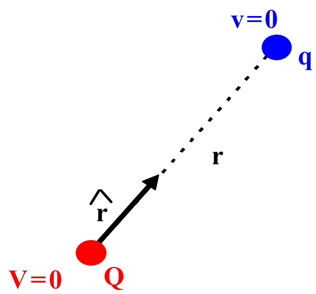
The Electromagnetic Force

The Force Between Two-Charged Particles (at rest):

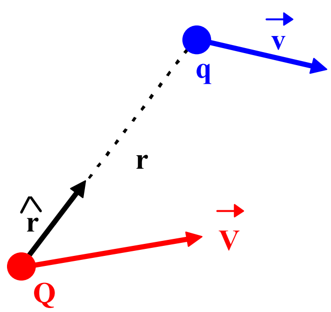


The force between two charged particles **at rest** is the **electrostatic force** and is given by

$$\vec{F}_E = \frac{KQq}{r^2}\hat{r} \quad \textbf{(electrostatic force)},$$

where K = 8.99x10^9 Nm^2/C^2.

The Force Between Two Moving Charged Particles:



The force between two **moving** charged particles is the **electromagnetic force** and is given by

$$\vec{F}_{EM} = \frac{KQq}{r^2}\hat{r} + \frac{KQq}{c^2r^2}\vec{v}\quad\vec{V}\quad\hat{r}$$

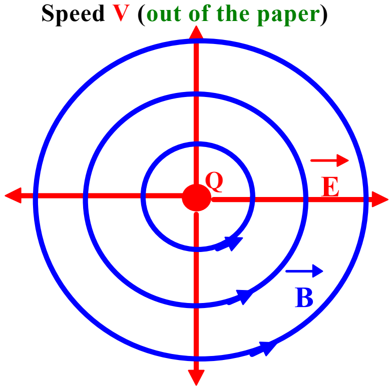
(electromagnetic force)

where K = 8.99x10^9 Nm^2/C^2 and **c = 3x10^8 m/s** (**speed of light in a vacuum**). The first term is the **electric force** and the second (**new**) term is the called the **magnetic force** so that $\vec{F}_{EM} = \vec{F}_E + \vec{F}_B$, with

$$\vec{F}_E = \frac{KQq}{r^2}\hat{r} = q\left(\frac{KQ}{r^2}\right)\hat{r} = q\vec{E}$$

$$\vec{F}_B = \frac{KQq}{c^2r^2}\vec{v}\quad\vec{V}\quad\hat{r} = q\vec{v}\quad\left(\frac{KQ}{c^2r^2}\vec{V}\quad\hat{r}\right) = q\vec{v}\quad\vec{B}$$

Electric and Magnetic Fields of a Charged Particle Q moving with Speed V (out of the paper)



The **electric** and **magnetic fields** due to the particle **Q** are

$$\vec{E} = \frac{KQ}{r^2}\hat{r}$$

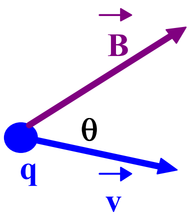
$$\vec{B} = \frac{KQ}{c^2r^2}\vec{V}\quad\hat{r}$$

The **electromagnetic force** on **q** is given by

$$\vec{F}_{EM} = q\vec{E} + q\vec{v}\quad\vec{B} \quad \textbf{(Lorenz Force)}.$$

The Magnetic Force

The Force on Charged Particle in a Magnetic Field:



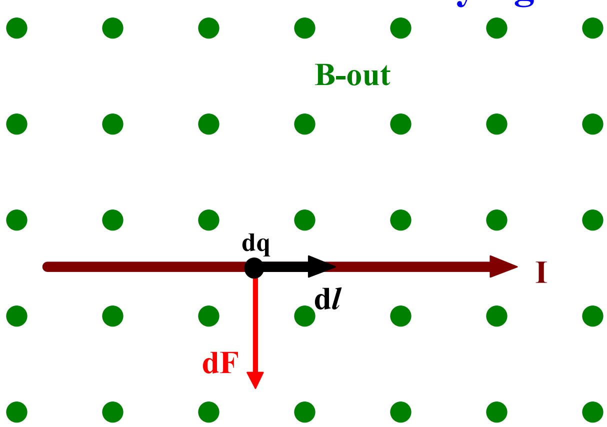
The **magnetic force** an a charged particle **q** in a **magnetic field B** is given by

$$\vec{F}_B = q\vec{v} \quad \vec{B} \ .$$

The magnitude of the magnetic force is **F_B = qvB sinθ** and **$B = F_B/(qv\ \sin\theta)$** is the definition of the magnetic field. (**The units for B are Tesla, T, where 1 T = 1 N/(C m/s)**). The **magnetic force** an infinitesimal charged particle **dq** in a **magnetic field B** is given by

$$d\vec{F}_B = dq\vec{v} \quad \vec{B} \ .$$

The Force on Wire Carrying a Current in a Magnetic Field:



A current in a wire corresponds to moving charged particles with **I = dq/dt**. The magnetic force on the charge **dq** is

$$d\vec{F}_B = dq\vec{v} \quad \vec{B} \ ,$$

and the speed **v=d*l*/dt**. Hence,

$$dq\vec{v} = dq\frac{d\vec{l}}{dt} = Id\vec{l} \ ,$$

and the **magnetic force** on a infinitesimal length **d*l*** of the wire becomes $d\vec{F}_B = Id\vec{l} \quad \vec{B}$. The total **magnetic force** on the **wire** is

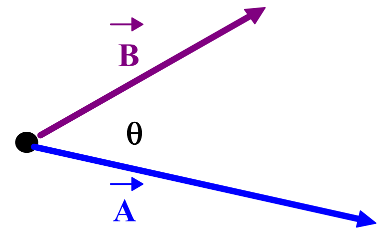
$$\vec{F}_B = \int d\vec{F}_B = \int Id\vec{l} \quad \vec{B} \ ,$$

which for a **straight wire of length L** in a **uniform magnetic field** becomes

$$\vec{F}_B = I\vec{L} \quad \vec{B} \ .$$

Vector Multiplication: Dot & Cross

Two Vectors:



Define two vectors according to

$$\vec{A} = A_x\hat{x} + A_y\hat{y} + A_z\hat{z}$$
$$\vec{B} = B_x\hat{x} + B_y\hat{y} + B_z\hat{z}\ .$$

The magnitudes of the vectors is given by

$$\left|\vec{A}\right| = A = \sqrt{A_x^2 + A_y^2 + A_z^2}$$
$$\left|\vec{B}\right| = B = \sqrt{B_x^2 + B_y^2 + B_z^2}$$

Dot Product (Scalar Product):

The dot product, S, is a **scalar** and is given by

$$S = \vec{A}\cdot\vec{B} = \left|\vec{A}\right|\left|\vec{B}\right|\cos\theta = A_xB_x + A_yB_y + A_zB_z$$

Cross Product (Vector Product):

The cross product, $\vec{C}$, is a **vector** and is given by

$$\vec{C} = \vec{A} \quad \vec{B} = (A_yB_z - A_zB_y)\hat{x} - (A_xB_z - A_zB_x)\hat{y} + (A_xB_y - A_yB_x)\hat{z}$$

The **magnitude of the cross product** is given by

$$\left|\vec{C}\right| = \vec{A} \quad \vec{B} = \left|\vec{A}\right|\left|\vec{B}\right|\sin\theta$$

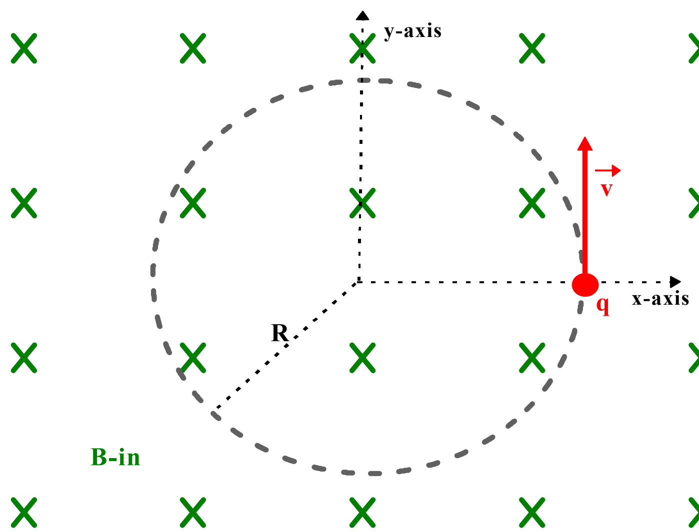
The **direction of the cross product** can be determined from the **"right hand rule"**.

Determinant Method:

The cross product can be constructed by evaluating the following determinant:

$$\vec{C} = \vec{A} \quad \vec{B} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

Motion of a Charged Particle in a Magnetic Field



Consider a charged particle **q** with **velocity**

$$\vec{v} = v_x\hat{x} + v_y\hat{y},$$

and **kinetic energy**

$$E_{kin} = \frac{1}{2}mv^2 = \frac{1}{2}m\vec{v}\cdot\vec{v},$$

in a **uniform magnetic field**

$$\vec{B} = -B\hat{z}.$$

The magnetic force on the particle is given by

$$\vec{F}_B = q\vec{v} \quad \vec{B}.$$

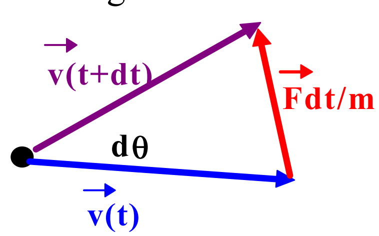
The magnetic force does not change the speed (kinetic energy) of the charged particle. The magnetic force does **no work on the charged particle** since the force is always perpendicular to the path of the particle. There is no change in the particle's kinetic energy and no change in its speed.

Proof: We know that $\vec{F}_B = q\vec{v} \quad \vec{B} = m\frac{d\vec{v}}{dt} \quad m\frac{d\vec{v}}{dt} = q\vec{v} \quad \vec{B}$. Hence

$$\frac{dE_{kin}}{dt} = \frac{1}{2}m\frac{dv^2}{dt} = \frac{1}{2}m\frac{d(\vec{v}\cdot\vec{v})}{dt} = m\vec{v}\cdot\frac{d\vec{v}}{dt} = q\vec{v}\cdot\vec{v} \quad \vec{B} = 0,$$

and thus $\mathbf{E_{kin}}$ (and **v**) are constant in time.

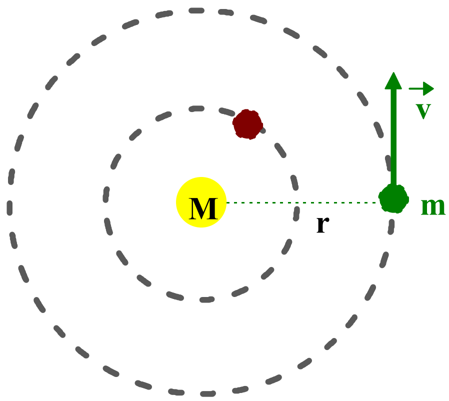
The magnetic force can **change the direction a charged particle** but not its speed. The particle undergoes **circular motion** with angular velocity **ω = qB/m**.



$$v\,d\theta = \frac{F}{m}dt = \frac{qvB}{m}dt$$

$$\omega = \frac{d\theta}{dt} = \frac{qB}{m}$$

Circular Motion: Magnetic vs Gravitational

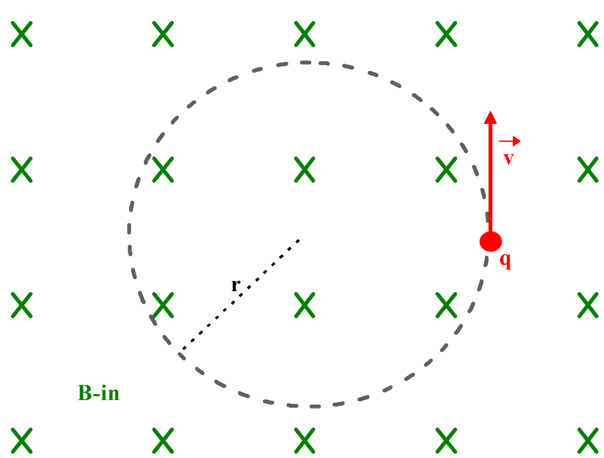


Planetary Motion:

For circular planetary motion the force on the orbiting planet is equal the mass times the **centripetal acceleration**, $\mathbf{a = v^2/r}$, as follows:

$$F_G = GmM/r^2 = mv^2/r$$

Solving for the radius and speed gives, $\mathbf{r = GM/v^2}$ and $\mathbf{v = (GM/r)^{1/2}}$. The **period of the rotation** (**time it takes to go around once**) is given by $\mathbf{T=2\pi r/v=2\pi GM/v^3}$ or $T = \frac{2\pi}{\sqrt{GM}} r^{3/2}$. The **angular velocity**, $\boldsymbol{\omega = d\theta/dt}$, and linear velocity $\mathbf{v = ds/dt}$ are related by $\mathbf{v = r\omega}$, since $\mathbf{s = r\theta}$. Thus, $\omega = \sqrt{GM} / r^{3/2}$. The angular velocity an period are related by $\mathbf{T = 2\pi/\omega}$ and the linear frequency **f** and $\boldsymbol{\omega}$ are related by $\boldsymbol{\omega = 2\pi f}$ with $\mathbf{T = 1/f}$. **Planets further from the sum travel slower and thus have a longer period T**.



Magnetism:

For magnetic circular motion the force on the charged particle is equal its mass times the **centripetal acceleration**, $\mathbf{a = v^2/r}$, as follows:

$$F_B = qvB = mv^2/r.$$

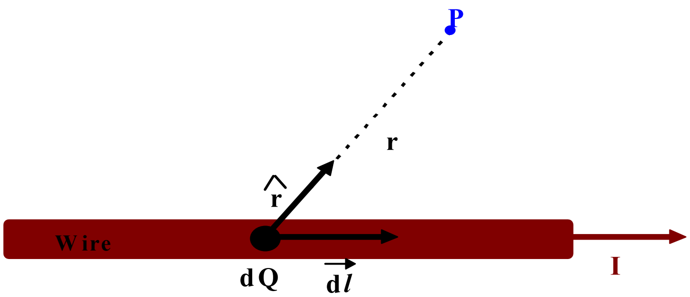
Solving for the radius and speed gives,

$$r = mv/(qB) = p/(qB)\ ,$$

and $\mathbf{v = qBr/m}$. The **period of the rotation** is given by $\mathbf{T = 2\pi r/v = 2\pi m/(qB)}$ and is **independent of the radius**! The frequency (**called the cyclotron frequency**) is given by $\mathbf{f = 1/T= qB/(2\pi m)}$ is the **same for all particles with the same charge and mass** ($\boldsymbol{\omega = qB/m}$).

The Magnetic Field Produced by a Current

The Law of Biot-Savart:



The magnetic field at the point **P** due to a charge **dQ** moving with speed **V** within a **wire carrying a current I** is given by

$$d\vec{B} = \frac{KdQ}{c^2r^2}\vec{V} \quad \hat{r}$$

where K = 8.99x10^9 Nm^2/C^2 and **c = 3x10^8 m/s** (**speed of light in a vacuum**).

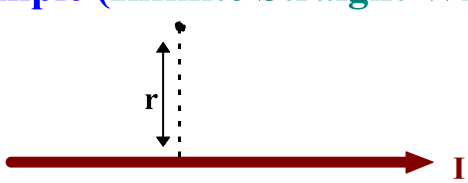
However, we know that **I = dQ/dt** and $\vec{V} = \frac{d\vec{l}}{dt}$ so that $dQ\vec{V} = Id\vec{l}$ and,

$$d\vec{B} = \frac{kI}{r^2}d\vec{l} \quad \hat{r} \qquad \text{(Law of Biot-Savart),}$$

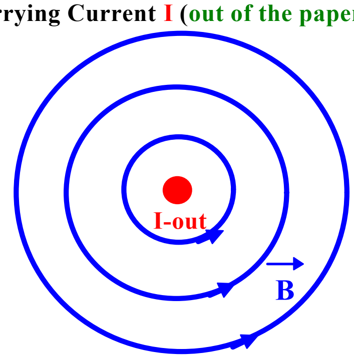
where **k = K/c^2 = 10^{-7} Tm/A**. For historical reasons we define **0** as follows:

$$k = \frac{\mu_0}{4\pi} = \frac{K}{c^2}, \qquad (\ _0 = 4\pi \text{ x } 10^{-7} \text{ Tm/A}).$$

Example (Infinite Straight Wire):



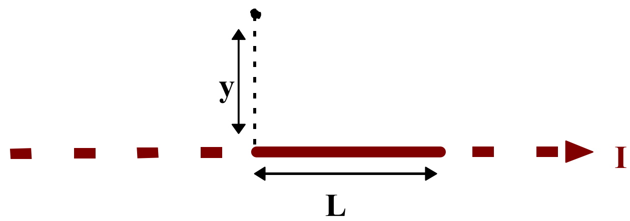
Magnetic Field of an Infinite Wire Carrying Current I (out of the paper)



An infinitely long straight wire carries a steady current I. What is the magnetic field at a distance r from the wire?

Answer: $B(r) = \frac{2kI}{r}$

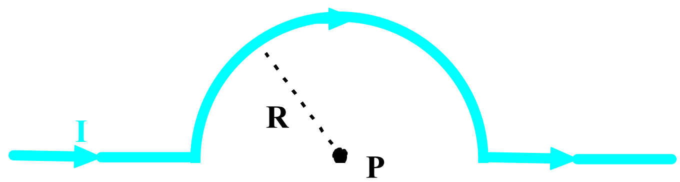
Calculating the Magnetic Field (1)



Example (Straight Wire Segment):
An infinitely long straight wire carries a steady current **I**. What is the magnetic field at a distance **y** from the wire due to the **segment** **0 <x < L**?

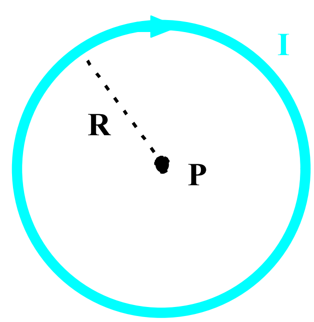
Answer: $$B(r) = \frac{kI}{y}\frac{L}{\sqrt{y^2 + L^2}}$$

Example (Semi-Circle):
A thin wire carrying a current **I** is bent into a **semi-circle** of radius **R**. What is the magnitude of magnetic field at the center of the semi-circle?



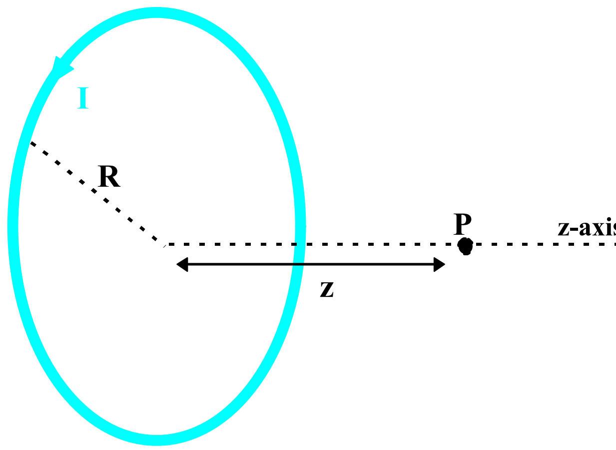
Answer: $$B = \frac{\pi kI}{R}$$

Example (Circle):
A thin wire carrying a current **I** is forms a **circle** of radius **R**. What is the magnitude of magnetic field at the center of the semi-circle?



Answer: $$B = \frac{2\pi kI}{R}$$

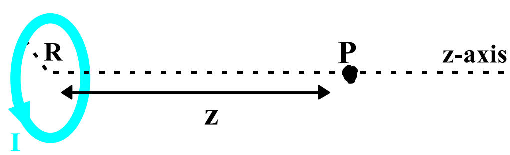
Calculating the Magnetic Field (2)



Example (Current Loop):
A thin **ring** of radius **R** carries a current **I**. What is the magnetic field at a point **P** on the z-axis a distance **z** from the center of the ring?

Answer:

$$B_z(z) = \frac{2kI\pi R^2}{\left(z^2 + R^2\right)^{3/2}}$$



Example (Magnetic Dipole):
A thin **ring** of radius **R** carries a current **I**. What is the magnetic field at a point **P** on the z-axis a distance **z >> R** from the center of the ring?

Answer: $B_z(z) = \dfrac{2k\mu_B}{z^3}$ $\quad \mu_B = I\pi R^2 = IA$

The quantity $_B$ is called the **magnetic dipole moment**,

$_B$ **= NIA**,

where **N** is the number of loops, **I** is the current and **A** is the area.

Ampere's Law

Gauss' Law for Magnetism:

The **net magnetic flux** emanating from a closed surface **S** is proportional to the amount of **magnetic charge** enclosed by the surface as follows:

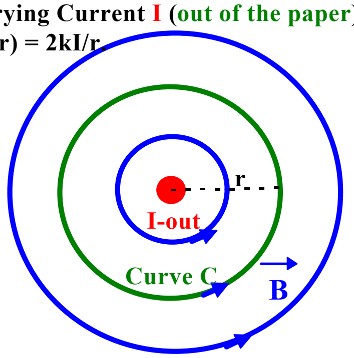
$$\Phi_B = \oint_S \vec{B} \cdot d\vec{A} \propto Q_{enclosed}^{Magnetic} .$$

However, there are **no magnetic charges** (**no magnetic monopoles**) so the **net magnetic flux** emanating from a closed surface **S** is always zero,

$$\Phi_B = \oint_S \vec{B} \cdot d\vec{A} = 0$$ **(Gauss's Law for Magnetism).**

Ampere's Law:

Magnetic Field of an Infinite Wire Carrying Current I (out of the paper) is B(r) = 2kI/r.



The line integral of the magnetic field around a closed loop (**circle**) of radius **r** around a current carrying wire is given by

$$\oint_{Loop} B \cdot d\vec{l} = 2\pi r B(r) = 4\pi k I = \mu_0 I .$$

This result is true for **any closed loop** that encloses the current **I**.

The line integral of the magnetic field around any closed path C is equal to ${}_0$ times the current intercepted by the area spanning the path:

$$\boxed{\oint_C B \cdot d\vec{l} = \mu_0 I_{enclosed}}$$ **Ampere's Law**

The **current enclosed by the closed curve C** is given by the integral over the **surface S (bounded by the curve C) of the current density J** as follows:

$$I_{enclosed} = \int_S \vec{J} \cdot d\vec{A}$$

Ampere's Law Examples

Example (Infinite Straight Wire with radius R):

An infinitely long straight wire has a circular cross section of radius **R** and carries a uniform current density **J** along the wire. The total current carried by the wire is **I**. What is the magnitude of the magnetic field inside and outside the wire?

Answer:

$$B_{out}(r) = \frac{2kI}{r}$$

$$B_{in}(r) = \frac{2krI}{R^2}.$$

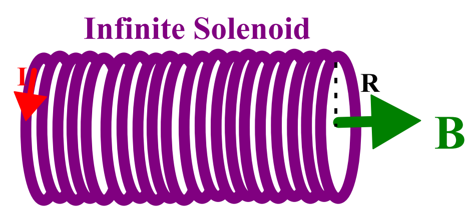
Example (Infinite Solenoid):

An infinitely long thin straight wire carrying current **I** is tightly wound into helical coil of wire (**solenoid**) of radius **R** and infinite length and with **n** turns of wire per unit length. What is the magnitude and direction of the magnetic field inside and outside the solenoid (**assume zero pitch**)?

Answer:

$$B_{out}(r) = 0$$

$$B_{in}(r) = \mu_0 nI.$$



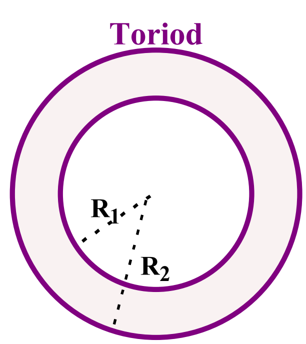
Example (Toroid):

A solenoid bent into the shape of a doughnut is called a **toriod**. What is the magnitude and direction of the magnetic field inside and outside a toriod of inner radius $\mathbf{R_1}$ and outer radius $\mathbf{R_2}$ and **N** turns of wire carrying a current **I** (**assume zero pitch**)?

Answer:

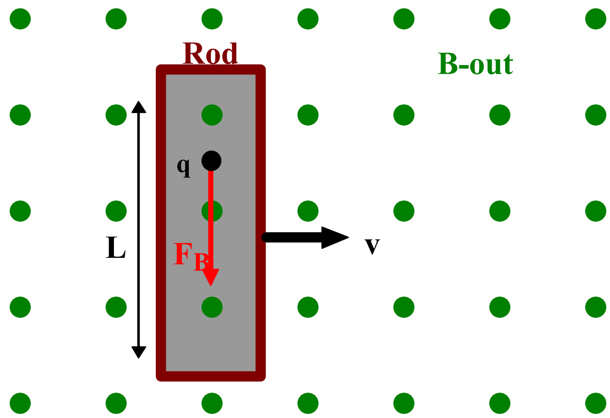
$$B_{out}(r) = 0$$

$$B_{in}(r) = \frac{2kNI}{r}$$



Electromagnetic Induction (1)

Conducting Rod Moving through a Uniform Magnetic Field:



The **magnetic force** on the charge **q** in the rod is

$$\vec{F}_B = q\vec{v} \quad \vec{B} \,.$$

The **induced EMF, ε,** is equal to the amount of **work done by the magnetic field in moving a unit charge across the rod**,

$$\varepsilon = \frac{1}{q}\int \vec{F}_B \cdot d\vec{l} = \int \vec{v} \quad \vec{B} \cdot d\vec{l} = vLB \,.$$

In Steady State:

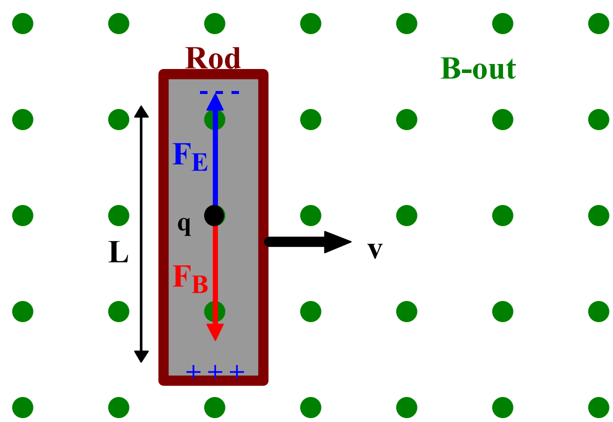
In steady state a charge **q** in the rod experiences **no net force** since,

$$\vec{F}_E + \vec{F}_B = 0 \,,$$

and thus,

$$\vec{E} = -\vec{v} \quad \vec{B} \,.$$

The **induced EMF** (**change in electric potential across the rod**) is calculated from the electric field in the usual way,

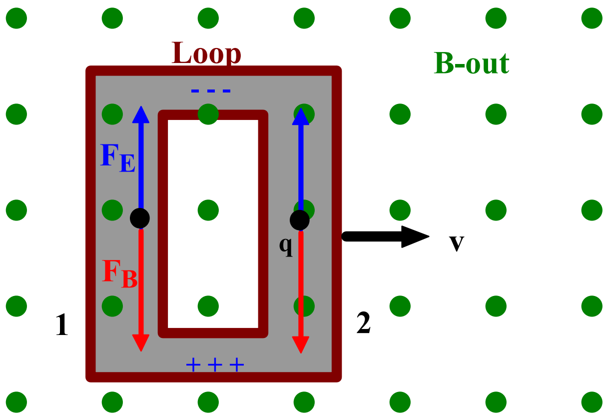


$$\varepsilon = \int \vec{E} \cdot d\vec{l} = -\int \vec{v} \quad \vec{B} \cdot d\vec{l} = vLB \,,$$

which is the **same as the work done per unit charge by the magnetic field**.

Electromagnetic Induction (2)

Conducting Loop Moving through a Uniform Magnetic Field:



The **magnetic force** on the charge **q** in the loop on **side 1** is,

$$\vec{F}_{B1} = q\vec{v} \quad \vec{B}_1 ,$$

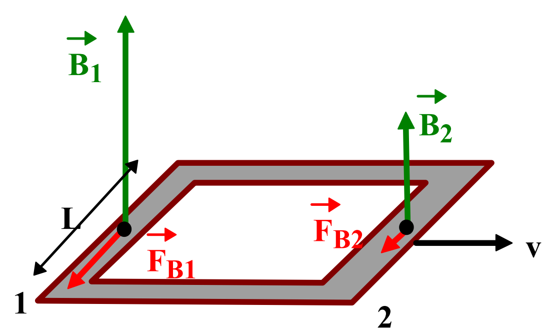
and for a charge **q** on **side 2** to it is,

$$\vec{F}_{B2} = q\vec{v} \quad \vec{B}_2 .$$

However, because the magnetic field is uniform, $\vec{B}_1 = \vec{B}_2$, and the induced EMF's on **side 1** and **side 2** are equal, $\varepsilon_1 = \varepsilon_2$, and the **net EMF around the loop** (**counterclockwise**) is zero,

$$\varepsilon = \frac{1}{q}\int_{Loop} \vec{F}_B \cdot d\vec{l} = \varepsilon_1 - \varepsilon_2 = 0 .$$

Conducting Loop Moving through a Non-Uniform Magnetic Field:



If we move a conducting loop through a **non-uniform magnetic field** then induced EMF's on **side 1** and **side 2** are not equal, $\varepsilon_1 = vLB_1$, $\varepsilon_2 = vLB_2$, and the **net EMF around the loop** (**counterclockwise**) is,

$$\varepsilon = \frac{1}{q}\int_{Loop} \vec{F}_B \cdot d\vec{l} = \varepsilon_1 - \varepsilon_2 = vL(B_1 - B_2) .$$

This induced EMF will cause a current to flow around the loop in a counterclockwise direction (if $B_1 > B_2$)!

Faraday's Law of Induction

Magnetic Flux:

The magnetic flux through the surface **S** is defined by,

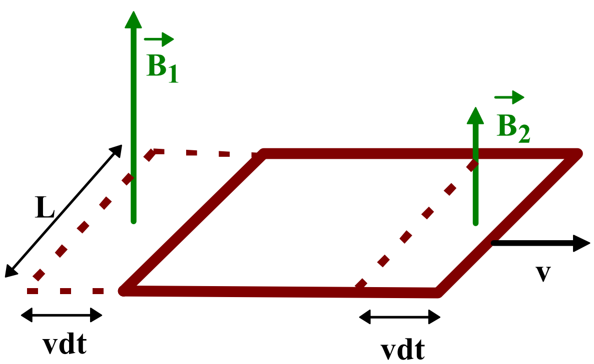
$$\Phi_B = \int_S \vec{B} \cdot d\vec{A} \; .$$

In the simple case where **B is constant and normal to the surface** then

$$\mathbf{\Phi_B = BA}.$$

The units for magnetic flux are webbers (1 Wb = 1 Tm^2).

Rate of Change of the Magnetic Flux through Moving Loop:



The **change in magnetic flux**, **dΦ_B**, in a time **dt** through the moving loop is,

$$\mathbf{d\Phi_B = B_2 dA - B_1 dA},$$

with **dA = vdtL** so that

$$\frac{d\Phi_B}{dt} = -vL(B_1 - B_2) = -\varepsilon$$

where ε is the induced EMF. Hence,

$$\varepsilon = -\frac{d\Phi_B}{dt}$$ (**Faraday's Law of Induction**).

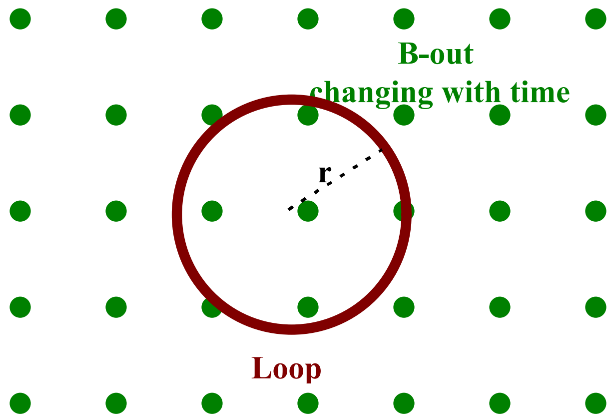
Substituting in the definition of the **induced EMF** and the **magnetic flux** yields,

$$\varepsilon = \oint_{\substack{Closed\\Loop}} \vec{E} \cdot d\vec{l} = -\frac{d\Phi_B}{dt} = -\frac{d}{dt}\left(\int_{Surface} \vec{B} \cdot d\vec{A}\right) = -\int_{Surface} \frac{\partial \vec{B}}{\partial t} \cdot d\vec{A}$$

We see that a changing magnetic field (with time) **can produce an electric field!**

Lenz's Law

Example (Loop of Wire in a Changing Magnetic Field):



A wire loop with a radius, **r**, of 1 meter is placed in a uniform magnetic field. Suppose that the electromagnetic is suddenly switched off and the strength of the **magnetic field decreases at a rate of 20 Tesla per second**. **What is the induced EMF in the loop (in Volts)? If the resistance of the loop, R, is 5 Ohms, what is the induced current in the loop (in Amps)? What is the direction of the induced current? What is the magnitude and direction of the magnetic field produced by the induced current (the induced magnetic field) at the center of the circle?**

Answers: If I choose my **orientation to be counterclockwise** then $\mathbf{\Phi_B = BA}$ and

$$\boldsymbol{\varepsilon = -d\Phi_B/dt} = -A\, dB/dt = -(\pi r^2)(-20T/s) = \mathbf{62.8\ V}.$$

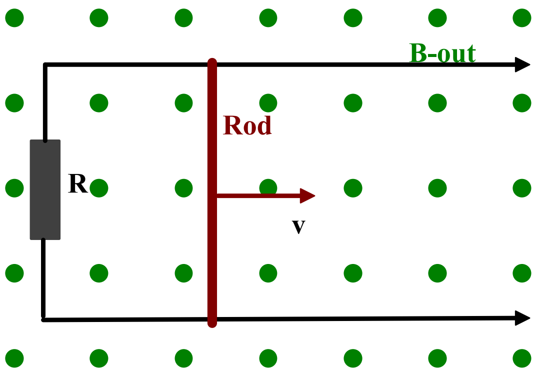
The induced current is $\mathbf{I = \varepsilon/R}$ = (62.8 V)/(5 Ω) = **12.6 A**. Since ε is positive the current is flowing in the direction of my chosen orientation (**counterclockwise**). The induced magnetic field at the center of the circle is given by $\mathbf{B_{ind} = 2\pi kI/r}$ = $(2\pi \times 10^{-7}$ Tm/A)(12.6 A)/(1 m) = **7.9 T** and points **out of the paper**.

Lenz's Law: **It is a physical fact not a law or not a consequence of sign conventions that an electromagnetic system tends to resist change. Traditionally this is referred to as Lenz's Law:**

> **Induced EMF's are always in such a direction as to oppose the change that generated them.**

Induction Examples

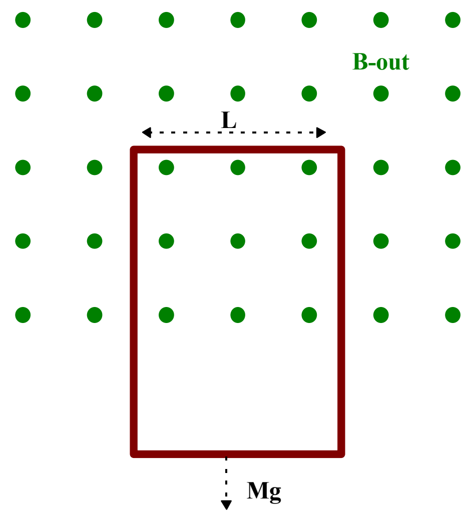
Example (simple generator):



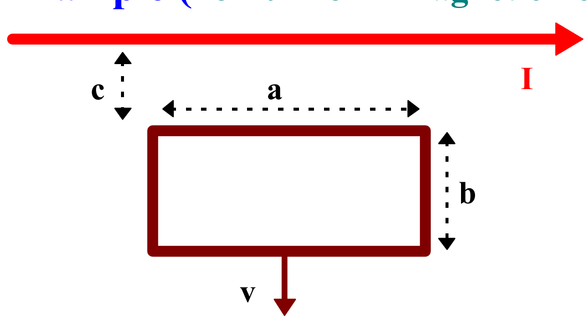
A conducting rod of length **L** is pulled along horizontal, frictionless, conducting rails at a constant speed **v**. A uniform magnetic field (**out of the paper**) fills the region in which the rod moves. The rails and the rod have negligible resistance but are connected by a resistor **R**. **What is the induced EMF in the loop? What is the induced current in the loop? At what rate is thermal energy being generated in the resistor? What force must be applied to the rod by an external agent to keep it in uniform motion? At what rate does this external agent do work on the system?**

Example (terminal velocity):

A long rectangular loop of wire of width **L**, mass **M**, and resistance **R**, falls vertically due to gravity **out of a uniform magnetic field**. Instead of falling with an acceleration, **g**, the loop falls a constant velocity (called **the terminal velocity**). **What is the terminal velocity of the loop?**

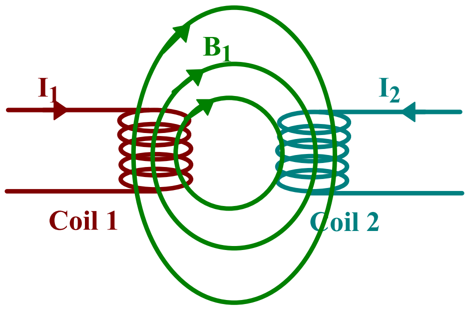


Example (non-uniform magnetic field):



A rectangular loop of wire with length **a**, width **b**, and resistance **R** is moved with velocity **v** away from an infinitely long wire carrying a current **I**. **What is the induced current in the loop when it is a distance c from the wire?**

Mutual & Self Inductance



Mutual Inductance (M):

Consider **two fixed coils** with a varying current I_1 in **coil 1** producing a magnetic field B_1. The **induced EMF** in **coil 2** due to B_1 is proportional to the magnetic flux through **coil 2**, $\Phi_2 = \int_{coil2} \vec{B}_1 \cdot d\vec{A}_2 = N_2\phi_2$,

where N_2 is the number of loops in **coil 2** and ϕ_2 is the flux through a **single loop in coil 2**. However, we know that B_1 is proportional to I_1 which means that Φ_2 is proportional to I_1. The **mutual inductance M** is defined to be the **constant of proportionality between** Φ_2 and I_1 and depends on the geometry of the situation,

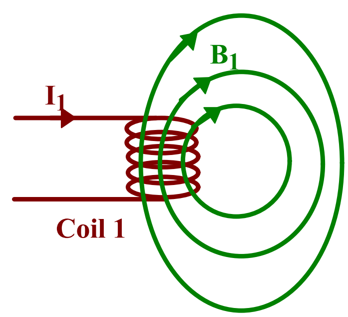
$$M = \frac{\Phi_2}{I_1} = \frac{N_2\phi_2}{I_1} \qquad \Phi_2 = N_2\phi_2 = MI_1$$

. The **induced EMF** in **coil 2** due to the varying current in **coil 1** is given by,

$$\boxed{\varepsilon_2 = -\frac{d\Phi_2}{dt} = -M\frac{dI_1}{dt}}$$

The units for inductance is a Henry ($1\ H = Tm^2/A = Vs/A$).

Self Inductance (L):



When the current I_1 in **coil 1** is varying there is a **changing magnetic flux** due to B_1 in **coil 1** itself! The **self inductance L** is defined to be the **constant of proportionality between** Φ_1 and I_1 and depends on the geometry of the situation,

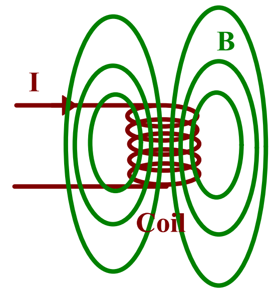
$$L = \frac{\Phi_1}{I_1} = \frac{N_1\phi_1}{I_1} \qquad \Phi_1 = N_1\phi_1 = LI_1,$$

where N_1 is the number of loops in **coil 1** and ϕ_1 is the flux through a **single loop in coil 1**. The **induced EMF** in **coil 2** due to the varying current in **coil 1** is given by,

$$\boxed{\varepsilon_1 = -\frac{d\Phi_1}{dt} = -L\frac{dI_1}{dt}}$$

Energy Stored in a Magnetic Field

When an **external source of EMF** is connected to an inductor and current begins to flow, the **induced EMF** (called **back EMF**) will oppose the increasing current and the **external EMF must do work** in order to overcome this opposition. This work is **stored in the magnetic field** and can be recovered by removing the external EMF.



Energy Stored in an Inductor L:

The rate at which work is done by the back EMF (power) is

$$P_{back} = \varepsilon I = -LI\frac{dI}{dt},$$

since **ε = -LdI/dt**. The power supplied by the external EMF (rate at which work is done against the back EMF) is

$$P = \frac{dW}{dt} = LI\frac{dI}{dt},$$

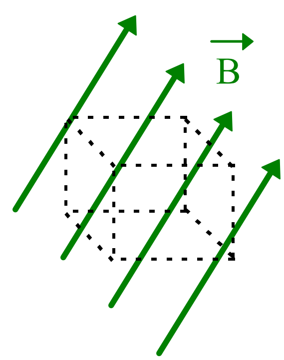
and the **energy stored in the magnetic field of the inductor** is

$$U = \int P dt = \int_0^t LI\frac{dI}{dt}dt = \int_0^I LIdI = \frac{1}{2}LI^2.$$

Energy Density of the Magnetic Field u:

Magnetic field line contain energy! The amount of energy per unit volume is

$$\boxed{u_B = \frac{1}{2\mu_0}B^2}$$



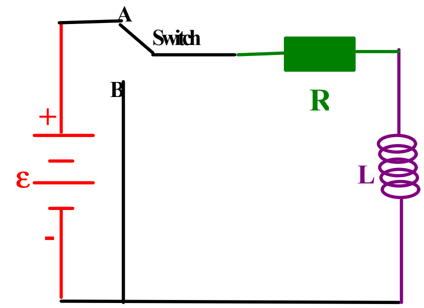
where **B** is the magnitude of the magnetic field. **The magnetic energy density has units of Joules/m^3.** The total amount of energy in an infinitesimal volume dV is **dU = u_BdV** and

$$U = \int_{Volume} u_B dV.$$

If **B** is constant through the volume, **V**, then **U = u_B V**.

RL Circuits

"Building-Up" Phase:



Connecting the switch to **position A** corresponds to the **"building up" phase of an RL circuit**. Summing all the potential changes in going around the loop gives

$$\varepsilon - IR - L\frac{dI}{dt} = 0 \ ,$$

where **I(t)** is a function of time. If the switch is closed (**position A**) at t=0 and **I(0)=0** (**assuming the current is zero at t=0**) then

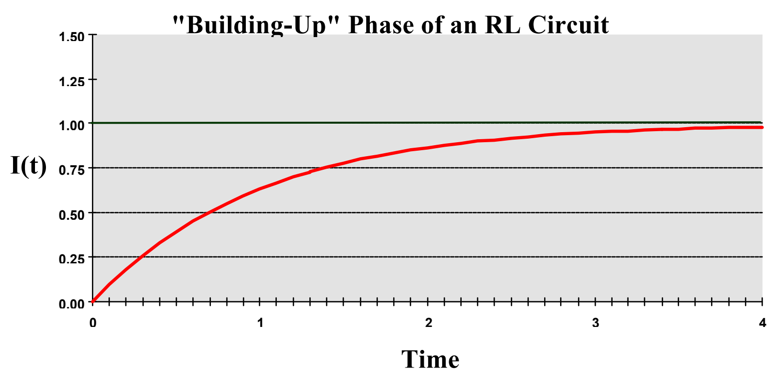
$$\frac{dI}{dt} = -\frac{1}{\tau}\left(I - \frac{\varepsilon}{R}\right) \ ,$$ where I have define **τ=L/R**.

Dividing by (I-ε/R) and multiplying by dt and integrating gives

$$\int_0^I \frac{dI}{(I - \varepsilon / R)} = -\int_0^t \frac{1}{\tau}dt$$, which implies $$\ln\left(\frac{I - \varepsilon / R}{-\varepsilon / R}\right) = -\frac{t}{\tau} \ .$$

Solving for **I(t)** gives

$$I(t) = \frac{\varepsilon}{R}\left(1 - e^{-t/\tau}\right) \ .$$



The potential change across the inductor is given by **ΔV_L(t)=-LdI/dt** which yields

$$\Delta V_L(t) = -\varepsilon e^{-t/\tau} \ .$$

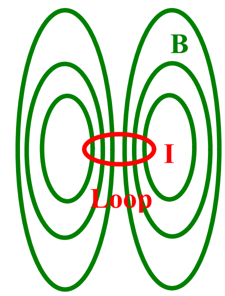
The quantity **τ=L/R** is call the **time constant** and has dimensions of time.

"Collapsing" Phase:

Connecting the switch to **position B** corresponds to the **"collapsing" phase of an RL circuit**. Summing all the potential changes in going around the loop gives $-IR - L\frac{dI}{dt} = 0$, where **I(t)** is a function of time. If the switch is closed (**position B**) at t=0 then **I(0)=I_0** and

$$\frac{dI}{dt} = -\frac{1}{\tau}I \quad \text{and} \quad I(t) = I_0 e^{-t/\tau} \ .$$

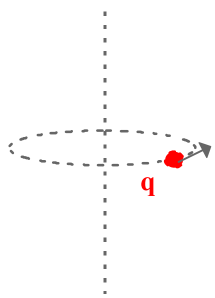
Electrons and Magnetism



Magnetic Dipole:

The magnetic field on the z-axis of a **current loop** with area **A=πR²** and current **I** is given by $\mathbf{B_z(z) = 2k\ \ /z^3}$, when **z >> R**, where the **magnetic dipole moment** **= IA**.

Orbital Magnetic Moment:



Consider a single particle with charge **q** and mass **m** undergoing **uniform circular motion** with radius **R** about the z-axis. The period of the orbit is given by **T = 2πR/v**, where **v** is the particles speed. The magnetic moment (**called the orbital magnetic moment**) is

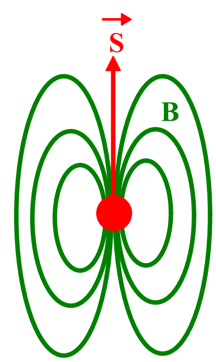
$$\mu_{orb} = IA = \frac{q}{T}\pi R^2 = \frac{q}{2}vR\ ,$$

since **I = q/T**. The orbital magnetic moment can be written in terms of the orbital angular momentum, $\vec{L} = \vec{r}\ \ \vec{p}$, as follows

$$\boxed{\mu_{orb} = \frac{q}{2m}L_{orb}},$$

where $\mathbf{L_{orb} = Rmv}$. For an electron,

$$\mu_{orb} = -\frac{e}{2m_e}L_{orb}\ .$$



"Spin" Magnetic Moment (Quantum Mechanics):

Certain elementary parrticles (such as electrons) carry **intrensic angular momentum** (called **"spin" angular momentum**) and an **intrensic magnetic moment** (called **"spin" magnetic moment**),

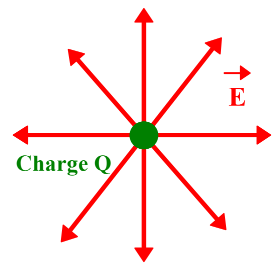
$$\mu_{spin} = -\frac{e}{2m_e}gS = -\frac{e\hbar}{2m_e}\ , \quad (\textbf{electron})$$

where $S = \hbar/2$ is the spin angular momentum of the electron and **g = 2** is the **gyromagnetic ratio**. ($\hbar = h/2\pi$ and **h** is **Plank's Constant**.). **Here the units are Bohr Magnitons,** $\mu_{Bohr} = \dfrac{e\hbar}{2m_e}$**, with** $\mathbf{\ \ _{Bohr} = 9.27x10^{-24}}$ **J/T.**

Maxwell's Equations

I. (Gauss' Law):

$$\Phi_E = \oint_{Surface} \vec{E}\cdot d\vec{A} = \frac{Q_{enclosed}}{\varepsilon_0} = \frac{1}{\varepsilon_0}\int_{Volume} \rho\, dV$$



Volume Enclosed by Surface

Two Sources of Electric Fields

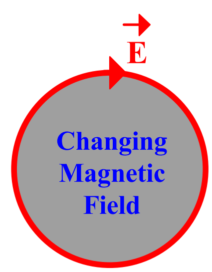
II. (Gauss' Law for Magnetism):

$$\Phi_B = \oint_{Surface} \vec{B}\cdot d\vec{A} = 0$$

No Magnetic Charges!

III. (Faraday's Law of Induction):

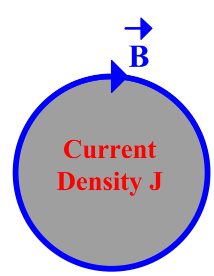
$$\varepsilon = \oint_{Curve} \vec{E}\cdot d\vec{l} = -\frac{d\Phi_B}{dt} = -\int_{Surface} \frac{\partial \vec{B}}{\partial t}\cdot d\vec{A}$$



Surface Bounded by Curve

IV. (Ampere's Law):

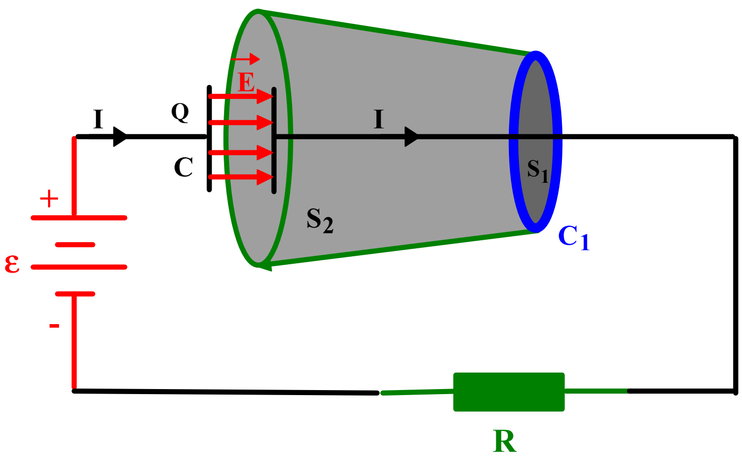
$$\oint_{Curve} \vec{B}\cdot d\vec{l} = \mu_0 I_{enclosed} = \mu_0 \int_{Surface} \vec{J}\cdot d\vec{A}$$



One Source of Magnetic Fields

Surface Bounded by Curve

Finding the Missing Term



We are looking for a **new term in Ampere's Law** of the form,

$$\oint_{C1} \vec{B}\cdot d\vec{l} = \mu_0 I + \delta\frac{d\Phi_E}{dt},$$

where δ is an unknown constant and

$$I = \int_S \vec{J}\cdot d\vec{A} \qquad \Phi_E = \int_S \vec{E}\cdot d\vec{A},$$

where **S** is any surface bounded by the curve $\mathbf{C_1}$.

Case I (use surface S_1):

If we use the surface $\mathbf{S_1}$ which is bounded by the curve $\mathbf{C_1}$ then

$$\oint_{C1} \vec{B}\cdot d\vec{l} = \mu_0 I + \delta\frac{d\Phi_E}{dt} = \int_{S1}\left(\mu_0\vec{J} + \delta\cancelto{0}{\frac{\partial\vec{E}}{\partial t}}\right)\cdot d\vec{A} = \mu_0 I,$$

since **E = 0 through the surface S_1**.

Must be equal, hence δ= $_0\varepsilon_0$.

Case II (use surface S_2):

If we use the surface $\mathbf{S_2}$ which is bounded by the curve $\mathbf{C_1}$ then

$$\oint_{C1} \vec{B}\cdot d\vec{l} = \mu_0 I + \delta\frac{d\Phi_E}{dt} = \int_{S1}\left(\cancelto{0}{\mu_0\vec{J}} + \delta\frac{\partial\vec{E}}{\partial t}\right)\cdot d\vec{A} = \frac{\delta I}{\varepsilon_0},$$

since **J = 0 through the surface S_2** and

$$E = \frac{\sigma}{\varepsilon_0} = \frac{Q}{\varepsilon_0 A} \qquad \frac{\partial E}{\partial t} = \frac{1}{\varepsilon_0 A}\frac{dQ}{dt} = \frac{I}{\varepsilon_0 A}.$$

Ampere's Law (complete):

$$\oint_{Curve} \vec{B}\cdot d\vec{l} = \mu_0 I + \mu_0\varepsilon_0\frac{d\Phi_E}{dt} = \mu_0\int_{Surface}\left(\vec{J} + \varepsilon_0\frac{\partial\vec{E}}{\partial t}\right)\cdot d\vec{A} = \mu_0\left(I + I_d\right),$$

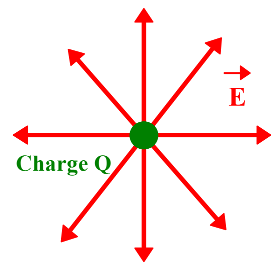
$$I_d = \int_S \vec{J}_d\cdot d\vec{A} \qquad \vec{J}_d = \varepsilon_0\frac{\partial\vec{E}}{\partial t}.$$

"Displacement Current"

"Displacement Current" Density

Complete Maxwell's Equations

I. (Gauss' Law):



$$\Phi_E = \oint_{Surface} \vec{E} \cdot d\vec{A} = \frac{Q_{enclosed}}{\varepsilon_0} = \frac{1}{\varepsilon_0} \int_{Volume} \rho\, dV$$

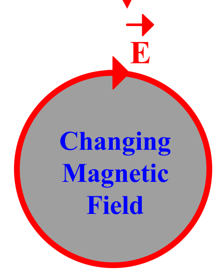
Volume Enclosed by Surface

Two Sources of Electric Fields

II. (Gauss' Law for Magnetism):

$$\Phi_B = \oint_{Surface} \vec{B} \cdot d\vec{A} = 0$$

No Magnetic Charges!

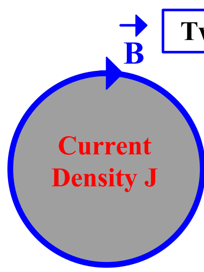


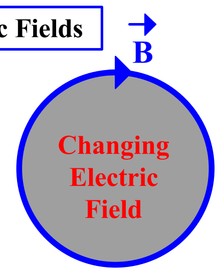
III. (Faraday's Law of Induction):

$$\varepsilon = \oint_{Curve} \vec{E} \cdot d\vec{l} = -\frac{d\Phi_B}{dt} = -\int_{Surface} \frac{\partial \vec{B}}{\partial t} \cdot d\vec{A}$$

Surface Bounded by Curve

Two Sources of Magnetic Fields





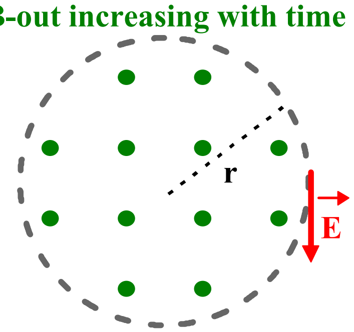
IV. (Ampere's Law):

$$\oint_{Curve} \vec{B} \cdot d\vec{l} = \mu_0 I_{enc} + \mu_0 \varepsilon_0 \frac{d\Phi_E}{dt} = \mu_0 \int_{Surface} \left(\vec{J} + \varepsilon_0 \frac{\partial \vec{E}}{\partial t} \right) \cdot d\vec{A}$$

Surface Bounded by Curve

Electric & Magnetic Fields that Change with Time

Changing Magnetic Field Produces an Electric Field:



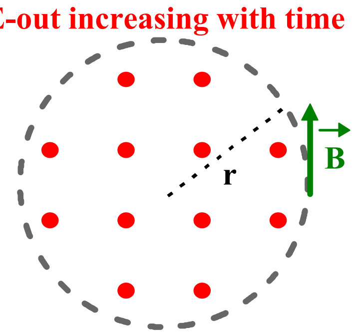
A **uniform magnetic field** is confined to a circular region of radius, **r**, and is **increasing with time**. **What is the direction and magnitude of the induced electric field at the radius r?**

Answer: If I choose my **orientation to be counterclockwise** then $\Phi_B = B(t)A$ with $A = \pi r^2$. **Faraday's Law of Induction** tells us that

$$\oint_{Circle} \vec{E} \cdot d\vec{l} = 2\pi r E(r) = -\frac{d\Phi_B}{dt} = -\pi r^2 \frac{dB}{dt},$$

and hence **E(r) = -(r/2) dB/dt**. Since **dB/dt > 0** (**increasing with time**), **E** is negative which means that it points opposite my chosen orientation.

Changing Electric Field Produces a Magnetic Field:



A **uniform electric field** is confined to a circular region of radius, **r**, and is **increasing with time. What is the direction and magnitude of the induced magnetic field at the radius r?**

Answer: If I choose my **orientation to be counterclockwise** then $\Phi_E = E(t)A$ with $A = \pi r^2$. **Ampere's Law** (**with J = 0**) tells us that

$$\oint_{Circle} \vec{B} \cdot d\vec{l} = 2\pi r B(r) = \varepsilon_0 \mu_0 \frac{d\Phi_E}{dt} = \frac{\pi r^2}{c^2}\frac{dE}{dt},$$

and hence **B(r) = (r/2c²) dE/dt**. Since **dE/dt > 0** (**increasing with time**), **B** is positive which means that it points in the direction of my chosen orientation.

Simple Harmonic Motion

Hooke's Law Spring:

For a **Hooke's Law spring** the restoring force is linearly proportional to the distance from equilibrium, $\mathbf{F_x = -kx}$, where **k** is the spring constant. Since, $\mathbf{F_x = ma_x}$ we have

$$-kx = m\frac{d^2x}{dt^2} \quad \text{or} \quad \frac{d^2x}{dt^2} + \frac{k}{m}x = 0, \text{ where } x = x(t).$$

General Form of SHM Differential Equation:

The general for of the **simple harmonic motion** (**SHM**) differential equation is

$$\frac{d^2x(t)}{dt^2} + Cx(t) = 0,$$

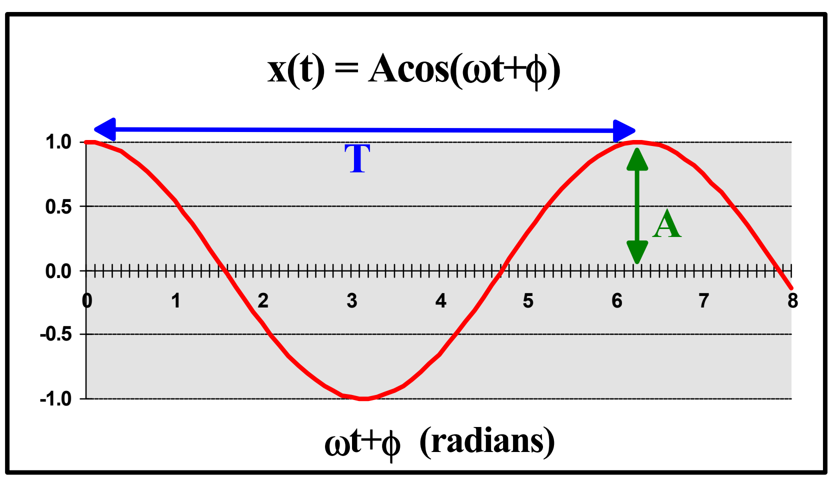
where **C** is a positive constant (**for the Hooke's Law spring C=k/m**). The most general solution of this **2nd order differential equation** can be written in the following four ways:

$$x(t) = Ae^{i\omega t} + Be^{-i\omega t}$$

$$x(t) = A\cos(\omega t) + B\sin(\omega t)$$

$$x(t) = A\sin(\omega t + \phi)$$

$$x(t) = A\cos(\omega t + \phi)$$



where **A**, **B**, and ϕ are arbitrary constants and $\omega = \sqrt{C}$. In the chart, **A is the amplitude** of the oscillations and **T is the period**. The **linear frequency** **f = 1/T** is measured in cycles per second (**1 Hz = 1/sec**). The **angular frequency** $\boldsymbol{\omega}$ **=** **2πf** and is measured in radians/second. For the Hooke's Law Spring **C = k/m** and thus $\omega = \sqrt{C} = \sqrt{k/m}$.

SHM Differential Equation

The general for of the **simple harmonic motion** (**SHM**) differential equation is

$$\frac{d^2x(t)}{dt^2} + Cx(t) = 0,$$

where **C** is a constant. One way to solve this equation is to turn it into an **algebraic equation** by looking for a solution of the form

$$x(t) = Ae^{at}.$$

Substituting this into the differential equation yields,

$$a^2Ae^{at} + CAe^{at} = 0 \quad \text{or} \quad \boxed{a^2 = -C}.$$

Case I (C > 0, oscillatory solution):

For positive **C**, $a = \pm i\sqrt{C} = \pm i\omega$, where $\omega = \sqrt{C}$. In this case the most general solution of this **2nd order differential equation** can be written in the following four ways:

$$x(t) = Ae^{i\omega t} + Be^{-i\omega t}$$

$$x(t) = A\cos(\omega t) + B\sin(\omega t)$$

$$x(t) = A\sin(\omega t + \phi)$$

$$x(t) = A\cos(\omega t + \phi)$$

where **A**, **B**, and ϕ are arbitrary constants (**two arbitrary constants for a 2nd order differential equation).** Remember that $e^{\pm i\theta} = \cos\theta \pm i\sin\theta$ where $i = \sqrt{-1}$.

Case II (C < 0, exponential solution):

For negative **C**, $a = \pm\sqrt{-C} = \pm\gamma$, where $\gamma = \sqrt{-C}$. In this case, the most general solution of this **2nd order differential equation** can be written as follows:

$$x(t) = Ae^{\gamma t} + Be^{-\gamma t},$$

where **A** and **B** arbitrary constants.