1st order diff equ.

Exo 1:

- $y' - 2xy = 2x \Rightarrow y' = 2x + 2xy$
$\Rightarrow y' = 2x(1+y)$.
$\Rightarrow \frac{y'}{1+y} = 2x$ (separated variables)
$\Rightarrow \frac{dy}{1+y} = 2x\,dx$.
$\Rightarrow \ln(|1+y|) = x^2 + C \Rightarrow |1+y| = Ke^{x^2}$.
$\Rightarrow \boxed{y = \pm K e^{x^2} - 1}$

- $y' = xy^2 - x - y^2 + 1 \Rightarrow y' = y^2(x-1) - (x-1)$.
$\Rightarrow y' = (y^2-1)(x-1)$.
$\Rightarrow \frac{y'}{y^2-1} = x-1 \Rightarrow \frac{dy}{y^2-1} = (x-1)\,dx$.
$\Rightarrow \int \frac{1}{y^2-1}\,dy = \int x-1\,dx$.
$\Rightarrow \int \frac{a}{y-1} + \frac{b}{y+1} = \frac{1}{2}x^2 - x + C$ / $a =$ $b =$
$\Rightarrow \frac{1}{2}\int \frac{1}{y-1} + \frac{1}{y+1}\,dy = \frac{1}{2}x^2 - x + C$
$\Rightarrow \ln(|y^2-1|) = x^2 - 2x + C$
$\Rightarrow |y^2-1| = e^{x^2-2x+C}$.
$y^2 - 1 = \begin{cases} e^{x^2-2x+C} & \text{if } y \in [-1,1] \\ -e^{x^2-2x+C} & \text{if } y \notin [-1,1] \end{cases}$
$\Rightarrow y^2 = \begin{cases} e^{P_2}+1 \\ e^{P_2}+1 \end{cases}$

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- $y' = \frac{y^2+1}{xy+y} \Rightarrow y' = \frac{y^2+1}{y} \times \frac{1}{x+1} \Rightarrow \frac{y}{y^2+1} y' = \frac{1}{x+1}$

$\Rightarrow \frac{1}{2}\int \frac{2y}{y^2+1}\,dy = \int \frac{1}{x+1}\,dx.$

$\Rightarrow \frac{1}{2}\ln(y^2+1) = \ln(|x+1|) + C.$

$\Rightarrow \ln(y^2+1) = 2\ln(|x+1|) + C.$

$\Rightarrow y^2+1 = K(x+1)^2.$

$\Rightarrow y^2 = K(x+1)^2 - 1 \Rightarrow \boxed{y = \pm\sqrt{|K(x+1)^2-1|}}$

- $(2xy^3+4x)y' = xy^2+y^2 \Rightarrow 2x(y^3+2)y' = y^2(x+1).$

$\Rightarrow \frac{y^3+2}{y^2} y' = \frac{x+1}{2x} \Rightarrow \left(y+\frac{2}{y^2}\right)y' = \frac{1}{2} + \frac{1}{2x}.$

$\Rightarrow \int y + \frac{2}{y^2}\,dy = \int \frac{1}{2} + \frac{1}{2x}\,dx.$

$\Rightarrow \boxed{\frac{1}{2}y^2 - \frac{2}{y} = \frac{1}{2}x + \ln(\sqrt{|x|}) + C.}$

- $y' = \frac{e^{x-y}}{1+e^x} \Rightarrow y' = \frac{e^x}{1+e^x} e^{-y} \Rightarrow e^y y' = \frac{e^x}{1+e^x}dx$

$\Rightarrow \int e^y dy = \int \frac{e^x}{1+e^x}dx \Rightarrow e^y = \int \frac{1}{1+t}dt \,/\, t = e^x.$

$\Rightarrow e^y = \ln(1+t) + C. \Rightarrow y = \ln(|\ln(1+t)| + C)$

we have $y(1) = 0$, $x = 1 \Rightarrow t = e$

$0 = \ln(|\ln(1+e)| + C) \Rightarrow 1 = \ln(1+e) + C$

$\Rightarrow \boxed{C = 1 - \ln(1+e)}$

$y = \ln(|\ln(1+t) + 1 - \ln(1+e)|).$

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- $y' = \frac{x^2y - y}{1+y} \Rightarrow y' = \frac{y}{1+y} * (x^2-1)$

$$\Rightarrow \left(\frac{1+y}{y}\right) y' = x^2 - 1 \Rightarrow \int \frac{1}{y} + 1 \, dy = \int x^2 - 1 \, dx.$$

$$\Rightarrow \ln(|y|) + y = \tfrac{1}{3}x^3 - x + C.$$

we have $y(3) = 1$ so $\underbrace{\ln(1)}_{=0} + 1 = 9 - 3 + C.$

$$\Rightarrow \boxed{C = -5}$$

<u>EX02</u>:

- $xy' - 2y = x \Rightarrow \begin{cases} xy_p' - 2y_p = 0 \\ y \end{cases}$

$xy' - 2y = 0 \Rightarrow xy' = 2y \Rightarrow \frac{y'}{y} = 2/x.$

$$\Rightarrow \ln(|y|) = \ln(x^2) + C.$$

$$\Rightarrow \boxed{y = K x^2}$$

Let put $K \equiv K(x)$ so $y = K(x)x^2$ and $y' = K'(x)x^2 + 2xK(x).$

thus, $x\left(K'x^2 + 2xK(x)\right) - 2Kx^2 = x.$

$$K'(x)x^3 + \cancel{2x^2K} - \cancel{2Kx^2} = x \Rightarrow K'(x) = \frac{1}{x^2}$$

$$\Rightarrow \boxed{K(x) = -\frac{1}{x} + C.}$$ consequently

$$y_G = \left(-\frac{1}{x} + C\right) * x^2 \text{ ie } \boxed{y_G = -x + Cx^2}$$

- $xy' + 2y = \frac{\cos(x)}{x}.$

$xy' + 2y = 0 \Rightarrow xy' = -2y$

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$$\Rightarrow \frac{y'}{y} = -\frac{2}{x} \Rightarrow \ln(y) = \ln\left(\frac{1}{x^2}\right) + c.$$

$$\Rightarrow \boxed{y = \frac{k}{x^2}} \quad \text{with } k = e^c \ / c \in \mathbb{R}.$$

Let $y = \frac{k(x)}{x^2} \Rightarrow y' = \frac{k'(x)x^2 - 2xk(x)}{x^4}.$

$$\Rightarrow \cancel{x}\left(\frac{x^2k'(x) - 2xk(x)}{x^4}\right) + 2\frac{k}{x^2} = \frac{\cos(x)}{x}.$$

$$\frac{k'(x)}{\cancel{x}} - \cancel{\frac{2k(x)}{x^2}} + \cancel{\frac{2k(x)}{x^2}} = \frac{\cos(x)}{\cancel{x}}$$

$$\Rightarrow k'(x) = \cos(x). \Rightarrow \boxed{k(x) = \sin(x) + C}$$

so $\boxed{y = \frac{\sin(x) + C}{x^2}}$ $/ C \in \mathbb{R}.$

- $y' + \frac{2y}{x} = \frac{4}{x}. \Rightarrow y' + \frac{2y}{x} = 0$

$$\Rightarrow y' = \frac{-2y}{x} \Rightarrow \frac{y'}{y} = \frac{-2}{x}.$$

$$\Rightarrow \ln(y) = \frac{1}{x^2} + c \Rightarrow \boxed{y = \frac{k}{x^2}} \quad k \in \mathbb{R}_+ \ / k = e^c / c \in \mathbb{R}$$

$$y = \frac{k(x)}{x^2} \Rightarrow y' = \frac{x^2k'(x) - 2xk(x)}{x^4}$$

$$\frac{x^2k'(x) - \cancel{2xk(x)}}{x^4} + \cancel{\frac{2k(x)}{x^3}} = \frac{4}{x}.$$

$$k'(x) = 4x \Rightarrow \boxed{k(x) = 2x^2 + C.}$$

$\Rightarrow \boxed{y = \frac{2x^2 + C}{x^2}.}$ we have $y(1) = 6$ so

$$6 = 2 + C \Rightarrow \boxed{C = 4}$$

$$\boxed{y = \frac{2x^2 + 4}{x^2}}$$

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• $xy' - 2y = x^3 e^x$.

$xy' - 2y = 0 \Rightarrow xy' = 2y$.

$\Rightarrow \frac{y'}{y} = \frac{2}{x} \Rightarrow \ln(y) = \ln(x^2) + C$.

$\Rightarrow \boxed{y = Kx^2}$

we put $K \equiv K(x) \Rightarrow y = K(x)x^2$ and $y' = K'(x)x^2 + 2xK(x)$.

So,

$x(K'(x)x^2 + \cancel{2xK(x)}) - \cancel{2K(x)x^2} = x^3 e^x$.

$x^3 K'(x) = x^3 e^x \Rightarrow K'(x) = e^x \Rightarrow \boxed{K(x) = e^x + C}$ thus.

$\boxed{y = x^2(e^x + C)}$.

we have $y(1) = 0$ so $e + C = 0 \Rightarrow \boxed{C = -e}$

$\Rightarrow \boxed{y = x^2(e^x + e)}$

EXO3:

we say that a diff eq $y' = f(x,y)$ is homogeneous iff: $f(\lambda x, \lambda y) = f(x,y)$ $\forall \lambda \in \mathbb{R}^*$.

• $y' = \frac{x^2 e^{y/x} + y^2}{xy}$, $f(x,y) = \frac{x^2 e^{y/x} + y^2}{xy}$.

$f(\lambda x, \lambda y) = \frac{\lambda^2 x^2 e^{\lambda y/\lambda x} + \lambda^2 y^2}{\lambda^2 xy} = \frac{x^2 e^{y/x} + y^2}{xy} = f(x,y) \Rightarrow$ homogen.

So to solve the equation we put $z = y/x \Rightarrow y = zx$.

$\Rightarrow y' = z'x + z$ after substitution we obtain:

$z'x + z = \frac{e^z + z^2}{z} \Rightarrow xz'z + z^2 = e^z + \cancel{z^2}$

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$$\Rightarrow z e^{-z} z' = \frac{1}{x} \Rightarrow \underbrace{\int z e^{-z} dz}_{\text{by parts}} = \ln(x) + C$$

$$\int z e^{-z} dz = -z e^{-z} + \int e^{-z} dz = -(z+1) e^{-z} + C$$

$$\begin{matrix} u = z \\ v' = e^{-z} \end{matrix} \Rightarrow \begin{cases} u' = 1 \\ v = -e^{-z} \end{cases}$$

so $-(z+1) e^{-z} = \ln(x) + C$

$$\Rightarrow \boxed{-\left(\frac{y}{x} + 1\right) e^{-y/x} = \ln(x) + C}$$

$$y' = \frac{\sqrt{x^2 - y^2} + y}{x} \Rightarrow f(x,y) = \frac{\sqrt{x^2 - y^2} + y}{x}$$

$$f(\lambda x, \lambda y) = \frac{\sqrt{\lambda^2 (x^2 - y^2)} + \lambda y}{\lambda x} = f(x,y) \checkmark$$

we put $\begin{cases} z = y/x \Rightarrow y' = x z' + z \\ y = zx \end{cases}$

$$x z' + z = \frac{\sqrt{x^2 - z^2 x^2} + zx}{x} \Rightarrow \sqrt{}$$

$$\Rightarrow x z' + \cancel{z} = \sqrt{1 - z^2} + \cancel{z}$$

$$\frac{z'}{\sqrt{1 - z^2}} = \frac{1}{x} \Rightarrow \arcsin(z) = \ln(|x|) + C$$

$$\Rightarrow \boxed{z = \sin(\ln(|x|) + C)}$$

$$\Rightarrow \boxed{y = x \sin(\ln(|x|) + C)} \checkmark$$

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• $y' = \frac{x^4+2y^4}{xy^3} \Rightarrow f(x,y) = \frac{x^4+2y^4}{xy^3}$.

$f(\lambda x, \lambda y) = \frac{\lambda^4 x^4 + 2\lambda^4 y^4}{\lambda^4 x y^3} = f(x,y)$

$z = y/x \Rightarrow y = xz \Rightarrow y' = xz' + z$

$\cancel{x} z' + z = \frac{\cancel{x^4} + 2\cancel{x^4} z^4}{\cancel{x^4} z^3} = \frac{1+2z^4}{z^3}$.

$\Rightarrow xz' + \cancel{z} = \frac{1}{z^3} + \cancel{2z} \Rightarrow xz' = \frac{1+z^4}{z^3}$.

$\Rightarrow z' * \frac{z^3}{1+z^4} = \frac{1}{x} \Rightarrow \frac{1}{4} \frac{4z^3}{1+z^4} z' = \frac{1}{x}$.

$\Rightarrow \ln(1+z^4) = \ln(x^4) + C$.

$\Rightarrow 1+z^4 = Kx^4 \Rightarrow z^4 = Kx^4 - 1$.

$\Rightarrow z = \sqrt[4]{|Kx^4-1|}$

$\Rightarrow \boxed{y = x\left(|Kx^4-1|\right)^{\frac{1}{4}}}$.

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