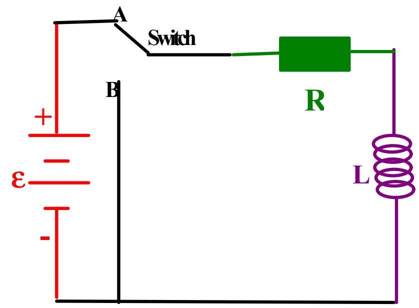
RL Circuits

"Building-Up" Phase:



Connecting the switch to **position A** corresponds to the **"building up" phase of an RL circuit**. Summing all the potential changes in going around the loop gives

$$\varepsilon - IR - L\frac{dI}{dt} = 0 ,$$

where **I(t)** is a function of time. If the switch is closed (**position A**) at t=0 and **I(0)=0** (**assuming the current is zero at t=0**) then

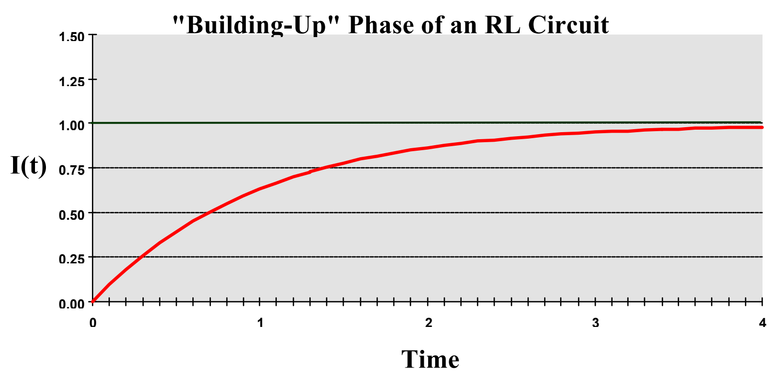
$$\frac{dI}{dt} = -\frac{1}{\tau}\left(I - \frac{\varepsilon}{R}\right) ,$$ where I have define **τ=L/R**.

Dividing by (I-ε/R) and multiplying by dt and integrating gives

$$\int_0^I \frac{dI}{(I-\varepsilon/R)} = -\int_0^t \frac{1}{\tau}dt$$, which implies $$\ln\left(\frac{I-\varepsilon/R}{-\varepsilon/R}\right) = -\frac{t}{\tau} .$$

Solving for **I(t)** gives

$$I(t) = \frac{\varepsilon}{R}\left(1 - e^{-t/\tau}\right) .$$



The potential change across the inductor is given by **$\Delta V_L(t)$=-LdI/dt** which yields

$$\Delta V_L(t) = -\varepsilon e^{-t/\tau} .$$

The quantity **τ=L/R** is call the **time constant** and has dimensions of time.

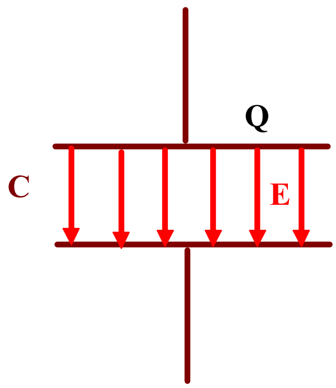
"Collapsing" Phase:

Connecting the switch to **position B** corresponds to the **"collapsing" phase of an RL circuit**. Summing all the potential changes in going around the loop gives $-IR - L\frac{dI}{dt} = 0$, where **I(t)** is a function of time. If the switch is closed (**position B**) at t=0 then **I(0)=I_0** and

$$\frac{dI}{dt} = -\frac{1}{\tau}I \quad \text{and} \quad I(t) = I_0 e^{-t/\tau} .$$

Capacitors and Inductors

Capacitors Store Electric Potential Energy:

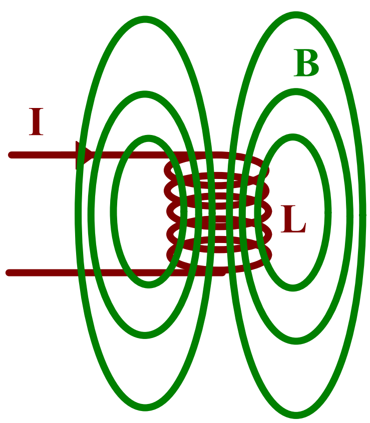


$$U_E = \frac{Q^2}{2C}$$

$$Q = C\Delta V_C \qquad \Delta V_C = Q / C$$

$$u_E = \frac{1}{2}\varepsilon_0 E^2$$ **(E-field energy density)**

Inductors Store Magnetic Potential Energy:



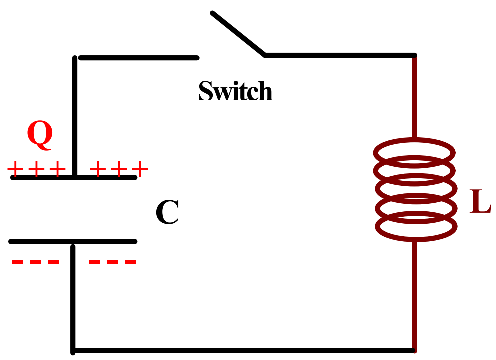
$$U_B = \frac{1}{2}LI^2$$

$$\Phi_B = LI \qquad L = \Phi_B / I$$

$$\varepsilon_L = -L\frac{dI}{dt}$$

$$u_B = \frac{1}{2\mu_0}B^2$$ **(B-field energy density)**

An LC Circuit



At **t = 0** the switch is closed and a capacitor with initial charge $\mathbf{Q_0}$ is connected in series across a inductor (assume there is no resistance). The initial conditions are $\mathbf{Q(0) = Q_0}$ and **I(0) = 0**. Moving around the circuit in the direction of the current flow yields

$$\frac{Q}{C} - L\frac{dI}{dt} = 0 .$$

Since I is flowing out of the capacitor, $I = -dQ/dt$, so that

$$\frac{d^2Q}{dt^2} + \frac{1}{LC}Q = 0 .$$

This differential equation for **Q(t)** is the **SHM differential equation** we studied earlier with $\omega = \sqrt{1/LC}$ and solution

$$Q(t) = A\cos\omega t + B\sin\omega t .$$

The current is thus,

$$I(t) = -\frac{dQ}{dt} = A\omega\sin\omega t - B\omega\cos\omega t .$$

Applying the initial conditions yields

$$Q(t) = Q_0\cos\omega t$$

$$I(t) = Q_0\omega\sin\omega t$$

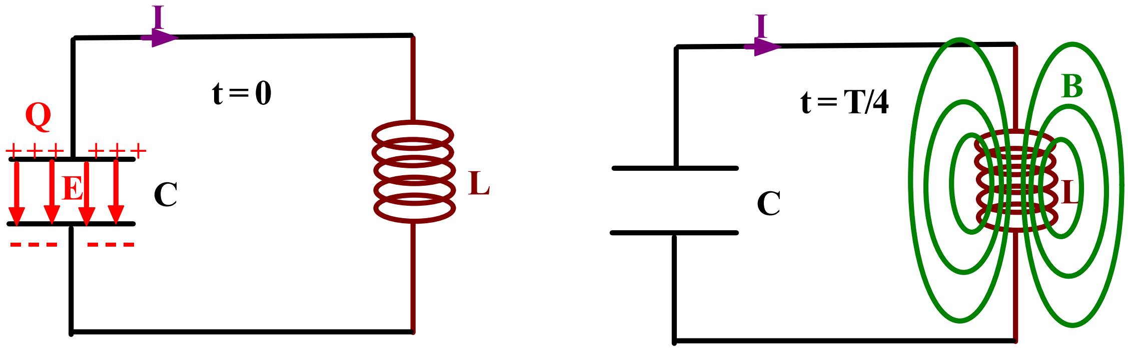
Thus, **Q(t)** and **I(t)** **oscillate with SHM with angular frequency** $\omega = \sqrt{1/LC}$. The stored energy oscillates between **electric** and **magnetic** according to

$$U_E(t) = \frac{Q^2(t)}{2C} = \frac{Q_0^2}{2C}\cos^2\omega t$$

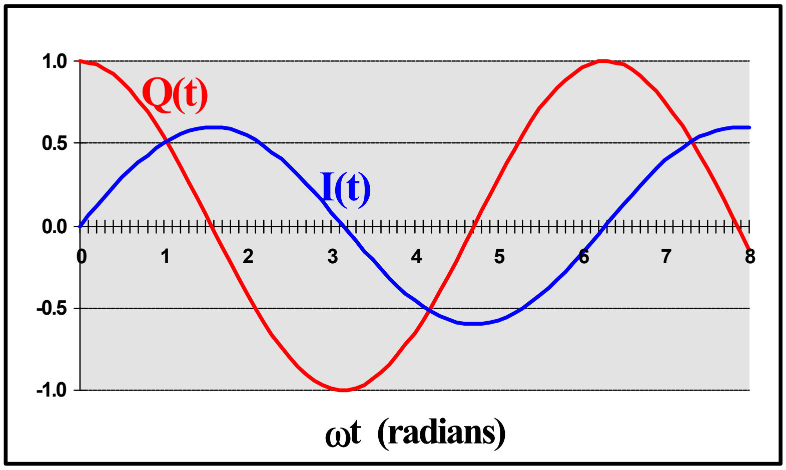
$$U_B(t) = \frac{1}{2}LI^2(t) = \frac{1}{2}LQ_0^2\omega^2\sin^2\omega t$$

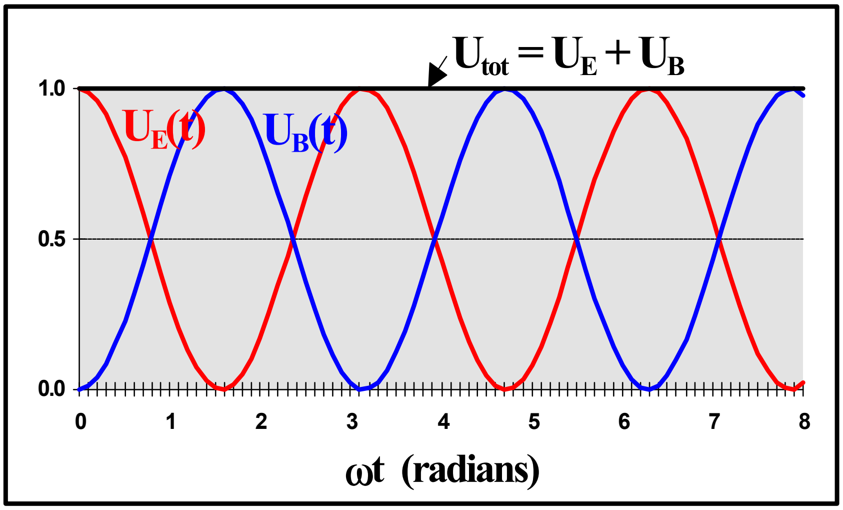
Energy is conserved since $\mathbf{U_{tot}(t) = U_E(t) + U_B(t) = Q_0^2/2C}$ is **constant**.

LC Oscillations



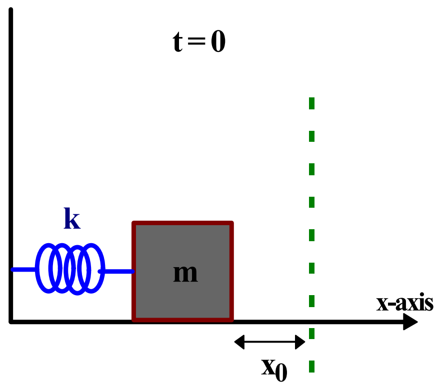
$$Q(t) = Q_0 \cos\omega t$$
$$I(t) = Q_0\omega \sin\omega t$$

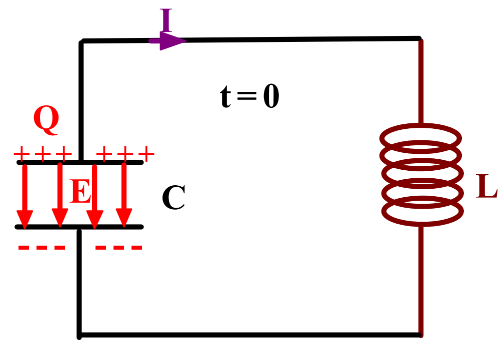




$$U_E(t) = \frac{Q_0^2}{2C}\cos^2\omega t$$
$$U_B(t) = \frac{Q_0^2}{2C}\sin^2\omega t$$

Mechanical Analogy





At t = 0:

$$E = \frac{1}{2}kx_0^2$$
$$v = 0$$

At Later t:

$$v = \frac{dx}{dt}$$
$$x(t) = x_0 \cos\omega t$$
$$\omega = \sqrt{\frac{k}{m}}$$
$$E = \frac{1}{2}mv^2 + \frac{1}{2}kx^2$$

At t = 0:

$$U = \frac{1}{2C}Q_0^2$$
$$I = 0$$

At Later t:

$$I = -\frac{dQ}{dt}$$
$$Q(t) = Q_0 \cos\omega t$$
$$\omega = \sqrt{\frac{1}{LC}}$$
$$E = \frac{1}{2}LI^2 + \frac{1}{2C}Q^2$$

Constant

Correspondence:

$$x(t) \leftrightarrow Q(t)$$
$$v(t) \leftrightarrow I(t)$$
$$m \leftrightarrow L$$
$$k \leftrightarrow 1/C$$

Another Differential Equation

Consider the **2nd order** differential equation

$$\frac{d^2x(t)}{dt^2} + D\frac{dx(t)}{dt} + Cx(t) = 0\,,$$

where **C** and **D** are constants. We solve this equation by turning it into an **algebraic equation** by looking for a solution of the form $x(t) = Ae^{at}$. Substituting this into the differential equation yields,

$$a^2 + Da + C = 0 \quad \text{or} \quad \boxed{a = -\frac{D}{2} \pm \sqrt{\left(\frac{D}{2}\right)^2 - C}}\,.$$

Case I (C > (D/2)^2, damped oscillations):

For **C > (D/2)^2**, $a = -D/2 \pm i\sqrt{C-(D/2)^2} = -D/2 \pm i\omega'$, where $\omega' = \sqrt{C-(D/2)^2}$, and the most general solution has the form:

$$x(t) = e^{-Dt/2}\left(Ae^{i\omega' t} + Be^{-i\omega' t}\right)$$

$$x(t) = e^{-Dt/2}\left(A\cos(\omega' t) + B\sin(\omega' t)\right)$$

$$x(t) = Ae^{-Dt/2}\sin(\omega' t + \phi)$$

$$x(t) = Ae^{-Dt/2}\cos(\omega' t + \phi)$$

where **A**, **B**, and ϕ are arbitrary constants.

Case II (C < (D/2)^2, over damped):

For **C < (D/2)^2**, $a = -D/2 \pm \sqrt{(D/2)^2 - C} = -D/2 \pm \gamma$, where $\gamma = \sqrt{(D/2)^2 - C}$. In this case,

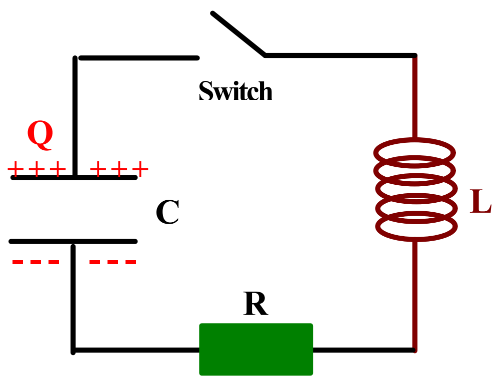
$$x(t) = e^{-Dt/2}\left(Ae^{\gamma t} + Be^{-\gamma t}\right).$$

Case III (C = (D/2)^2, critically damped):

For **C = (D/2)^2**, $a = -D/2$, and

$$x(t) = Ae^{-Dt/2}.$$

An LRC Circuit



At **t = 0** the switch is closed and a capacitor with initial charge $\mathbf{Q_0}$ is connected in series across an inductor and a resistor. The initial conditions are $\mathbf{Q(0) = Q_0}$ and **I(0) = 0**. Moving around the circuit in the direction of the current flow yields

$$\frac{Q}{C} - L\frac{dI}{dt} - IR = 0 .$$

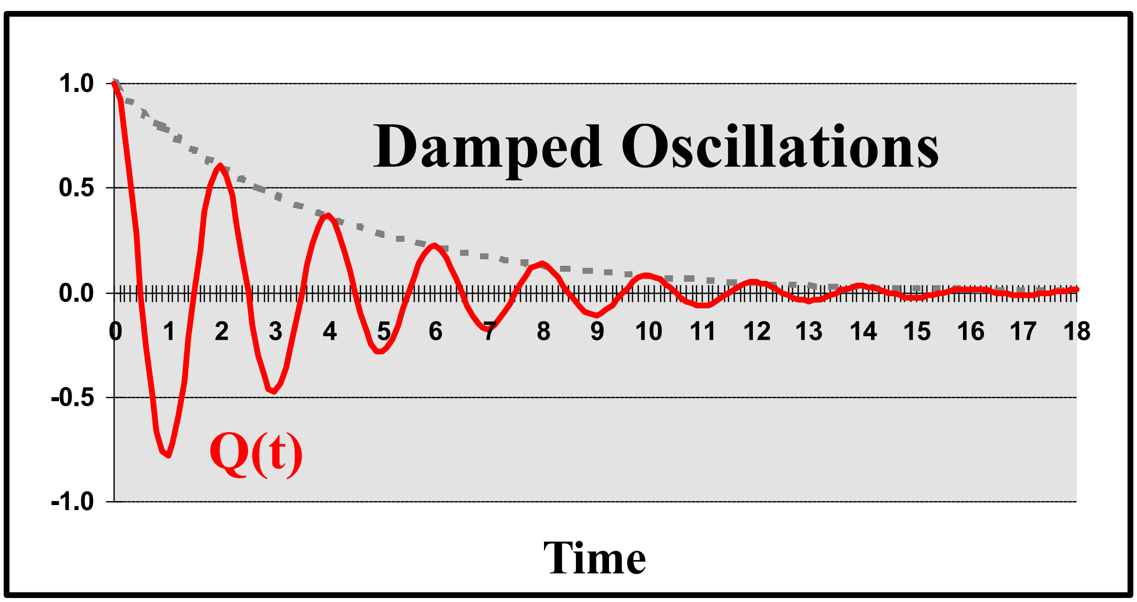
Since I is flowing out of the capacitor, $I = -dQ/dt$, so that

$$\frac{d^2Q}{dt^2} + \frac{R}{L}\frac{dQ}{dt} + \frac{1}{LC}Q = 0 .$$

This differential equation for **Q(t)** is the differential equation we studied earlier. If we take the case where $\mathbf{R^2 < 4L/C}$ (**damped oscillations**) then

$$Q(t) = Q_0 e^{-Rt/2L}\cos\omega' t ,$$

with $\omega' = \sqrt{\omega^2 - (R/2L)^2}$ and $\omega = \sqrt{1/LC}$.

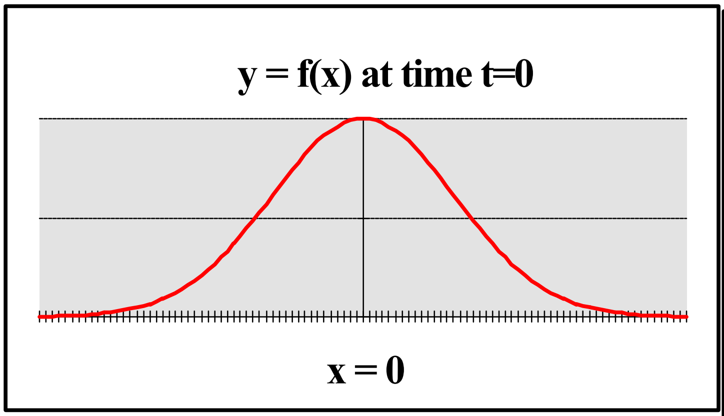


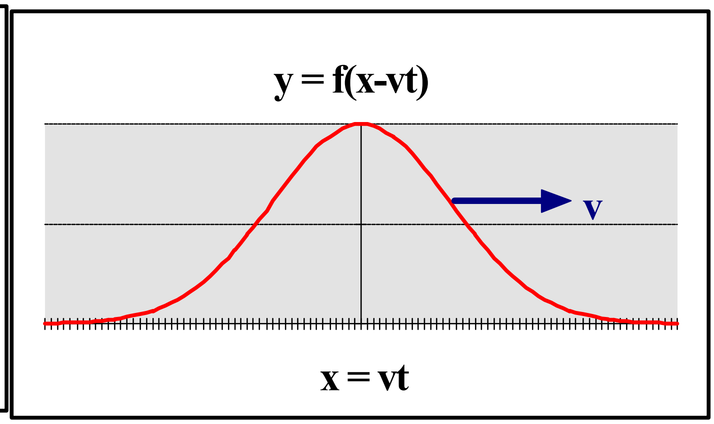
Traveling Waves

A "wave" is a traveling disturbance that transports energy but not matter.

Constructing Traveling Waves:

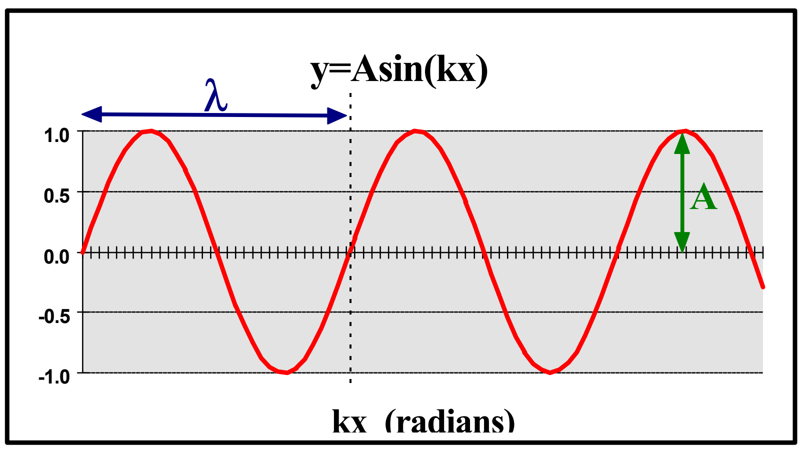
To construct a wave with shape **y = f(x)** at **time t = 0** traveling to the **right with speed v** simply make the replacement $x \to x - vt$.





Traveling Harmonic Waves:

Harmonic waves have the form **y = A sin(kx)** or **y = Acos(kx)** at time **t = 0**, where **k** is the **"wave number"** (**k = 2π/λ** where **λ** is the **"wave length"**) and **A** is the **"amplitude"**. To construct an harmonic wave traveling to the right with speed **v**, replace **x** by **x-vt** as follows:



y = Asin(k(x-vt) = Asin(kx-ωt) where **ω = kv** (**v = ω/k**). The period of the oscillation, **T = 2π/ω = 1/f**, where **f** is the **linear frequency** (measured in **Hertz** where **1Hz = 1/sec**) and **ω** is the **angular frequency** (**ω = 2πf**). The speed of propagation is given by **v = ω/k = λf** .

y = y(x,t) = Asin(kx-ωt) right moving harmonic wave
y = y(x,t) = Asin(kx+ωt) left moving harmonic wave

The Wave Equation

$$\frac{\partial^2 y(x,t)}{\partial x^2} - \frac{1}{v^2}\frac{\partial^2 y(x,t)}{\partial t^2} = 0$$

Whenever analysis of a system results in an equation of the form given above then we know that the system supports traveling waves propagating at speed v.

General Proof:

If **y = y(x,t) = f(x-vt)** then

$$\frac{\partial y}{\partial x} = f' \qquad \frac{\partial^2 y}{\partial x^2} = f''$$

$$\frac{\partial y}{\partial t} = -vf' \qquad \frac{\partial^2 y}{\partial t^2} = v^2 f''$$

and

$$\frac{\partial^2 y(x,t)}{\partial x^2} - \frac{1}{v^2}\frac{\partial^2 y(x,t)}{\partial t^2} = f'' - f'' = 0 .$$

Proof for Harmonic Wave:

If **y = y(x,t) = Asin(kx-ωt)** then

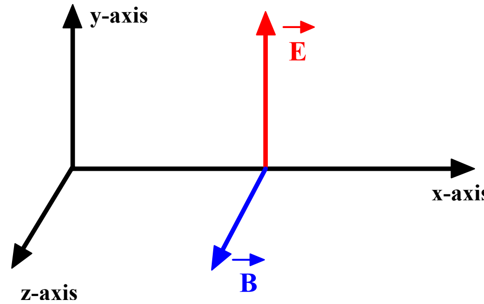
$$\frac{\partial^2 y}{\partial x^2} = -k^2 A\sin(kx - \omega t) \qquad \frac{\partial^2 y}{\partial t^2} = -\omega^2 A\sin(kx - \omega t)$$

and

$$\frac{\partial^2 y(x,t)}{\partial x^2} - \frac{1}{v^2}\frac{\partial^2 y(x,t)}{\partial t^2} = \left(-k^2 + \frac{\omega^2}{v^2}\right)A\sin(kx - \omega t) = 0 ,$$

since **ω = kv**.

Light Propagating in Empty Space



Since there are no charges and no current in empty space, Faraday's Law and Ampere's Law take the form

$$\oint \vec{E}\cdot d\vec{l} = -\frac{d\Phi_B}{dt} \qquad \oint \vec{B}\cdot d\vec{l} = \mu_0\varepsilon_0\frac{d\Phi_E}{dt}.$$

Look for a solution of the form

$$\vec{E}(x,t) = E_y(x,t)\hat{y}$$
$$\vec{B}(x,t) = B_z(x,t)\hat{z}$$

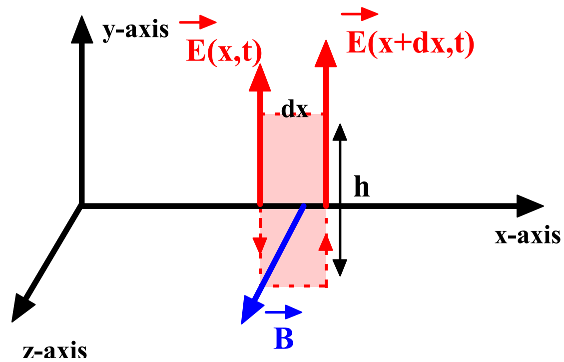
Faraday's Law:

Computing the left and right hand side of **Faraday's Law** using a rectangle (**in the xy-plane**) with width **dx** and **height h** (**counterclockwise**) gives

$$E_y(x+dx,t)h - E_y(x,t)h = -\frac{\partial B_z}{\partial t}hdx$$

or

$$\boxed{\frac{\partial E_y}{\partial x} = -\frac{\partial B_z}{\partial t}}$$



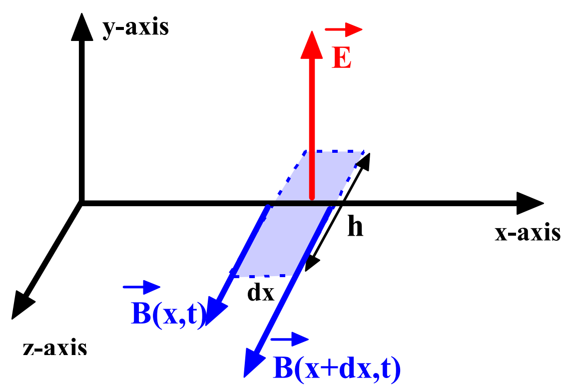
Ampere's Law:

Computing the left and right hand side of **Ampere's Law** using a rectangle (**in the xz-plane**) with width **dx** and **height h** (**counterclockwise**) gives

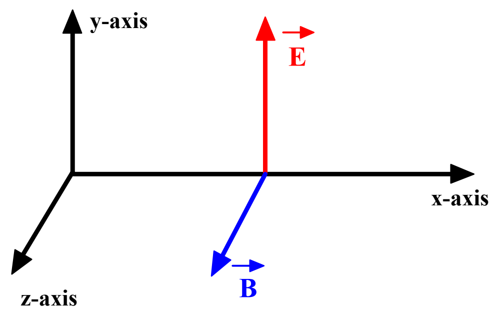
$$B_z(x,t)h - B_z(x+dx,t)h = \mu_0\varepsilon_0\frac{\partial E_y}{\partial t}hdx$$

or

$$\boxed{-\frac{\partial B_z}{\partial x} = \mu_0\varepsilon_0\frac{\partial E_y}{\partial t}}$$



Electromagnetic Plane Waves (1)



We have the following two **differential equations** for $\mathbf{E_y(x,t)}$ and $\mathbf{B_z(x,t)}$:

$$\frac{\partial B_z}{\partial t} = -\frac{\partial E_y}{\partial x} \qquad \mathbf{(1)}$$

and

$$\frac{\partial E_y}{\partial t} = -\frac{1}{\mu_0\varepsilon_0}\frac{\partial B_z}{\partial x} \qquad \mathbf{(2)}$$

Taking the time derivative of **(2)** and using **(1)** gives

$$\frac{\partial^2 E_y}{\partial t^2} = -\frac{1}{\mu_0\varepsilon_0}\frac{\partial}{\partial t}\left(\frac{\partial B_z}{\partial x}\right) = -\frac{1}{\mu_0\varepsilon_0}\frac{\partial}{\partial x}\left(\frac{\partial B_z}{\partial t}\right) = \frac{1}{\mu_0\varepsilon_0}\frac{\partial^2 E_y}{\partial x^2}$$

which implies

$$\frac{\partial^2 E_y}{\partial x^2} - \mu_0\varepsilon_0\frac{\partial^2 E_y}{\partial t^2} = 0\ .$$

Thus $\mathbf{E_y(x,t)}$ satisfies the wave equation with speed $v = 1/\sqrt{\varepsilon_0\mu_0}$ and has a solution in the form of traveling waves as follows:

$$\mathbf{E_y(x,t) = E_0 \sin(kx-\omega t)},$$

where $\mathbf{E_0}$ is the **amplitude of the electric field oscillations** and where the wave has a **unique speed**

$$v = c = \frac{\omega}{k} = \lambda f = \frac{1}{\sqrt{\varepsilon_0\mu_0}} = 2.99792 \quad 10^8 m/s \ \ (\textbf{speed of light}).$$

From **(1)** we see that

$$\frac{\partial B_z}{\partial t} = -\frac{\partial E_y}{\partial x} = -E_0 k\cos(kx-\omega t)\,,$$

which has a solution given by

$$B_z(x,t) = E_0\frac{k}{\omega}\sin(kx-\omega t) = \frac{E_0}{c}\sin(kx-\omega t)\,,$$

so that

$$\mathbf{B_z(x,t) = B_0 \sin(kx-\omega t)},$$

where $\mathbf{B_0 = E_0/c}$ is the **amplitude of the magnetic field oscillations**.

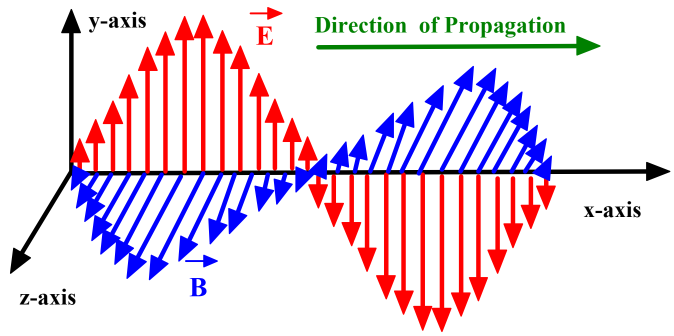
Electromagnetic Plane Waves (2)

The **plane harmonic wave solution** for light with frequency **f** and wavelength λ and speed **c = fλ** is given by

$$\vec{E}(x,t) = E_0 \sin(kx - \omega t)\hat{y}$$

$$\vec{B}(x,t) = B_0 \sin(kx - \omega t)\hat{z}$$

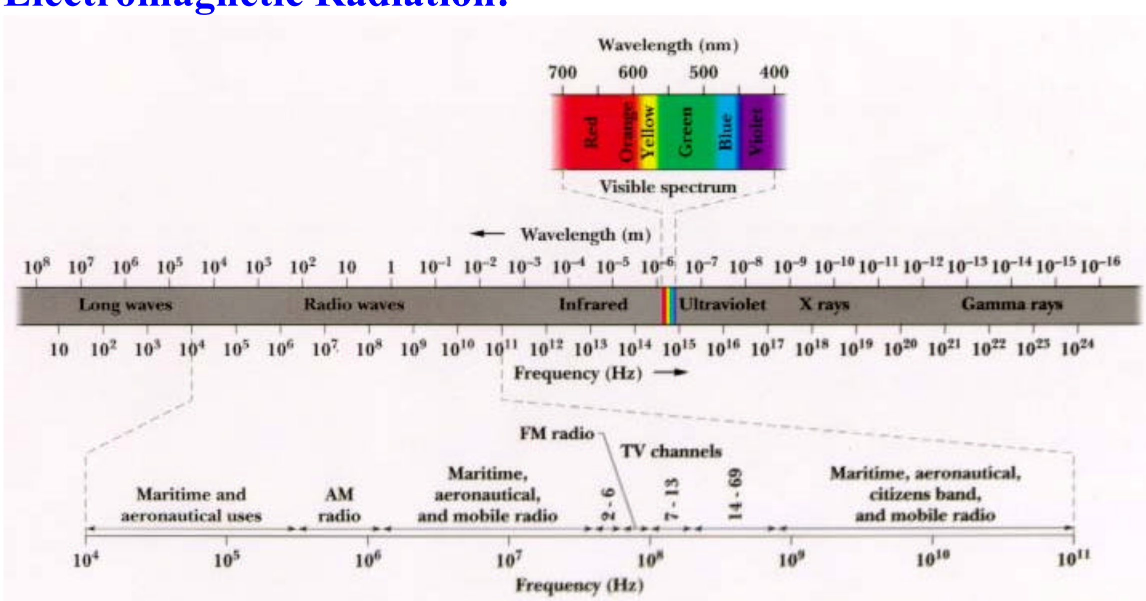
where $\mathbf{k = 2\pi/\lambda}$, $\mathbf{\omega = 2\pi f}$, and $\mathbf{E_0 = cB_0}$.



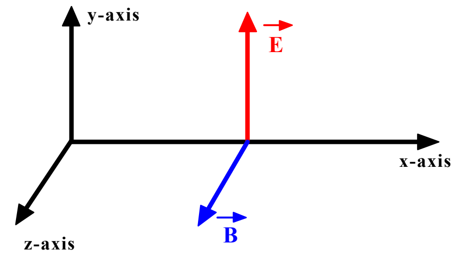
Properties of the Electromagnetic Plane Wave:

- **Wave travels at speed c ($c = 1/\sqrt{\mu_0\varepsilon_0}$).**
- **E and B are perpendicular ($\vec{E}\cdot\vec{B} = 0$).**
- **The wave travels in the direction of $\vec{E}\ \ \vec{B}$.**
- **At any point and time E = cB.**

Electromagnetic Radiation:



Energy Transport - Poynting Vector



Electric and Magnetic Energy Density:

For an **electromagnetic plane wave**

$$E_y(x,t) = E_0\sin(kx-\omega t),$$

$$B_z(x,t) = B_0\sin(kx-\omega t),$$

where $B_0 = E_0/c$. The **electric energy density** is given by

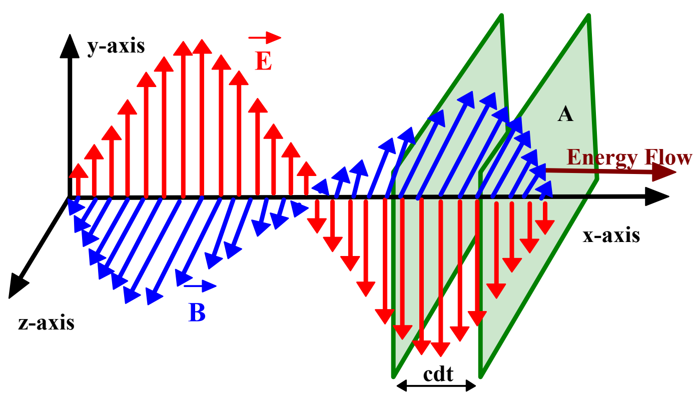
$u_E = \frac{1}{2}\varepsilon_0 E^2 = \frac{1}{2}\varepsilon_0 E_0^2\sin^2(kx-\omega t)$ and the **magnetic energy density** is

$$u_B = \frac{1}{2\mu_0}B^2 = \frac{1}{2\mu_0 c^2}E^2 = \frac{1}{2}\varepsilon_0 E^2 = u_E,$$

where I used $\mathbf{E} = \mathbf{cB}$. Thus, **for light the electric and magnetic field energy densities are equal** and the **total energy density** is

$$u_{tot} = u_E + u_B = \varepsilon_0 E^2 = \frac{1}{\mu_0}B^2 = \varepsilon_0 E_0^2\sin^2(kx-\omega t).$$

Poynting Vector ($\vec{S} = \frac{1}{\mu_0}\vec{E} \quad \vec{B}$):



The **direction** of the **Poynting Vector** is the **direction of energy flow** and the **magnitude**

$$S = \frac{1}{\mu_0}EB = \frac{E^2}{\mu_0 c} = \frac{1}{A}\frac{dU}{dt}$$

is the **energy per unit time per unit area** (units of **Watts/m^2**).

Proof:

$dU_{tot} = u_{tot}V = \varepsilon_0 E^2 Acdt$ so

$$S = \frac{1}{A}\frac{dU}{dt} = \varepsilon_0 c E^2 = \frac{E^2}{\mu_0 c} = \frac{E_0^2}{\mu_0 c}\sin^2(kx-\omega t).$$

Intensity of the Radiation (Watts/m^2):

The intensity, **I**, is the **average of S** as follows:

$$I = \overline{S} = \frac{1}{A}\frac{d\overline{U}}{dt} = \frac{E_0^2}{\mu_0 c}\langle\sin^2(kx-\omega t)\rangle = \frac{E_0^2}{2\mu_0 c}.$$

Momentum Transport - Radiation Pressure

Relativistic Energy and Momentum:

$$E^2 = (cp)^2 + (m_0c^2)^2$$

energy momentum rest mass

For light $\mathbf{m_0} = \mathbf{0}$ and

$$E = cp \quad \text{(for light)}$$

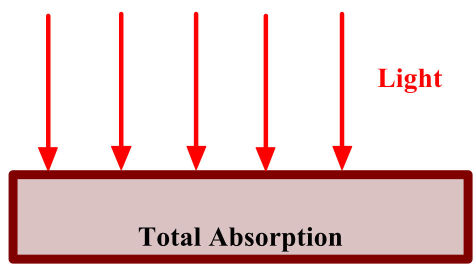
For light the **average momentum per unit time per unit area** is equal to the intensity of the light, **I**, divided by speed of light, **c**, as follows:

$$\frac{1}{A}\frac{d\overline{p}}{dt} = \frac{1}{c}\frac{1}{A}\frac{d\overline{U}}{dt} = \frac{1}{c}I \,.$$

Total Absorption:

$$\overline{F} = \frac{d\overline{p}}{dt} = \frac{1}{c}\frac{d\overline{U}}{dt} = \frac{1}{c}IA$$

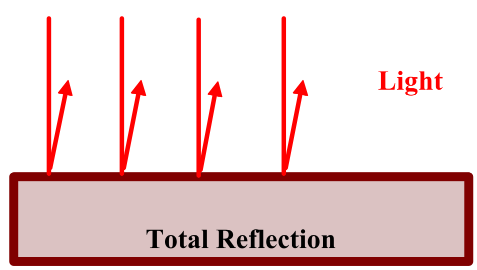
$$P = \frac{\overline{F}}{A} = \frac{1}{c}I \quad \text{(radiation pressure)}$$



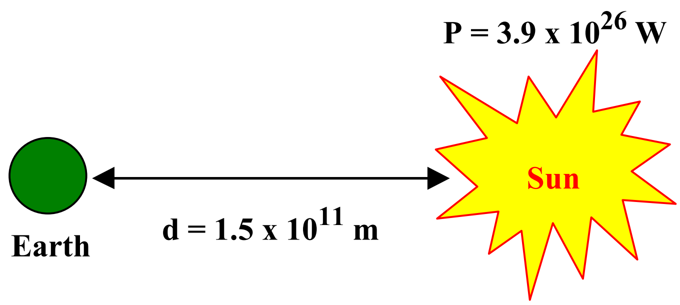
Total Reflection:

$$\overline{F} = \frac{d\overline{p}}{dt} = \frac{2}{c}\frac{d\overline{U}}{dt} = \frac{2}{c}IA \,.$$

$$P = \frac{\overline{F}}{A} = \frac{2}{c}I \quad \text{(radiation pressure)}$$



The Radiation Power of the Sun



Problem:

The **radiation power** of the sun is **3.9×10^{26} W** and the distance from the Earth to the sun is **1.5×10^{11} m**.

(a) What is the **intensity** of the electromagnetic radiation from the sun at the surface of the Earth (outside the atmosphere)? (**answer: 1.4 kW/m^2**)

(b) What is the **maximum value of the electric field** in the light coming from the sun? (**answer: 1,020 V/m**)

(c) What is the **maximum energy density of the electric field** in the light coming from the sun? (**answer: 4.6×10^{-6} J/m^3**)

(d) What is the **maximum value of the magnetic field** in the light coming from the sun? (**answer: 3.4   T**)

(e) What is the **maximum energy density of the magnetic field** in the light coming from the sun? (**answer: 4.6×10^{-6} J/m^3**)

(f) Assuming complete absorption what is the **radiation pressure on the Earth** from the light coming from the sun? (**answer: 4.7×10^{-6} N/m^2**)

(g) Assuming complete absorption what is the **radiation force on the Earth** from the light coming from the sun? The radius of the Earth is about **6.4×10^{6} m**. (**answer: 6×10^{8} N**)

(h) What is the **gravitational force** on the Earth due to the sun. The mass of the Earth and the sun are **5.98×10^{24} kg** and **1.99×10^{30} kg**, respectively, and **$G = 6.67 \times 10^{-11}$ Nm^2/kg^2**. (**answer: 3.5×10^{22} N**)