Modeling and Simulation

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What are Dynamic Systems?

- Dynamic systems are systems that change over time, where the state of the system at any given time depends on past and current inputs, as well as the internal dynamics of the system itself.
- Examples:
- Mechanical systems: A swinging pendulum, a vibrating mass-spring-damper system.
- Electrical circuits: RLC circuits, operational amplifiers.
- Biological systems: Population growth models, biochemical reaction networks.

Static System vs Dynamic System (1)

- Definition: A system is called static if the knowledge of an input value uniquely determines the knowledge of the output value regardless of the time interval.
- For a static system: no change in the output value will occur if the input value is constant.

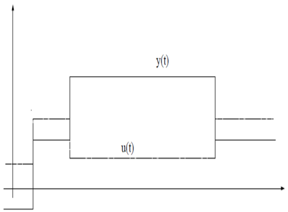

Figure 1: Static system

Static System vs Dynamic System (2)

- Definition: A system is called dynamic if the knowledge of an input value does not uniquely determine the knowledge of the output value in different time intervals.
- For a dynamic system: the output value changes not only depending on the input value.

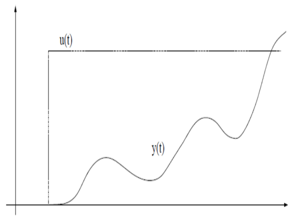

Figure 2: Dynamic system

Components of Dynamic Systems

- Inputs: These are the signals or forces applied to the system that influence its behavior. Inputs can be constant, time-varying, or stochastic.
- Outputs: These are the measurable quantities or responses of the system that are of interest. Outputs are typically influenced by the inputs and the internal dynamics of the system.
- State Variables: These are variables that define the current state of the system and are necessary to predict its future behavior. State variables fully describe the system's dynamics.
- Parameters: These are constants or coefficients that characterize the behavior of the system. Parameters can include physical properties, such as mass and stiffness, as well as coefficients in mathematical models.

Basic Notions

- Time: is the set T. Represents the instants of sampling of the studied system. Such that each value $t \in T$ is associated with an input value and an output value.
- Input(s): Represented by a set U, With values $u(t) \in U$ which, Define the evolution of inputs at time t.
- Output(s): Represented by a set Y, With values $y(t) \in Y$ which, Define the evolution of outputs at time t. NB. The evolution of a value is the variation of this quantity over a very small interval [t, t+dt].

Types of Dynamic Systems

- Continuous-time systems: These are systems in which the state variables and inputs are defined and evolve continuously over time. The dynamics of continuous-time systems are typically described using differential equations.
- Discrete-time systems: These are systems in which the state variables and inputs are defined at discrete time intervals. The dynamics of discrete-time systems are described using the difference equations.
- Hybrid systems: These are systems that exhibit both continuous and discrete behavior. Hybrid systems are common in control systems with both continuous and digital components, as well as in cyber-physical systems where continuous physical processes are controlled by discrete computational algorithms.

Types of discrete-time systems

- Two types of discrete-time systems are distinguished:
- A synchronous discrete-time system: in which the system variables take their values at a predetermined frequency.
- Example: a system that describes the monthly evolution of sales dates in a country. Here, the set T is the set of integers.
- An asynchronous discrete-time system: in which the system variables take their values according to a random distribution.
- Example: a system that describes the number of items purchased from a supermarket is an asynchronous discrete-time system. Here, an item sold is counted only when a customer takes it.

Input Functions and Output Function (1)

- Example: filling a water tank:
- Consider a cylindrical water tank with radius R and height H. This tank is filled by a water source located above and pours water from a circular hole of radius r, located at the bottom of the tank.

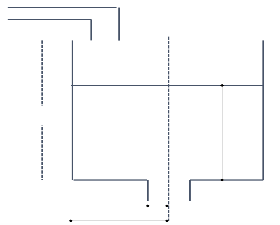

Figure 3: Water Tank

Input Functions and Output Function (2)

- Inputs:
- u1(t): the flow rate of water into the tank at time t (measured in m^3/s)
- u2(t): the area of the hole at the bottom of the tank at time t (measured in m^2) Output:
- y(t): the water level at time t (measured in m) Modeling of this dynamic system: We seek a relationship between the inputs and the output, but as dynamic systems are characterized by non-determinism, we cannot find an instantaneous relationship between U and Y.

Input Functions and Output Function (3)

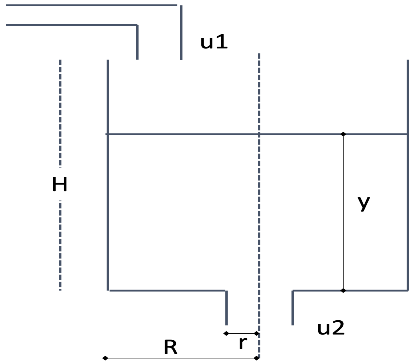

Figure 4: Water Tank

Input Functions and Output Function

- Modeling Idea:
- We seek a non-instantaneous relationship between U and Y. For this example:
- Let x(t) denote the volume of water in the tank at time t (measured in m^3). We have:
- Flow into the tank: $d(t) = din(t) - dout(t)$
- According to Bernoulli's law: Flow out: dout
- $dout(t) = u2(t) - \sqrt{((2 * g * (x(t))/(\pi R^2)))}$

Input Functions and Output Function

- Modeling Idea:
- If we consider Δ as the sampling period, then:
- The volume of water in the tank at time t $x(t)$ is:
- $x(t) = max(0, x(t-\Delta) + d(t-\Delta).\Delta)$
- $x(t) = max(0, x(t-\Delta) + u1(t-\Delta) - (u2(t-\Delta) - u2(t) - \sqrt{((2 * g * (x(t))/(R^2)))}) \times \Delta)$
  We can subsequently deduce:
- $y(t) = x(t)/(\pi R^2)$

The State of the Dynamic System

- Conclusion from the reservoir example:
- The past history of a system helps determine the output based on the input by formalizing it.
- The system's history can be represented by a new quantity called the state of the dynamic system (symbolized by: $x(t)$).

Axiomatic Definition of a Dynamic System (1)

- Set T: Time
- Set U: input values where $u(t) \in U$ is an input value
- Set Ω: input functions where $u(.) \in \Omega$
- Set X: system states
- Set Y: output values where $y(t) \in Y$ is an output value
- Set Γ: output functions

Axiomatic Definition of a Dynamic System (2)

- State transition function $\Phi : x(t) = \Phi(t, t0, x(t0), u(.))$
- This function allows finding the new state of the system at time t, knowing that the system was in state $x(t0)$ at time t_0, this is ensured by the application of the input function $u(.)$ which corresponds to the interval $[t0, t]$
- Output transformation function $\eta : y(t) = \eta(t, x(t))$. This function allows finding the output value at time t

Axiomatic Definition of a Dynamic System (3)

- A dynamic system can be represented by the following tuple:
- $S = (T, U, \Omega, X, Y, \lceil, \Phi, \eta)$

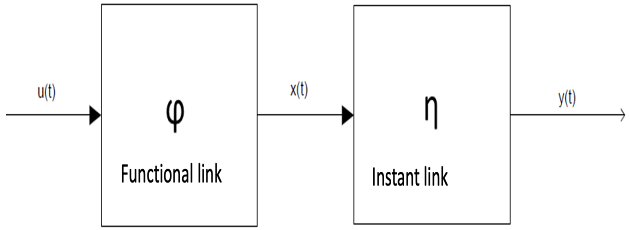

Figure 5: Functional and instant link

Proprietaries on the system dynamics and the transition function

- Equilibrium state (1):
- Definition:
- A state $\overline{x} \in X$ is called an equilibrium state in infinite time if for each initial time $t \in T$, there exists an input function $u(.) \in \Omega$ such that: $\Phi(t, t0, \overline{x}, u(.)) = \overline{x} \forall t \in T$

Proprietaries on the system dynamics and the transition function

- Equilibrium State (2):
- Example 1 (Tank):
- Let an initial state $x(t) = 0$. If $d_{\text{in}} \leq d_{\text{out}}$ for all $t \in T$, then $\phi(t, t_0, x(t) = 0, u(.)) = 0$, thus, $x(t) = 0$ is an equilibrium state denoted $\bar{x}$.
- Example 2 (Tank):
- Let an initial state $x(t) = \text{Val}$. If $d_{\text{in}} = d_{\text{out}}$ for all $t \in T$, then $\phi(t, t_0, x(t) = \text{Val}, u(.)) = \text{Val}$, thus, $x(t) = \text{Val}$ is also an equilibrium state denoted $\bar{x}$.

Solution (1)

- To model the dynamics of the water tank system, we can use principles of fluid mechanics and differential equations. Here's a simplified model of the system:
  Let:
- $h(t)$ be the height of the water in the tank at time t.
- $A(t)$ be the cross-sectional area of the water at height $h(t)$.
- Q_{in} be the flow rate of water into the tank from the source.
- Q_{out} be the flow rate of water out of the tank through the hole at the bottom.
- r be the radius of the hole at the bottom.
- g be the acceleration due to gravity.

Solution (2)

- The change in the volume of water in the tank with respect to time is given by the difference between the inflow and outflow rates:

$$\frac{dV}{dt} = Q_{\text{in}} - Q_{\text{out}}$$

Since the tank is a cylinder, the cross-sectional area of the water at height $h(t)$ is given by:

$$A(t) = \pi(R^2 - (R - h(t))^2)$$

The inflow rate can be assumed to be constant, so $Q_{\text{in}} = C_{\text{in}}$, where C_{in} is a constant.

The outflow rate depends on the velocity of the water flowing out of the hole. Using Torricelli's law, the velocity $v(t)$ of the water flowing out of the hole at the bottom of the tank is given by: