \frac{1}{2}\times\frac{2}{\sqrt{7}}\arctan\left(\frac{2x+1}{\sqrt{7}}\right) + c$$

$$\boxed{I = \ln\left(\sqrt{x^2+x+2}\right) + \frac{\sqrt{7}}{7}\arctan\left(\frac{2x+1}{\sqrt{7}}\right) + c.}$$

$$I_2 = \int \frac{x+1}{x^2+x-2}\,dx = ?$$

$x^2+x-2 = 0 \Rightarrow$ As $\Delta = 9 \Rightarrow \begin{cases} x_1 = \frac{-1-3}{2} = -2 \\ x_2 = \frac{-1+3}{2} = 1 \end{cases}$

$\Rightarrow x^2+x-2 = (x+2)(x-1)$

$$= \int \frac{x+1}{(x+2)(x-1)}\,dx$$

$$= \int \frac{a}{x+2} + \frac{b}{x-1}\,dx$$

$a = ?,\ b = ?$

$\left(\frac{x+1}{(x-1)(x+2)} = \frac{b}{x-1} + \frac{a}{x+2}\right) * (x-1)$

$\Rightarrow \frac{x+1}{x+2} = b + \frac{a(x-1)}{x+2}$

so for $x=1$ we have : $\boxed{\frac{2}{3} = b}$

$\left(\frac{x+1}{(x-1)(x+2)} = \frac{b}{x-1} + \frac{a}{x+2}\right) * (x+2)$

$\Rightarrow \frac{x+1}{x-1} = \frac{b(x+2)}{x-1} + a.$

so for $x=-2$ we have $\boxed{a = -\frac{1}{3}}$

hence,

$I_2 = \int \frac{x+1}{(x-1)(x+2)}\,dx = \int \frac{2/3}{x-1} - \frac{1}{3}\frac{1}{(x+2)}\,dx$

$= \frac{2}{3}\ln(|x-1|) - \frac{1}{3}\ln(|x+2|) + C$ ✓

$$\boxed{I_2 = \frac{1}{3}\ln(|x-1|) + \frac{1}{3}\ln\left(\left|\frac{x-1}{x+2}\right|\right) + C.}$$

$I_3 = \int \frac{x+1}{(x-1)(x-2)}\,dx = \int \frac{a}{x-1} + \frac{b}{x-2}\,dx.$

$\frac{x+1}{(x-1)(x-2)} = \frac{a}{x-1} + \frac{b}{x-2}. \Rightarrow \frac{x+1}{x-2} = a + \frac{b(x-1)}{x-2}.$

$\xRightarrow{x=1} \boxed{a=-2}$

$\frac{x+1}{(x-1)(x-2)} = \frac{a}{x-1} + \frac{b}{x-2} \Rightarrow \frac{x+1}{x-1} = \frac{a(x-2)}{x-1} + b.$

$\xRightarrow{x=2} \boxed{b=3}$

hence $I_3 = \int \frac{-2}{x-1} + \frac{3}{x-2}\,dx = -2\ln(|x-1|) + 3\ln(|x-2|) + C$ ✓

$$\boxed{I_3 = \ln\left(\frac{|x-2|}{|x-1|}\right)^2 + \ln(|x-2|) + C.}$$

$I_4 = \int \frac{x+1}{(x-1)^2(x-2)}\,dx = \int \frac{a}{x-1} + \frac{b}{(x-1)^2} + \frac{c}{x-2}\,dx.$

$\frac{x+1}{(x-1)^2(x-2)} = \frac{a}{x-1} + \frac{b}{(x-1)^2} + \frac{c}{x-2}$ —— (*)

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$\Rightarrow \frac{x+1}{(x-2)} = a(x-1) + b + \frac{c(x-1)^2}{x-2} \xrightarrow{x=1} \boxed{b=-2}$

$\Rightarrow \frac{x+1}{(x-1)^2} = \frac{a(x-2)}{(x-1)} + \frac{b(x-2)}{(x-1)^2} + c \xrightarrow{x=2} \boxed{c=3}$

if we put $x=0$ in $(*)$ we obtain $-\frac{1}{2} = -a + b - \frac{c}{2}$

$\Rightarrow \boxed{a=-3}$

So

$$I_4 = \int \frac{-3}{x-1} - \frac{2}{(x-1)^2} + \frac{3}{x-2}\, dx.$$

$$= -3\ln(|x-1|) + \frac{2}{(x-1)} + 3\ln(|x-2|) + C.$$

$$\boxed{I_4 = 3\ln\left(\left|\frac{x-2}{x-1}\right|\right) + \frac{2}{x-1} + C.}$$

$$I_5 = \int \frac{x^3+2x^2+3x-1}{(x^2+1)(x^2-4)}\, dx = \int \frac{x^3+2x^2+3x-1}{(x^2+1)(x-2)(x+2)}\, dx.$$

$$= \int \frac{ax+b}{x^2+1} + \frac{c}{x-2} + \frac{d}{x+2}\, dx.$$

$$\frac{x^3+2x^2+3x-1}{(x^2+1)(x^2-4)} = \frac{ax+b}{x^2+1} + \frac{c}{x-2} + \frac{d}{x+2}$$

$\Rightarrow \frac{(x^3+2x^2+3x-1)}{(x^2+1)(x+2)} = \frac{(ax+b)(x-2)}{x^2+1} + c + \frac{d(x-2)}{x+2}$

$\xrightarrow{x=2} \boxed{c = \frac{21}{20}}$

$\Rightarrow \frac{x^3+2x+3x-1}{(x^2+1)(x-2)} = \frac{(ax+b)(x+2)}{x^2+1} + \frac{c(x+2)}{(x-2)} + d.$

$\xrightarrow{x=-2} d = \frac{7}{20}$

$\xrightarrow{x=0} \frac{-1}{4} = b - \frac{c}{2} + \frac{d}{2} \Rightarrow \boxed{b = \frac{1}{10}}$

$$\Rightarrow \lim_{x \to \infty} x \left(\frac{x^3+2x^2+3x-1}{(x^2+1)(x^2-4} \right) = a+c+d$$

$$\Rightarrow a+c+d=1 \Rightarrow a = 1-c-d = \boxed{-\tfrac{2}{5}=a}$$

so $$I_5 = \int \frac{-\frac{2}{5}x+\frac{1}{10}}{x^2+1} + \frac{21/20}{x-2} + \frac{7/20}{x+2}\,dx.$$

$$= -\frac{1}{5}\int \frac{2x}{x^2+1}dx + \frac{1}{10}\int \frac{1}{x^2+1}dx + \frac{21}{20}\int \frac{1}{x-2}dx + \frac{7}{20}\int \frac{1}{x+2}dx$$

$$-\frac{1}{5}\ln(x^2+1) + \frac{1}{10}\arctan(x) + \frac{21}{20}\ln(|x-2|) + \frac{7}{20}\ln(|x+2|) + C$$

$$I_6 = \int \frac{x^5+x^4-8}{x^2(x-4)}dx = ?$$

$$\frac{x^5+x^4-8}{x^2(x-4)} = ax^2+bx+c + \frac{d}{x} + \frac{e}{x^2} + \frac{f}{x-4}.$$

by Ecludian division we obtain the exact expression of ax^2+bx+c.

$$\begin{array}{r|l} x^5+x^4-8 & x^3-4x \\ \hline \vdots & x^2+x+4 \\ 4x^2+16x-8 & \end{array}$$

$$\Rightarrow \frac{x^5+x^4-8}{x^2(x-8)} = x^2+x+4 + \frac{4(x^2+4x-2)}{x^2(x-4)}.$$

$$\Rightarrow I_6 = \int x^2+x+4\,dx + 4\int \frac{x^2+4x-2}{x^2(x-4)}dx.$$

$$= \frac{1}{3}x^3+\frac{1}{2}x^2+4x + 4\left[\int \frac{d}{x}dx + \int \frac{e}{x^2}dx + \int \frac{f}{x-4}dx\right]$$

$$= \frac{1}{3}x^3+\frac{1}{2}x^2+4x + 4\ln(|x|) - \frac{e}{x} + f\ln(|x-4|) + C$$

$$\Rightarrow \frac{4(x^2+4x-2)}{x^2(x-4)} = \frac{d}{x} + \frac{e}{x^2} + \frac{f}{x-4}$$

$$\begin{cases} \Rightarrow \dfrac{4(x^2+4x-2)}{x-4} = dx + e + \dfrac{f x^2}{x-4} \\ \xrightarrow{x=0} \boxed{2 = e} \end{cases}$$

$$\begin{cases} \dfrac{4(x^2+4x-2)}{x^2} = \dfrac{d(x-4)}{x} + \dfrac{e(x-4)}{x^2} + f \\ \xrightarrow{x=4} \boxed{f = \frac{15}{2}} \end{cases}$$

$$\xrightarrow{x=1} -4 = d + e - \frac{f}{3} \Rightarrow d = -4 - e + f/3$$

$$\Rightarrow \boxed{d = \frac{3}{2}}$$

$$I_6 = \tfrac{1}{3}x^3 + \tfrac{1}{2}x^2 + 4x + \tfrac{3}{2}\ln(|x|) - \frac{2}{x} + \frac{15}{2}\ln(|x-4|) + C.$$

$$I_7 = \int \frac{x+1}{\sqrt{(x-1)(x-2)}}\,dx = \int \frac{x+1}{\sqrt{x^2-3x+2}}\,dx = \int \frac{2x-3}{2\sqrt{x^2-3x+2}}\,dx + \int \frac{5}{\sqrt{x^2-3x+2}}\,dx$$

$$= \int \frac{(x^2-3x+2)'}{2\sqrt{x^2-3x+2}}\,dx + 5\int \frac{1}{(x-\frac{3}{2})^2 - (\frac{1}{2})^2}\,dx$$

$$= \sqrt{x^2-3x+2} + 5\ln\left(|(x-\tfrac{3}{2})^2 - \sqrt{(x-\tfrac{3}{2})^2 - (\tfrac{1}{2})^2}|\right) + C$$

Recall that: $\int \frac{1}{\sqrt{x^2 \pm a^2}}\,dx = \ln\left(x + \sqrt{x^2 \pm a^2}\right) + C.$

$$I_8 = \int \frac{x+1}{\sqrt{(1-x)(x-2)}}\,dx = \int \frac{x+1}{\sqrt{-(x-1)(x-2)}}\,dx.$$

$$= -\int \frac{-2x+3}{2\sqrt{f(x)}}\,dx + 5\int \frac{1}{\sqrt{(\frac{1}{2})^2 - (x-\frac{3}{2})^2}}\,dx$$

$$= -\sqrt{f(x)} + 5\arcsin\left(\frac{x-3/2}{1/2}\right) + C \quad 5$$

$$I_9 = \int \frac{x^3+2x^2+3x-1}{x^2+2x+1}\,dx = \int \frac{P_3(x)}{P_2(x)}\,dx = \int ax+b+\frac{P_1(x)}{P_2(x)}\,dx.$$

Using the euclidian division we get

$$\frac{x^3+2x^2+3x-1}{x^2+2x+1} = x + \frac{2x-1}{x^2+2x+1} = x + \frac{2x+2}{x^2+2x+1} - \frac{3}{(x+1)^2}$$

So, $$I_9 = \int x + \frac{2x+2}{x^2+2x+1} - \frac{3}{(x+1)^2}\,dx$$

$$= \frac{1}{2}x^2 + \ln(x^2+2x+1) + \frac{3}{x+1} + C. \quad / C \in \mathbb{R}.$$

$$I_{10} = \int \frac{1}{2\cos^2(x)+\cos(x)\sin(x)+\sin^2(x)}\,dx = \int \frac{1}{2+\frac{\sin(x)}{\cos(x)}+\left(\frac{\sin(x)}{\cos(x)}\right)^2}\frac{dx}{\cos^2(x)}$$

$$= \int \frac{1}{2+\tan(x)+(\tan(x))^2} * \frac{dx}{\cos^2(x)}.$$ we put $t = \tan(x)$

$$\Rightarrow dt = \frac{dx}{\cos^2(x)}.$$

So, $$I_{10} = \int \frac{1}{t^2+t+2}\,dt = \int \frac{1}{(t+\frac{1}{2})^2+\left(\frac{\sqrt{7}}{2}\right)^2}\,dt$$

$$= \frac{2}{\sqrt{7}}\arctan\left((x+\tfrac{1}{2}) * \frac{2}{\sqrt{7}}\right) + C$$

$I_{11} = \int \ln(x^2+2x-3)\,dx$ using the Integration by part

$$\begin{cases} u' = 1 \\ v = \ln(x^2+2x-3) \end{cases} \Rightarrow \begin{cases} u = x \\ v' = \frac{2x+2}{x^2+2x-3} \end{cases}$$

$$I_{10} = x\ln(x^2+2x-3) - \underbrace{\int \frac{2x^2+2x}{x^2+2x-3}}_{I_{10.1}}$$

$$I_{10.1} = \int 2 + \frac{-2x+6}{x^2+2x-3}\,dx = \int 2 + \frac{-2x-2}{x^2+2x-3} + \frac{8}{x^2+2x-3}\,dx.$$

we have: $\frac{8}{x^2+2x-3} = \frac{8}{(x+3)(x-1)} = \frac{a}{x+3} + \frac{b}{x-1} = \frac{-2}{x+3} + \frac{2}{x-1}$

$$\Rightarrow I_{10.1} = 2x + \ln(|x^2+2x-3|) + 2\ln\left(\left|\frac{x-1}{x+3}\right|\right) + C.$$

thus:

$$I_{10} = (x+1)\ln(x^2+2x-3) + 2\ln\left(\left|\frac{x-1}{x+3}\right|\right) + 2x + C.$$

$I_{11} = \int \frac{2x^{1/2}+3x^{1/4}}{1+x^{1/4}}\,dx$ we put $y^4 = x \Rightarrow 4y^3dy = dx$.

$\Rightarrow y = x^{\frac{1}{4}}$. because $lcm(2,4) = 4$

$$I_{11} = \int \frac{2y^2+3y}{1+y} 4y^3 dy = 4\int \frac{2y^5+3y^4}{1+y}dy = 4\int \frac{P_5(y)}{P_1(y)}dy.$$

$$= 4\int ay^4+by^3+cy^2+dy+e+\frac{f}{1+y}\,dy.$$

by euclidian division we get

$$I_{11} = 4\int 2y^4+y^3-y^2+y-1+\frac{1}{y+1}\,dy.$$

$$= \frac{8}{5}y^5 + y^4 - \frac{4}{3}y^3 + 2y^2 - 4y + 4\ln(|y+1|) + C.$$

$$= \frac{8}{5}x^{5/4} + x - \frac{4}{3}x^{3/4} + 2x^{1/2} + 4\ln(|x^{1/4}+1|) + C.$$

(I) $I_{12} = \int \frac{2x^{1/2}+3x^{1/3}}{1+x^{1/3}}\,dx$ we put $y^6 = x \Rightarrow 6y^5dy = dX$.

(*) $y = x^{1/6}$. because $lcm(2,3) = 6$.

$$= \int \frac{2y^3+3y^2}{1+y^2} 6y^5 dy.$$

$$= \int \frac{12y^8+18y^7}{1+y^2}dy = \int \underbrace{12y^6+18y^5-12y^4-18y^3+12y^2+18y-12}_{I_{12.1}} + \underbrace{\frac{-18y+12}{1+y^2}}_{I_{12.2}}\,dy$$

$$I_{12.1} = \frac{12}{7}y^7 + \frac{18}{6}y^6 - \frac{12}{5}y^5 - \frac{18}{4}y^4 + 4y^3 + 9y^2 - 12y \longrightarrow (**)$$

$$I_{12.2} = \int \frac{-18y\,dy}{1+y^2} + 12\int \frac{1}{1+y^2}dy.$$

$$= -9\ln(1+y^2) + 12\arctan(1+y^2). \quad (***)$$

by sobsstitution of $(*)$, $(**)$ and $(***)$ in (I) we get I_{12}.

$$I_{14} = \int \frac{dx}{\sqrt{\sqrt{x}-2}} = ? \quad \text{we put } \sqrt{x}-2 = y^2 \Rightarrow \sqrt{x} = y^2+2$$

$$\Rightarrow x = (y^2+2)^2$$

$$\Rightarrow dx = 4y(y^2+2)dy$$

$$= \int \frac{4y(y^2+2)dy}{\sqrt{y^2}}$$

$$= \int 4y^2+8\,dy = \frac{4}{3}y^3 + 8y + C \quad \text{with } y = \sqrt{\sqrt{x}-2}$$

$$I_{13} = \int (5x-1)^{\frac{1}{3}}dx = \frac{1}{5}\int (5x-1)'(5x-1)^{\frac{1}{3}}dx.$$

$$= \frac{3}{5\times 4}\int \frac{4}{3}(5x-1)'(5x-1)^{\frac{4}{3}-1}dx.$$

$$= \frac{3}{20}\int n f' f^{n-1}dx = \frac{4}{15}(5x-1)^{4/3} + C$$

Or, we put $y^3 = 5x-1$.

$$\Rightarrow 3y^2dy = 5dx. \Rightarrow dx = \frac{3}{5}y^2dy.$$

So, $I_{13} = \int y * \frac{3}{5}y^2dy = \int \frac{3}{5}y^3dy = \frac{3}{20}y^4 + C$

with $y = (5x-1)^{\frac{1}{3}}$.

$$I_{15} = \int \left(\frac{x+1}{x-1}\right)^{\frac{1}{3}}dx, \quad \text{we put } y^3 = \frac{x+1}{x-1} \Rightarrow x = \frac{y^3+1}{y^3-1}$$

$$= \int \frac{-6y^3}{(y^3-1)^2}dy = \ldots \qquad \Rightarrow dx = \frac{-6y^2}{(y^3-1)^2}dy$$

Exo 1:

$$I_1 = \int_0^{+\infty} e^{-x}\,dx = -e^{-x}\Big|_0^{+\infty} = \lim_{x\to+\infty}(-e^{-x}) - (-e^{0}) = 1 \quad -(*)$$

$I_2 = \int_0^{+\infty} x^2 e^{-x}\,dx$ we can compute the integral by two ways namely: using the gamma function or direct computing

- $I_2 = \Gamma(2) = 2! = 2.$

$$\Rightarrow \int x^2 e^{-x}\,dx = -x^2 e^{-x} + \int 2x\,e^{-x}\,dx = -x^2 e^{-x} - 2x e^{-x} + 2\int_0 e^{-x}\,dx.$$

$u = x^2,\ v' = e^{-x} \Rightarrow \begin{cases} u' = 2x \\ v = -e^{-x} \end{cases}$ | $u = 2x,\ v' = e^{-x} \Rightarrow \begin{cases} u' = 2 \\ v = -e^{-x} \end{cases}$

from (*) we have $\int_0^{+\infty} e^{-x} = 1$ so

$$I_2 = \int_0^{+\infty} x^2 e^{-x}\,dx = -x^2 e^{-x}\Big|_0^{+\infty} - 2xe^{-x}\Big|_0^{+\infty} + 2.$$

$$= \Big[\lim_{x\to+\infty}(-x^2 e^{-x}) + 0\Big] - 2\Big[\lim_{x\to+\infty} x e^{-x} + 0\Big] + 2.$$

$$\boxed{I_2 = 2.}$$

- $I_3 = \int_1^e \ln(x)\,dx = x\ln(x)\Big|_1^e - \int_1^e 1\,dx = e\ln(e) - 1\ln(1) - x\Big|_1^e$

$u' = 1,\ v = \ln(x) \Rightarrow \begin{cases} u = x \\ v' = \frac{1}{x} \end{cases}$

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$$\Rightarrow I_3 = \cancel{e} - \cancel{e} + 1 = \boxed{1 = I_3}$$

$$I_4 = \int_e^{2e} x^2 \ln(x) = \tfrac{1}{3}x^3\ln(x)\Big|_e^{2e} - \int_e^{2e} \tfrac{1}{3}x^2\,dx = \tfrac{1}{3}x^3\ln(x) - \tfrac{1}{9}x^3\Big|_e^{2e}$$

$u' = x^2,\ v = \ln(x) \Rightarrow \begin{cases} u = \frac{1}{3}x^3 \\ v' = \frac{1}{x}. \end{cases}$

$$I_4 = \frac{1}{3} x^3 \left(\ln(x) - \frac{1}{3}\right)\Big|_e^{2e}$$

$$= \frac{8}{3} e^3 \left(\ln(2e) - \frac{1}{3}\right) - \frac{e^3}{3}\left(\ln(e) - \frac{1}{3}\right)$$

$$= \frac{e^3}{3}\left[8\ln(2e) - \frac{8}{3} - \ln(e) + \frac{1}{3}\right]$$

$$= \frac{e^3}{3}\left[8\ln(2e) - \frac{10}{3}\right] = \frac{e^3}{3}\left(8\ln(2) + \frac{14}{3}\right).$$

- $I_5 = \int_0^{2\pi} \sin(x)\cos^2(x)\,dx$

we put: $t = \cos(x) \Rightarrow dt = -\sin(x)\,dx$.

$x = 0 \Rightarrow t = \cos(0) = 1.$

$x = 2\pi \Rightarrow t = \cos(2\pi) = 1$

So, $I_5 = -\int_1^1 (1-t^2)\,dt = 0$. / As $\int_a^a f(x)\,dx = 0$ if f is continous.

$$I_5 = -\int_1^1 1 - t^2\,dt =$$

$$= -t + \frac{1}{3}t^3\Big|_1^1 = \left(-1 + \frac{1}{3}\right) - \left(-1 + \frac{1}{3}\right) = 0$$

$I_6 = \int_0^{\pi/2} \sin(x)\,e^x\,dx.$ we compute the integrale by parts.

$$\begin{cases} u = \sin(x) \\ v' = e^x \end{cases} \Rightarrow \begin{cases} u' = \cos(x) \\ v = e^x \end{cases}$$

$$I_6 = \sin(x)e^x\Big|_0^{\pi/2} - \int_0^{\pi/2} \cos(x)\,e^x = e^{\pi/2} - \cos(x)e^x\Big|_0^{\pi/2} - I_6 \quad (**)$$

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$$\begin{matrix} u = \cos(x) \\ v' = e^x \end{matrix} \Rightarrow \begin{cases} u' = -\sin(x) \\ v = e^x \end{cases}$$

from $(**)$ we will have $I_6 = \frac{1}{2}e^{\pi/2} - \frac{1}{2}(0-1) = \frac{1}{2}(e^{\pi/2}+1)$

Ex06:

$$I_n = \int_1^{+\infty} x^n f(x)\,dx = \int_1^{+\infty} \alpha\, x^{n-\alpha-1}\,dx.$$

1st case: $\begin{cases} n-\alpha-1 = -1 \\ n = \alpha \end{cases}$ in this case:

$$I_n = \alpha \ln(x)\Big|_1^{+\infty} = \alpha \lim_{x\to\infty} \cancel{(\ln(x))}^{+\infty} - \alpha. = \{+\infty.$$

2nd case: $n \neq \alpha$. so,

$$I_n = \frac{\alpha}{n-\alpha}\, x^{n-\alpha}\Big|_1^{+\infty}$$

In this case we distinct two sub-cases:

2.1 $n-\alpha > 0$ ie $n > \alpha$. so

$$I_n = \frac{\alpha}{n-\alpha} \cancel{\lim_{x\to\infty} x^{n-\alpha}}^{\to+\infty} - \frac{\alpha}{n-\alpha} = +\infty$$

2.2 $n < \alpha$. so

$$I_n = \frac{\alpha}{n-\alpha} \cancel{\lim_{x\to+\infty} x^{n-\alpha}}^{0} - \frac{\alpha}{n-\alpha} = \boxed{\frac{+\alpha}{\alpha-n} = I_n}$$