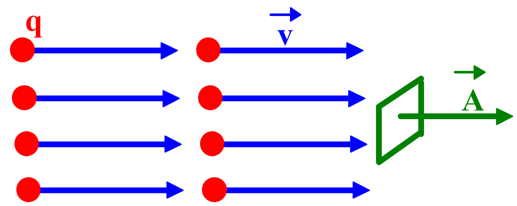
Charge Transport and Current Density



Consider **n** particles per unit volume all moving with velocity **v** and each carrying a charge **q**.

The number of particles, **ΔN**, passing through the (**directed**) area **A** in a time **Δt** is $\Delta N = n\vec{v}\cdot\vec{A}\Delta t$ and the amount of charge, **ΔQ**, passing through the (**directed**) area **A** in a time **Δt** is

$$\Delta Q = nq\vec{v}\cdot\vec{A}\Delta t\,.$$

The **current**, **I(A)**, is the amount of charge per unit time passing through the (**directed**) area **A**:

$$I(\vec{A}) = \frac{\Delta Q}{\Delta t} = nq\vec{v}\cdot\vec{A} = \vec{J}\cdot\vec{A}\,,$$

where the **"current density"** is given by $\vec{J} = nq\vec{v}_{drift}$.

The current **I** is measured in **Ampere's** where 1 Amp is equal to one Coulomb per second (**1A = 1C/s**).

For an infinitesimal area (**directed**) area **dA**:

$$dI = \vec{J}\cdot d\vec{A} \quad \text{and} \quad \vec{J}\cdot\hat{n} = \frac{dI}{dA}\,.$$

The **"current density"** is the amount of current per unit area and has units of **A/m^2**. The current passing through the surface S is given by

$$I = \int_S \vec{J}\cdot d\vec{A}\,.$$

The current, I, is the "flux" associated with the vector J.

Electrical Conductivity and Ohms Law

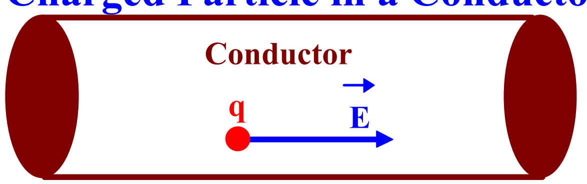
Free Charged Particle:

For a free charged particle in an electric field,

$$\vec{F} = m\vec{a} = q\vec{E} \quad \text{and thus} \quad \vec{a} = \frac{q}{m}\vec{E}.$$

The **acceleration is proportional to the electric field strength E** and the **velocity of the particle increases with time!**

Charged Particle in a Conductor:



However, for a charged particle in a conductor the **average velocity is proportional to the electric field** strength **E** and since $\vec{J} = nq\vec{v}_{ave}$ we have

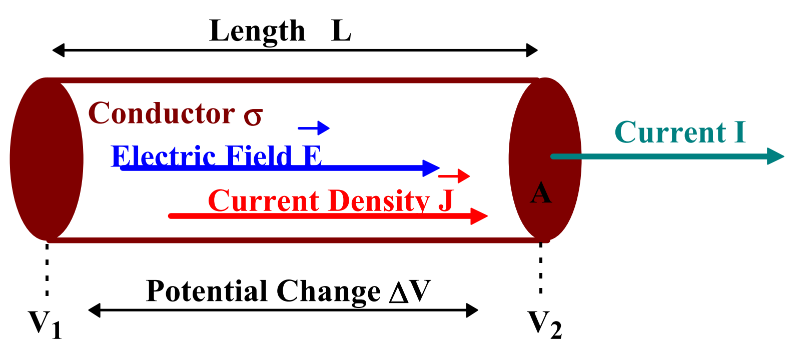
$$\vec{J} = \sigma\vec{E},$$

where **σ** is the **conductivity** of the material and is a property of the conductor. **The resistivity ρ = 1/σ.**

Ohm's Law:

$$\vec{J} = \sigma\vec{E}$$

$$I = JA = \sigma EA$$



$$\Delta V = EL = \frac{I}{\sigma A}L = \left(\frac{L}{\sigma A}\right)I = RI$$

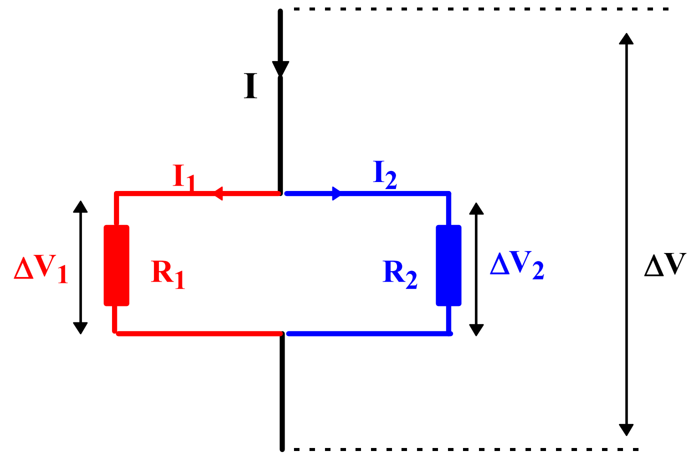
ΔV = IR (Ohm's Law) **R = L/(σA) = ρL/A (Resistance)**

Units for R are Ohms 1Ω = 1V/1A

Resistors in Series & Parallel

Parallel:

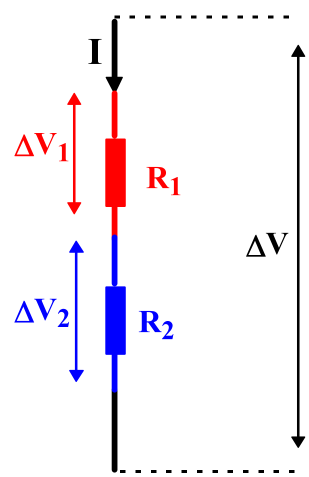
In this case $\Delta V_1 = \Delta V_2 = \Delta V$ and $I = I_1 + I_2$. Hence, $I = I_1 + I_2 = \Delta V_1/R_1 + \Delta V_2/R_2 = (1/R_1 + 1/R_2)\Delta V$ so $1/R = I/\Delta V = 1/R_1 + 1/R_2$, where I used $I_1 = \Delta V_1/R_1$ and $I_2 = \Delta V_2/R_2$. Also, $\Delta V = I_1R_1 = I_2R_2 = IR$ so $I_1 = R_2I/(R_1+R_2)$ and $I_2 = R_1I/(R_1+R_2)$.



Resistors in parallel add inverses.

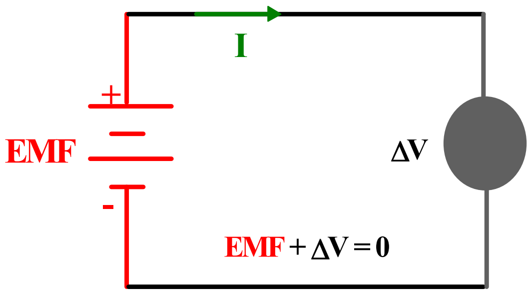
Series:

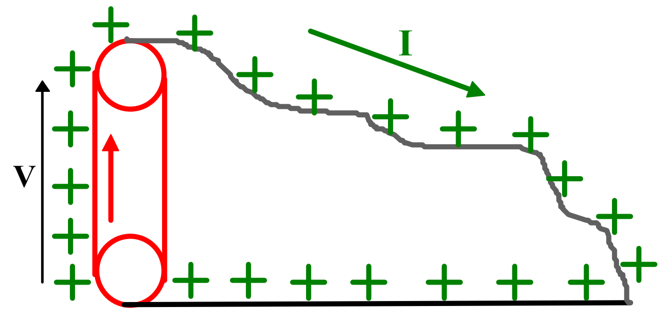
In this case $\Delta V = \Delta V_1 + \Delta V_2$ and $I = I_1 = I_2$. Hence, $\Delta V = \Delta V_1 + \Delta V_2 = I_1R_1 + I_2R_2 = (R_1+R_2)I$ so $R = \Delta V/I = R_1 + R_2$, where I used $\Delta V_1 = I_1R_1$ and $\Delta V_2 = I_2R_2$.



Resistors in series add.

Direct Current (DC) Circuits





Electromotive Force:

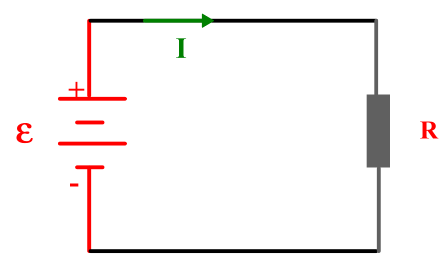
The **electromotive force EMF** of a source of electric potential energy is defined as the amount of **electric energy per Coulomb of positive charge** as the charge passes through the source from low potential to high potental.

$$\text{EMF} = \varepsilon = U/q$$ (The units for EMF is Volts)

Single Loop Circuits:

$\varepsilon - IR = 0$ and $I = \varepsilon / R$

(Kirchhoff's Rule)



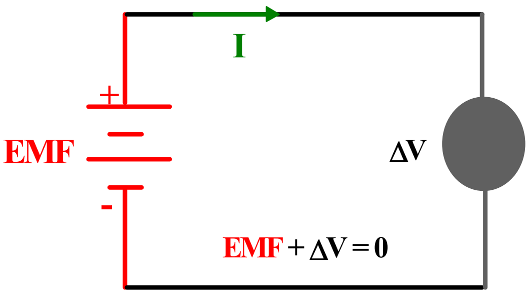
Power Delivered by EMF ($P = \varepsilon I$):

$$dW = \varepsilon dq \qquad P = \frac{dW}{dt} = \varepsilon \frac{dq}{dt} = \varepsilon I$$

Power Dissipated in Resistor ($P = I^2 R$):

$$dU = \Delta V_R dq \qquad P = \frac{dU}{dt} = \Delta V_R \frac{dq}{dt} = \Delta V_R I$$

DC Circuit Rules



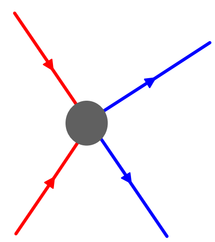
Loop Rule:

The algebraic sum of the **changes in potential** encountered in a complete traversal of any **loop** of a circuit must be **zero**.

$$\sum_{loop} \Delta V_i = 0 \quad .$$

Junction Rule:

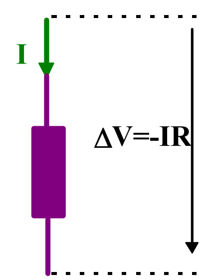
The sum of the currents entering any **junction** must be equal the sum of the currents leaving that junction.



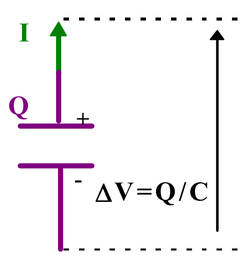
$$\sum_{in} I_i = \sum_{out} I_i$$

Resistor:

If you move across a **resistor** **in the direction of the current flow** then the potential change is **$\Delta V_R = - IR$**.



Capacitor:



If you move across a **capacitor** from **minus to plus** then the potential change is

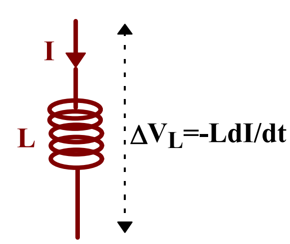
$$\Delta V_C = Q/C,$$

and the current **leaving the capacitor** is **$I = -dQ/dt$**.

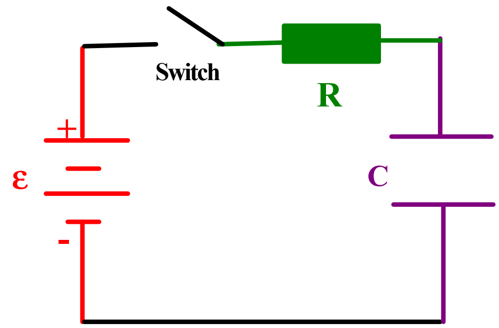
Inductor (Chapter 31):

If you move across an **inductor** **in the direction of the current flow** then the potential change is

$$\Delta V_L = - L\ dI/dt.$$



Charging a Capacitor



After the switch is closed the current is entering the capacitor so that **I = dQ/dt**, where **Q** is the charge on the capacitor and summing all the potential changes in going around the loop gives

$$\varepsilon - IR - \frac{Q}{C} = 0 \ ,$$

where **I(t)** and **Q(t)** are a function of time. If the switch is closed at t=0 then **Q(0)=0** and

$$\varepsilon - R\frac{dQ}{dt} - \frac{Q}{C} = 0 \ ,$$

which can be written in the form

$$\frac{dQ}{dt} = -\frac{1}{\tau}\left(Q - \varepsilon C\right) \ ,$$ where I have define **τ=RC**.

Dividing by (Q-εC) and multipling by dt and integrating gives

$$\int_0^Q \frac{dQ}{\left(Q - \varepsilon C\right)} = -\int_0^t \frac{1}{\tau}dt$$, which implies $$\ln\left(\frac{Q - \varepsilon C}{-\varepsilon C}\right) = -\frac{t}{\tau} \ .$$

Solving for **Q(t)** gives

$$Q(t) = \varepsilon C\left(1 - e^{-t/\tau}\right)$$

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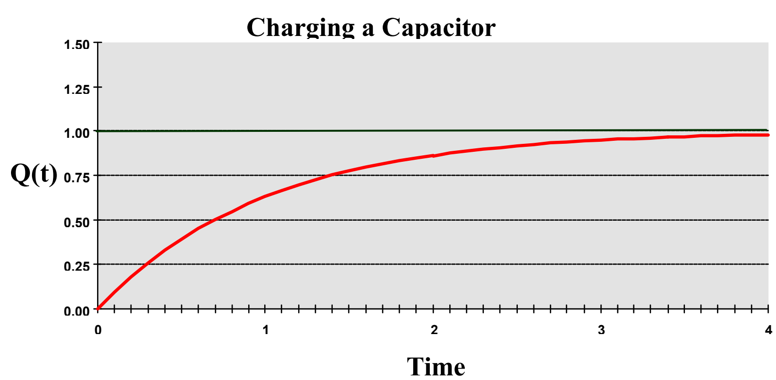
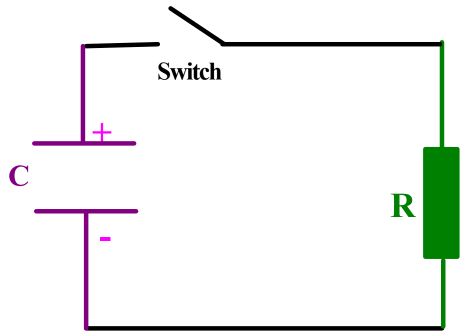


The curent is given by **I(t)=dQ/dt** which yields

$$I(t) = \frac{\varepsilon C}{\tau}e^{-t/\tau} = \frac{\varepsilon}{R}e^{-t/\tau}$$. The quantity **τ=RC** is call the **time constant** and has dimensions of time.

Discharging a Capacitor



After the switch is closed the current is leaving the capacitor so that **I = -dQ/dt**, where **Q** is the charge on the capacitor and summing all the potential changes in going around the loop gives

$$\frac{Q}{C} - IR = 0 \ ,$$

where **I(t)** and **Q(t)** are a function of time. If the switch is closed at t=0 then **$Q(0)=Q_0$** and

$$\frac{Q}{C} + R\frac{dQ}{dt} = 0 \ ,$$

which can be written in the form

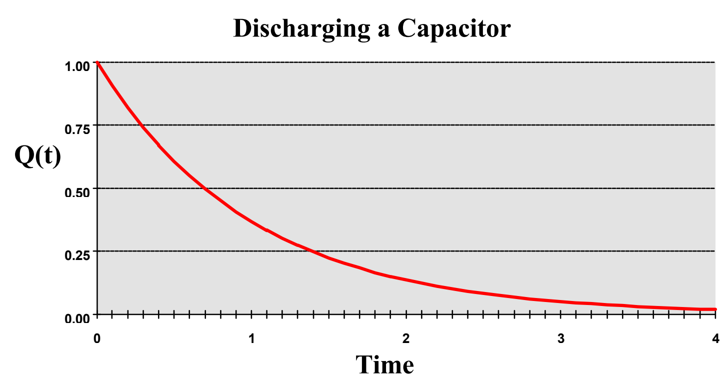
$$\frac{dQ}{dt} = -\frac{1}{\tau}Q \ , \quad \text{where I have defined } \tau = RC.$$

Dividing by Q and multiplying by dt and integrating gives

$$\int_{Q_0}^{Q}\frac{dQ}{Q} = -\int_0^t \frac{1}{\tau}dt \ , \text{ which implies } \quad \ln\left(\frac{Q}{Q_0}\right) = -\frac{t}{\tau} \ .$$

Solving for **Q(t)** gives

$$Q(t) = Q_0 e^{-t/\tau} \ .$$



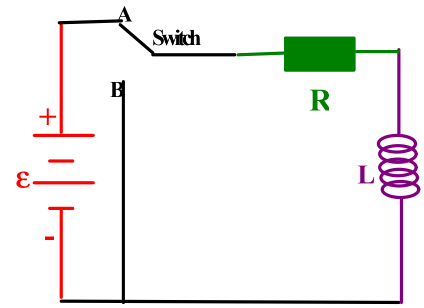
The current is given by **I(t)=-dQ/dt** which yields

$$I(t) = \frac{Q_0}{RC}e^{-t/\tau} \ .$$

The quantity **τ=RC** is call the **"time constant"** and has dimensions of **time**.

RL Circuits

"Building-Up" Phase:



Connecting the switch to **position A** corresponds to the **"building up" phase of an RL circuit**. Summing all the potential changes in going around the loop gives

$$\varepsilon - IR - L\frac{dI}{dt} = 0 \ ,$$

where **I(t)** is a function of time. If the switch is closed (**position A**) at t=0 and **I(0)=0** (**assuming the current is zero at t=0**) then

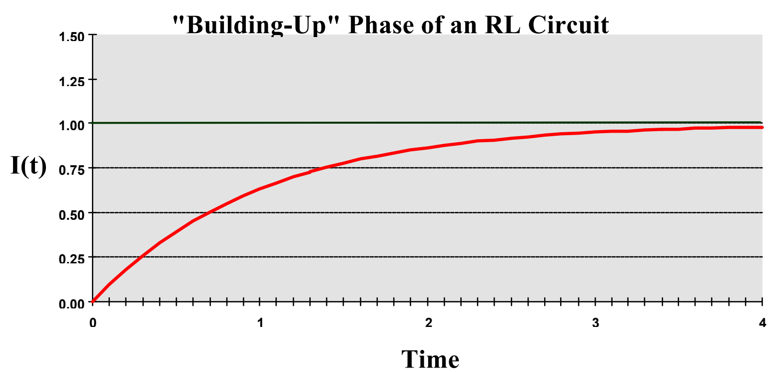
$$\frac{dI}{dt} = -\frac{1}{\tau}\left(I - \frac{\varepsilon}{R}\right) \ ,$$ where I have define **τ=L/R**.

Dividing by (I-ε/R) and multiplying by dt and integrating gives

$$\int_0^I \frac{dI}{(I-\varepsilon/R)} = -\int_0^t \frac{1}{\tau}dt$$, which implies $$\ln\left(\frac{I-\varepsilon/R}{-\varepsilon/R}\right) = -\frac{t}{\tau} .$$

Solving for **I(t)** gives

$$I(t) = \frac{\varepsilon}{R}\left(1 - e^{-t/\tau}\right) .$$



The potential change across the inductor is given by **$\Delta V_L(t)$=-LdI/dt** which yields

$$\Delta V_L(t) = -\varepsilon e^{-t/\tau} .$$

The quantity **τ=L/R** is call the **time constant** and has dimensions of time.

"Collapsing" Phase:

Connecting the switch to **position B** corresponds to the **"collapsing" phase of an RL circuit**. Summing all the potential changes in going around the loop gives $-IR - L\frac{dI}{dt} = 0$, where **I(t)** is a function of time. If the switch is closed (**position B**) at t=0 then **I(0)=I_0** and

$$\frac{dI}{dt} = -\frac{1}{\tau}I \quad \text{and} \quad I(t) = I_0 e^{-t/\tau} .$$