MATRICES

After studying this chapter you will acquire the skills in

- knowledge on matrices
- Knowledge on matrix operations.
- Matrix as a tool of solving linear equations with two or three unknowns.

List of References:

- Frank Ayres, JR, Theory and Problems of Matrices Sohaum's Outline Series
- Datta KB , Matrix and Linear Algebra
- Vatssa BS, Theory of Matrices, second Revise Edition
- Cooray TMJA, Advance Mathematics for Engineers, Chapter 1- 4

Chapter I: Introduction of Matrices

1.1 Definition 1:

A rectangular arrangement of mn numbers, in m rows and n columns and enclosed within a bracket is called a matrix. We shall denote matrices by capital letters as *A,B, C* etc.

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} = \left(a_{ij}\right)_{m \times n}$$

A is a matrix of order $m \times n$. i^{th} row j^{th} column element of the matrix denoted by a_{ij}

Remark: A matrix is not just a collection of elements but every element has assigned a definite position in a particular row and column.

1.2 Special Types of Matrices:

1. **Square matrix**:

A matrix in which numbers of rows are equal to number of columns is called a square matrix.

Example:

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \qquad B = \begin{pmatrix} 2 & 5 & -8 \\ 0 & -3 & -4 \\ 6 & 8 & 9 \end{pmatrix}$$

2. **Diagonal matrix**:

A square matrix $A = \left(a_{ij}\right)_{n \times n}$ is called a diagonal matrix if each of its non-diagonal element is zero.

That is $a_{ij} = 0 \ \ if \ i \neq j$ and at least one element $a_{ii} \neq 0$.

Example:

$$A = \begin{pmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{pmatrix} \qquad B = \begin{pmatrix} 2 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 9 \end{pmatrix}$$

3. **Identity Matrix**

A diagonal matrix whose diagonal elements are equal to 1 is called identity matrix and denoted by I_n.

That is $a_{ij} = \begin{cases} 0 & if \ i \neq j \\ 1 & if \ i = j \end{cases}$

Example:

$$I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

4. **Upper Triangular matrix**:

A square matrix said to be a Upper triangular matrix if $a_{ij} = 0 \ \ if \ i > j$.

Example:

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{pmatrix} \qquad B = \begin{pmatrix} 2 & 0 & 8 \\ 0 & -2 & 5 \\ 0 & 0 & 7 \end{pmatrix}$$

5. **Lower Triangular Matrix**:

A square matrix said to be a Lower triangular matrix if $a_{ij} = 0 \ \ if \ i < j$.

Example:

$$A = \begin{pmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \qquad B = \begin{pmatrix} -1 & 0 & 0 \\ 7 & 0 & 0 \\ 9 & 6 & 2 \end{pmatrix}$$

6. **Symmetric Matrix**:

A square matrix *A* = $(a_{ij})_{n \times n}$ said to be a symmetric if $a_{ij} = a_{ji}$ for all *i* and *j.*

Example:

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{13} & a_{23} & a_{33} \end{pmatrix} \qquad B = \begin{pmatrix} 8 & -2 & 7 \\ -2 & -9 & 3 \\ 7 & 3 & 5 \end{pmatrix}$$

7. Skew- Symmetric Matrix:

A square matrix $A = (a_{ij})_{n \times n}$ said to be a skew-symmetric if $a_{ij} = -a_{ji}$ for all *i* and *j.*

Example:

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ -a_{12} & a_{22} & a_{23} \\ -a_{13} & -a_{23} & a_{33} \end{pmatrix} \qquad B = \begin{pmatrix} 8 & -2 & 7 \\ 2 & -9 & 3 \\ -7 & -3 & 5 \end{pmatrix}$$

8. Zero Matrix:

A matrix whose all elements are zero is called as Zero Matrix and order $n \times m$ Zero matrix denoted by $0_{n \times m}$.

Example:

$$0_{3\times 2} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$$

9. Row Vector

A matrix consists a single row is called as a row vector or row matrix.

Example:

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \end{pmatrix} \qquad B = \begin{pmatrix} 7 & 4 & -3 \end{pmatrix}$$

10. Column Vector

A matrix consists a single column is called a column vector or column matrix.

Example:

$$A = \begin{pmatrix} a_{11} \\ a_{21} \\ a_{31} \end{pmatrix} \qquad B = \begin{pmatrix} 9 \\ -7 \\ 3 \end{pmatrix}$$

Chapter 2: Matrix Algebra

2.1. Equality of two matrices:

Two matrices A and B are said to be equal if

(i) They are of same order.
(ii) Their corresponding elements are equal.

That is if $A = (a_{ij})_{m \times n}$ and $B = (b_{ij})_{m \times n}$ then $a_{ij} = b_{ij}$ for all *i* and *j.*

2.2. Scalar multiple of a matrix

Let k be a scalar then scalar product of matrix *A* = $(a_{ij})_{m\times n}$ given denoted by *kA* and given by *kA* = $(ka_{ij})_{m\times n}$ or

$$kA = \begin{pmatrix} ka_{11} & ka_{12} & \cdots & ka_{1n} \\ \vdots & & \ddots & \vdots \\ ka_{m1} & ka_{m2} & \cdots & ka_{mn} \end{pmatrix}$$

2.3. Addition of two matrices:

Let *A* = $(a_{ij})_{m\times n}$ and $B = (b_{ij})_{m\times n}$ are two matrices with same order then sum of the two matrices are given by

$$A + B = (a_{ij})_{m\times n} + (b_{ij})_{m\times n} = (a_{ij} + b_{ij})_{m\times n}$$

Example 2.1: let

$$A = \begin{pmatrix} 1 & -2 & 3 \\ 4 & 5 & -4 \end{pmatrix} \text{ and } B = \begin{pmatrix} 3 & 0 & 2 \\ -1 & 1 & 8 \end{pmatrix}.$$

Find (i) 5B (ii) A + B (iii) 4A – 2B (iv) 0 A

2.4. Multiplication of two matrices:

Two matrices *A* and *B* are said to be confirmable for product *AB* if number of columns in *A* equals to the number of rows in matrix *B*. Let *A* = $(a_{ij})_{m\times n}$ $and\ B = (b_{ij})_{n\times r}$ be two matrices the product matrix *C*= *AB*, is matrix of order $m\times r$ where

$$c_{ij} = \sum_{k=1}^{n} a_{ik}\, b_{kj} = a_{i1}b_{1j} + a_{i2}b_{2j} + \dots\dots\dots\dots + a_{in}b_{nj}$$

Example 2.2: Let $A = \begin{pmatrix} 1 & 2 & -3 & 4 \\ 0 & -5 & -1 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 2 & 3 \\ -5 & 0 \\ 6 & -2 \\ -1 & -3 \end{pmatrix}$

Calculate (i) AB (ii) BA

(iii) is AB = BA ?

2.5. Integral power of Matrices:

Let A be a square matrix of order n, and m be positive integer then we define

$A^m = A \times A \times A \dots\dots. \times A$ (m times multiplication)

2.6. Properties of the Matrices

Let A, B and C are three matrices and λ and μ are scalars then

(i) $A + (B + C) = (A + B) + C$ Associative Law

(ii) $\lambda\,(A+B) = \lambda A + \lambda B$ Distributive law

(iii) $\lambda(\mu A) = (\lambda\mu)A$ Associative Law

(iv) $(\lambda A)B = \lambda(AB)$ Associative Law

(v) $A(BC) = (AB)C$ Associative Law

(vi) $A(B+C) = AB + AC$ Distributive law

2.7. Transpose:

The transpose of matrix $A = (a_{ij})_{m\times n}$, written A^t (A' or A^T) is the matrix obtained by writing the rows of *A* in order as columns.

That is $A^t = (a_{ji})_{n\times m}$.

Properties of Transpose:

(i) $(A+B)^t = (A^t + B^t)$
(ii) $(A^t)^t$ = A
(iii) $(kA)^t$ = k A^t for scalar k.
(iv) $(AB)^t = B^t A^t$

Example 2.3: Using the following matrices A and B, Verify the transpose properties

$$A = \begin{pmatrix} 1 & -1 & 2 \\ 5 & -4 & 3 \\ 1 & -2 & -3 \end{pmatrix},\quad B = \begin{pmatrix} -2 & 6 & -2 \\ -1 & 0 & 1 \\ -2 & 1 & 0 \end{pmatrix}$$

Proof: (i) Let a_{ij} and b_{ij} are the $(i,j)^{th}$ element of the matrix A and B respectively. Then $a_{ij} + b_{ij}$ is the $(i,j)^{th}$ element of matrix $A+B$ and it is $(j,i)^{th}$ element of the matrix $(A+B)^t$

Also a_{ij} and b_{ij} are the $(j,i)^{th}$ element of the matrix A^t and B^t respectively. Therefore $a_{ij} + b_{ij}$ is the $(j,i)^{th}$ element of the matrix A^t + B^t

(ii) Let $(i,j)^{th}$ element of the matrix A is a_{ij}, it is $(j,i)^{th}$ element of the A^t then it is $(i,j)^{th}$ element of the matrix $(A^t)^t$

(iii) try

(iv) $c_{ik} = \sum_{j=1}^{n} a_{ij}\, b_{jk} = a_{i1}b_{1j} + a_{i2}b_{2j} + \dots\dots\dots + a_{in}b_{nj}$ is the $(i,k)^{th}$ element of the AB It is result of the multiplication of the i^{th} row and k^{th} column and it is $(k,i)^{th}$ element of the matrix $(AB)^t$.

$B^t A^t$, $(k,i)^{th}$ element is the multiplication of k^{th} row of B^t with i^{th} column of A^t, That is k^{th} column of B with i^{th} row of A.

2.8 A square matrix A is said to be symmetric if $A = A^t$.

Example:

$$A = \begin{pmatrix} 1 & -1 & 1 \\ -1 & -4 & -2 \\ 1 & -2 & -3 \end{pmatrix}$$, A is symmetric by the definition of symmetric matrix.

Then

$$A^t = \begin{pmatrix} 1 & -1 & 1 \\ -1 & -4 & -2 \\ 1 & -2 & -3 \end{pmatrix}$$

That is $A = A^t$

2.9 A square matrix A is said to be skew- symmetric if $A = -A^t$

Example:

$$A = \begin{pmatrix} 1 & 3 & -1 \\ -3 & -5 & 8 \\ 1 & 8 & 9 \end{pmatrix}$$

(i) AA^t and A^tA are both symmetric.

(ii) $A + A^t$ is a symmetric matrix.

(iii) $A - A^t$ is a skew-symmetric matrix.

(iv) If A is a symmetric matrix and m is any positive integer then A^m is also symmetric.

(v) If A is skew symmetric matrix then odd integral powers of A is skew symmetric, while positive even integral powers of A is symmetric.

If *A* and *B* are symmetric matrices then

(vi) $(AB + BA)$ is symmetric.

(vii) $(AB - BA)$ is skew-symmetric.

Exercise 2.1: Verify the (i) , (ii) and (iii) using the following matrix A.

$$A = \begin{pmatrix} 1 & 3 & 5 \\ -3 & -5 & 10 \\ 1 & 8 & 9 \end{pmatrix}$$

Chapter 3: Determinant, Minor and Adjoint Matrices

Definition 3.1:

Let $A = (a_{ij})_{n\times n}$ be a square matrix of order n , then the number $|A|$ called determinant of the matrix A.

(i) Determinant of 2×2 matrix

Let A= $\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ then $|A|$ = $\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$ = $a_{11}a_{22} - a_{12}a_{21}$

(ii) Determinant of 3 $\times$ 3 matrix

Let B = $\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$

Then $|B|$ = $a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$

$$|B| = a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) - a_{13}(a_{21}a_{32} - a_{31}a_{22})$$

Exercise 3.1: Calculate the determinants of the following matrices

(i) $A = \begin{pmatrix} 1 & 3 & 4 \\ 2 & 6 & 8 \\ 1 & 9 & 5 \end{pmatrix}$ (ii) $B = \begin{pmatrix} 2 & -3 & 4 \\ 5 & 6 & 7 \\ 8 & 9 & 1 \end{pmatrix}$

3.1 Properties of the Determinant:

a. The determinant of a matrix A and its transpose A^t are equal.

$$|A| = |A^t|$$

b. Let A be a square matrix

(i) If A has a row (column) of zeros then $|A| = 0$.

(ii) If A has two identical rows (or columns) then $|A| = 0$.

c. If A is triangular matrix then $|A|$ is product of the diagonal elements.

d. If A is a square matrix of order n and k is a scalar then $|kA| = k^n|A|$

3.2 Singular Matrix

If A is square matrix of order n, the A is called singular matrix when $|A| = 0$ and non-singular otherwise.

3.3. Minor and Cofactors:

Let A = $(a_{ij})_{n\times n}$ is a square matrix. Then M_{ij} denote a sub matrix of A with order (n-1) $\times$ (n-1) obtained by deleting its i^{th}row and j^{th} column. The determinant $|M_{ij}|$ is called the minor of the element a_{ij} of A.

The cofactor of a_{ij} denoted by A_{ij} and is equal to$(-1)^{i+j}|M_{ij}|$.

Exercise 3.2: Let $A = \begin{pmatrix} 5 & 4 & 2 \\ 2 & 3 & 1 \\ 3 & -2 & -1 \end{pmatrix}$

(i) Compute determinant of A.

(ii) Find the cofactor matrix.

3.4. Adjoin Matrix:

The transpose of the matrix of cofactors of the element a_{ij} of A denoted by $adj\ A$ is called adjoin of matrix A.

Example 3.3: Find the adjoin matrix of the above example.

Theorem 3.1:

For any square matrix A,

$$A\ (adj\ A\ ) = (adj\ A\ )A = \ |A|\ I$$ where I is the identity matrix of same order.

Proof: Let A = $(a_{ij})_{n\times n}$

Since A is a square matrix of order n, then $adj\ A$ also in same order.

Consider

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots\dots & a_{1n} \\ a_{21} & a_{22} & \dots\dots & a_{2n} \\ \dots & \dots & \dots\dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix}$$ then

$$adj\ A = \begin{pmatrix} A_{11} & A_{12} & \dots\dots & A_{1n} \\ A_{21} & A_{22} & \dots\dots & A_{2n} \\ \dots & \dots & \dots\dots & \dots \\ A_{n1} & A_{n2} & \dots & A_{nn} \end{pmatrix}$$

Now consider the product $A\ (adj\ A\ )$

$$A\ (adj\ A\ ) = \begin{pmatrix} a_{11} & a_{12} & \dots\dots & a_{1n} \\ a_{21} & a_{22} & \dots\dots & a_{2n} \\ \dots & \dots & \dots\dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix} \begin{pmatrix} A_{11} & A_{12} & \dots\dots & A_{1n} \\ A_{21} & A_{22} & \dots\dots & A_{2n} \\ \dots & \dots & \dots\dots & \dots \\ A_{n1} & A_{n2} & \dots & A_{nn} \end{pmatrix}$$

$$= \begin{pmatrix} \sum_{j=1}^{n} a_{1j}\,A_{1j} & \sum_{j=1}^{n} a_{1j}\,A_{2j} & \dots\dots & \sum_{j=1}^{n} a_{1j}\,A_{nj} \\ \sum_{j=1}^{n} a_{2j}\,A_{1j} & \sum_{j=1}^{n} a_{2j}\,A_{2j} & \dots\dots & \sum_{j=1}^{n} a_{2j}\,A_{nj} \\ \dots\dots & \dots\dots & \dots\dots & \dots \\ \sum_{j=1}^{n} a_{nj}\,A_{1j} & \sum_{j=1}^{n} a_{nj}\,A_{2j} & \dots\dots & \sum_{j=1}^{n} a_{nj}\,A_{nj} \end{pmatrix}$$

$$= \begin{pmatrix} |A| & 0 & \dots & 0 \\ 0 & |A| & \dots & 0 \\ 0 & 0 & \dots & 0 \\ 0 & 0 & \dots\dots & |A| \end{pmatrix}$$

(as we know that $\sum_{j=1}^{n} a_{ij}\,A_{ij} = |A|$ and $\sum_{j=1}^{n} a_{ij}\,A_{kj} = 0$ when $i \neq k$)

$$= |A| \begin{pmatrix} 1 & 0 & \dots.. & 0 \\ 0 & 1 & \dots.. & 0 \\ 0 & 0 & \dots.. & 0 \\ 0 & 0 & \dots\dots & 1 \end{pmatrix}$$

$= |A|\, I_n$ Where I_n is unit matrix of order n.

Theorem 3.2: If *A* is a non-singular matrix of order n, then $|adj\ A| = |A|^{n-1}$.

Proof: By the theorem 1

$$A\,(adj\ A) = |A|\, I$$

$$|A\,(adj\ A)| = \big||A|\, I\big|$$

$$|A||\,adj\ A\,| = |A|^n$$

$$|\,adj\ A\,| = |A|^{n-1}$$

Theorem 3.3: If *A* and *B* are two square matrices of order n then

$$adj(AB) = (adj\ B)(adj\ A)$$

Proof: By the theorem 1 $A\,(adj\ A) = |A|\, I$

Therefore $(AB)\ adj\ (AB) = adj\ (AB) AB = |AB| I$

Consider $(AB)(adj\ B\ adj\ A)$,

$$(AB)(adj\ B\ adj\ A) = A\,(B\ adj\ B)\ adj\ A$$

$$= A(|B|\ I)\ adj\ A$$

$$= |B|\ (A\ adj\ A)$$

$$= |B|\ |A|\ I$$

$$= |A||B|\ I$$

$$= |AB| I \qquad \dots\dots\dots\dots\dots\dots \text{(i)}$$

Also consider $(adj\ B\ .\ adj\ A) A\, B$

$$(adj\ B\ .\ adj\ A) A\, B = adj\ B\ (adj\ A\ A) B$$

$$= adj\ B\ |A|\ I\ B$$

$$= |A| adj\ B\ B$$

$$= |A||B|\ I$$

$$= |AB| I \qquad \dots\dots\dots\dots\dots.. \text{(ii)}$$

Therefore from (i) and (ii) we conclude that

$$adj(AB) = (adj\ A)(adj\ B)$$

Some results of adjoint

(i) For any square matrix A $(adj\ A)^t = adj\ A^t$
(ii) The adjoint of an identity matrix is the identity matrix.
(iii) The adjoint of a symmetric matrix is a symmetric matrix.

.Chapter 4: Inverse of a Matrix and Elementary Row Operations

4.1 Inverse of a Matrix

Definition 4.1:

If A and B are two matrices such that $AB = BA = I$, then each is said to be inverse of the other. The inverse of A is denoted by A^{-1}.

Theorem 4.1: (Existence of the Inverse)

The necessary and sufficient condition for a square matrix A to have an inverse is that $|A| \neq 0$ (That is A is non singular).

Proof: (i) The necessary condition

Let A be a square matrix of order n and B is inverse of it, then

$$AB = I$$

$$|AB| = |A||B| = I$$

Therefore $|A| \neq 0$.

(ii) The sufficient condition:

If $|A| \neq 0$, the we define the matrix B such that

$$B = \frac{1}{|A|}\ (adj\ A)$$

Then $AB = A\,\frac{1}{|A|}\ (adj\ A) = \frac{1}{|A|}\,A(adj\ A)$

$$= \frac{1}{|A|}\,|A|I = I$$

Similarly $BA = \frac{1}{|A|}\ (adj\ A)A = \frac{1}{|A|}\,A(adj\ A) = \frac{1}{|A|}\,|A|I = I$

Thus $AB = BA = I$ hence B is inverse of A and is given by $A^{-1} = \frac{1}{|A|}\ (adj\ A)$

Theorem 4.2: (Uniqueness of the Inverse)

Inverse of a matrix if it exists is unique.

Proof: Let B and C are inverse s of the matrix A then

$$AB = BA = I \text{ and } AC = CA = I$$

$$B(AC) = BI$$

$$(BA)C = B$$

$$C = B$$

Example 6: Let $A = \begin{pmatrix} 2 & 3 & -4 \\ 0 & -4 & 2 \\ 1 & -1 & 5 \end{pmatrix}$ find A^{-1}

Theorem 4.3: (Reversal law of the inverse of product)

If A and B are two non-singular matrices of order n, then (AB) is also non singular and $(AB)^{-1} = B^{-1}A^{-1}$.

Proof:

Since A and B are non-singular $|A| \neq 0$ and $|B| \neq 0$, therefore $|A||B| \neq 0$, then $|AB| \neq 0$.

Consider $(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1}$

$$= AIA^{-1} = AA^{-1} = I \qquad \ldots\ldots\ldots\ldots(1)$$

Similarly $(B^{-1}A^{-1})\ (AB) = B^{-1}(A^{-1}A)B$

$$= B^{-1}IB = B^{-1}B = I \qquad \ldots\ldots\ldots\ldots(2)$$

From (1) and (2)

$$(AB)(B^{-1}A^{-1}) = (B^{-1}A^{-1})\ (AB) = I$$

Therefore by the definition and uniqueness of the inverse $(AB)^{-1} = B^{-1}A^{-1}$

Corollary4.1: If $A_1A_2 \ldots\ldots\ldots\ldots A_m$ are non singular matrices of order n, then $(A_1A_2 \ldots\ldots\ldots\ldots A_m)^{-1} = A_1^{-1}A_2^{-2} \ldots\ldots\ldots . A_m^{-1}$.

Theorem 4.4: If A is a non-singular matrix of order n then $(A^t)^{-1} = (A^{-1})^t$.

Proof: Since $|A^t| = |A| \neq 0$therefore the matrix A^t is non-singular and $(A^t)^{-1}$ exists.

Let $AA^{-1} = A^{-1}A = I$

Taking transpose on both sides we get

$$(AA^{-1})^t = (A^{-1})^tA^t = I^t{}_n = I_n$$

$$(A^{-1}A)^t = A^t(A^{-1})^t = I^t = I_n$$

Therefore $A^t(A^{-1})^t = (A^{-1})^tA^t = I_n$

That is $(A^{-1})^t = = (A^t)^{-1}$.

Theorem 4.5: If A is a non-singular matrix , k is non zero scalar, then $(kA)^{-1} = \frac{1}{k}A^{-1}$.

Proof: Since A is non-singular matrix A^{-1} exits.

Let consider $(kA) \left(\frac{1}{k}A^{-1}\right) = \left(k \times \frac{1}{k}\right) (A\, A^{-1}) = I$

Therefore $\left(\frac{1}{k}A^{-1}\right)$ is inverse of kA

By uniqueness if inverse $(kA)^{-1} = \frac{1}{k}A^{-1}$

Theorem 4.6: If A is a non-singular matrix then

$$|A^{-1}| = \frac{1}{|A|}.$$

Proof: Since A is non-singular matrix, A^{-1} exits and we have

$$AA^{-1} = I$$

Therefore $|AA^{-1}| = |A||A^{-1}| = |I| = 1$

Then $|A^{-1}| = \frac{1}{|A|}$

4.2 Elementary Transformations:

Some operations on matrices called as elementary transformations. There are six types of elementary transformations, three of then are row transformations and other three of them are column transformations. There are as follows

(i) Interchange of any two rows or columns.
(ii) Multiplication of the elements of any row (or column) by a non zero number k.
(iii) Multiplication to elements of any row or column by a scalar k and addition of it to the corresponding elements of any other row or column.

We adopt the following notations for above transformations

(i) Interchange of i^{th} row and j^{th} row is denoted by $R_i \leftrightarrow R_j$.
(ii) Multiplication by k to all elements in the i^{th} row $R_i \rightarrow kR_i$.
(iii) Multiplication to elements of j^{th} row by k and adding them to the corresponding elements of i^{th} row is denoted by $R_i \rightarrow R_i + kR_j$.

4.2.1 Equivalent Matrix:

A matrix B is said to be equivalent to a matrix A if B can be obtained from A, by for forming finitely many successive elementary transformations on a matrix A.

Denoted by A~ B.

4.3 Rank of a Matrix:

Definition:

A positive integer ‘r’ is said to be the rank of a non- zero matrix A if

(i) There exists at least one non-zero minor of order r of A and
(ii) Every minor of order greater than r of A is zero.

The rank of a matrix A is denoted by $\rho(A)$.

4.4 Echelon Matrices:

Definition 4.3:

A matrix $A = (a_{ij})$ is said to be echelon form (echelon matrix) if the number of zeros preceding the first non zero entry of a row increasing by row until zero rows remain.

In particular, an echelon matrix is called a row reduced echelon matrix if the distinguished elements are

(i) The only non- zero elements in their respective columns.
(ii) Each equal to 1.

Remark: The rank of a matrix in echelon form is equal to the number of non-zero rows of the matrix.

Example 4.1:

Reduce following matrices to row reduce echelon form

(i) $$A = \begin{pmatrix} 1 & -2 & 3 & -1 \\ 2 & -1 & 2 & 2 \\ 3 & 1 & 2 & 3 \end{pmatrix}$$

(ii) $$B = \begin{pmatrix} 1 & 2 & -1 & 2 & 1 \\ 2 & 4 & 1 & -2 & 3 \\ 3 & 6 & 2 & -6 & 5 \end{pmatrix}$$

Chapter 5: Solution of System of Linear Equation by Matrix Method

5.1 Solution of the linear system *AX= B*

We now study how to find the solution of system of m linear equations in n unknowns.

Consider the system of equations in unknowns $x_1, x_2, \dots\dots\dots\dots\dots, x_n$ as

$$a_{11}x_1 + a_{12}x_2 + \dots\dots\dots\dots\dots . a_{1n}x_n = b_1$$

$$a_{21}x_1 + a_{22}x_2 + \dots\dots\dots\dots\dots . a_{2n}x_n = b_2$$

…………………………………………………

$$a_{n1}x_1 + a_{n2}x_2 + \dots\dots\dots\dots\dots . a_{nn}x_n = b_n$$

is called system of linear equations with n unknowns $x_1, x_2, \ldots \ldots \ldots \ldots \ldots \ldots, x_n$. If the constants $b_1, b_2 \ldots \ldots \ldots \ldots \ldots \ldots b_n$ are all zero then the system is said to be homogeneous type.

The above system can be put in the matrix form as

$$AX = B$$

Where $A = \begin{pmatrix} a_{11}\ a_{12} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1}\ a_{n2} & \cdots & a_{nn} \end{pmatrix}$ $\quad X = \begin{pmatrix} x_1 \\ x_2 \\ \ldots \\ x_n \end{pmatrix}$ $\quad B = \begin{pmatrix} b_1 \\ b_2 \\ \ldots \\ b_n \end{pmatrix}$

The matrix $A = (a_{ij})_{n\times n}$ is called coefficient matrix, the matrix X is called matrix of unknowns and B is called as matrix of constants, matrices X and B are of order $n \times 1$.

Definition 5.1: (**consistent**)

A set of values of $x_1, x_2, \ldots \ldots \ldots \ldots \ldots \ldots, x_n$ which satisfy all these equations simultaneously is called the solution of the system. If the system has at least one solution then the equations are said to be consistent otherwise they are said to be inconsistent.

Theorem 5.2:

A system of m equations in n unknowns represented by the matrix equation AX= *B* is consistent if and only if $\rho(A) = \rho(A, B)$. That is the rank of matrix A is equal to rank of augment matrix (A, B)

Theorem 5.2:

If A be an non-singular matrix, X be an $n \times 1$ matrix and B be an $n \times 1$ *m*atrix then the system of equations *AX*= *B* has a unique solution.

(1) $AX = B$

Consistent if $\rho(A) = \rho(A, B) = r$

Inconsistent if $\rho(A) \neq \rho(A, B)$

Unique solution if r=n

Infinite solution if r<n

(2)

$$\boldsymbol{AX = 0} \text{ (All wayes consistant)}$$

If $\rho(A) = r = n$ Trivial solution

If $\rho(A) = r < n$ Infinite solutions

Therefore every system of linear equations solutions under one of the following:

(i) There is no solution
(ii) There is a unique solution
(iii) There are more than one solution

Methods of solving system of linear Equations:

5.1 Method of inversesion:

Consider the matrix equation

Consider the matrix equation

$AX = B$ Where $|A| \neq 0$

Pre multiplying by A^{-1}, we have

$$A^{-1}\ (AX\ ) = A^{-1}B$$

$$X = A^{-1}B$$

Thus $AX = B$, has only one solution if $|A| \neq 0$ and is given by $X = A^{-1}B$.

5.2 Using Elementary row operations: (Gaussian Elimination)

Suppose the coefficient matrix is of the type $m \times n$. That is we have m equations in n unknowns Write matrix $[A, B]$ and reduce it to Echelon augmented form by applying elementary row transformations only.

Example 5.1: Solve the following system of linear equations using matrix method

(i)

$$2x + y + -2z = 10$$
$$y + 10z = -28$$
$$3y + 16z = -42$$

(ii)

$$x + 2y - 3z = -1$$
$$3x - y + 2z = 7$$
$$5x + 3y - 4z = 2$$

Example 5.2: Determine the values of a so that the following system in unknowns x, y and z has

(i) No solutions
(ii) More than one solutions
(iii) A unique solution

$$x \quad + y \quad + z = 0$$

$$2x + 3y + az = 0$$

$$x + ay + 3z = 0$$

Chapter 6: Eigen values and Eigenvectors:

If A is a square matrix of order n and X is a vector in R^n, (X considered as $n \times 1$ column matrix), we are going to study the properties of non-zero X, where AX are scalar multiples of one another. Such vectors arise naturally in the study of vibrations, electrical systems, genetics, chemical reactions, quantum mechanics, economics and geometry.

Definition 6.1:

If A is a square matrix of order n , then a non-zero vector X in $R^n$is called eigenvector of A if $AX = \lambda X$ for some scalar λ. The scalar λ is called an eigenvalue of A, and X is said to be an eigenvector of A corresponding to λ.

Remark: Eigen values are also called proper values or characteristic values.

Example 6.1: The vector $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ is an eigenvector of A= $\begin{pmatrix} 3 & 0 \\ 8 & -1 \end{pmatrix}$

Theorem 6.1: If A is a square matrix of order n and λ is a real number, then λ is an eigenvalue of A if and only if $|\ \lambda I - A\ | = 0$.

Proof: If λ is an eigenvalue of A, the there exist a non-zero X a vector in R^n such that $\boldsymbol{AX} = \lambda X$.

$$\boldsymbol{AX} = \lambda X$$

$\boldsymbol{AX} = \lambda IX$ Where I is a identity matrix of order n.

$$(\lambda I - A)X = 0$$

The equation has trivial solution when if and only if $|A| = 0$. The equation has non-zero solution if and only if $|(A - \lambda I)|$ = 0.

Conversely , if $|(A - \lambda I)|$ = 0 then by the result there will be a non-zero solution for the equation,

$$(A - \lambda I)X = 0$$

That is, there will a non-zero X in R^n such that $AX = \lambda X$, which shows that λ is an eigenvalue of A.

Example 6.2: Find the eigen values of the matrixes

(i) A = $\begin{pmatrix} 2 & 7 \\ 1 & -2 \end{pmatrix}$ (ii) B = $\begin{pmatrix} 2 & 1 & 1 \\ 2 & 3 & 4 \\ -1 & -1 & -2 \end{pmatrix}$

Theorem 6.2:

If A is an $n \times n$ matrix and λ is a real number, then the following are equivalent:

(i) λ is an eigenvalue of A.
(ii) The system of equations $(\lambda I - A)X = 0$ has non-trivial solutions.
(iii) There is a non-zero vector X in R^n such that $AX = \lambda I$.
(iv) Is a solution of the characteristic equation $|(A - \lambda I)|$ = 0.

Definition 6.2:

Let A be an $n \times n$ matrix and λ be the eigen value of A. The set of all vectors X in R^n which satisfy the identity $\boldsymbol{AX} = \lambda X$ is called the eigen space of a corresponding to λ. This is denoted by $E(\lambda)$.

Remark:

The eigenvectors of A corresponding to an eigen value λ are the non-zero vectors of X that satisfy $AX = \lambda I$. Equivalently the eigen vectors corresponding to λ are the non zero in the solution space of $(\lambda I - A)X = 0$. Therefore, the eigen space is the set of all non-zero X that satisfy $(A - \lambda I)X = 0$ with trivial solution in addition.

Steps to obtain eigen values and eigen vectors

Step I : For all real numbers λ form the matrix $\lambda I - A$

Step II: Evaluate $|(A - \lambda I)|$ That is characteristic polynomial of *A*.

Step III: Consider the equation $|(A - \lambda I)| = 0$ (The characteristic equation of *A*) Solve the equation for Let $\lambda_1, \lambda_2, \lambda_3 \ldots \ldots \ldots \lambda_n$ be eigen values of A thus calculated.

Step IV: For each λ_i consider the equation $(\lambda_i I - A)X = 0$

Find the solution space of this system which an eigen space $E(\lambda_i)$ of A, corresponding to the eigen value λ_i of A . Repeat this for each λ_i $i = 1, 2, \ldots \ldots . n$

Step V: From step IV , we can find basis and dimension for each eigen space $E(\lambda_i)$ for $i = 1, 2, \ldots \ldots . n$

Example 6.3:

Find (i) Characteristic polynomial

(ii) Eigen values
(iii) Basis for the eigen space of a matrix

$$A = \begin{pmatrix} 3 & 0 \\ 8 & -1 \end{pmatrix}$$

Example 6.4:

Find eigen values of the matrix

$$A = \begin{pmatrix} 2 & 1 & 1 \\ 2 & 3 & 4 \\ -1 & -1 & -2 \end{pmatrix}$$

Also eigen space corresponding to each value of A. Further find basis and dimension for the same.

6.2 Diagonalization:

Definition 6.2.1: A square matrix *A* is called diagonalizable if there exists an invertible matrix P such that $P^{-1}AP$ is a diagonal matrix, the matrix P is said to diagonalizable A.

Theorem 6.2.1: If A is a square matrix of order n, then the following are equivalent.

(i) A is diagonizible.
(ii) A has n linearly independent eigenvectors.

Procedure for diagonalizing a matrix

Step I: Find n linearly independent eigenvectors of A, say $P_1, P_2, \dots \dots \dots . . P_n$

Step II: From the matrix P having $P_1, P_2, \dots \dots \dots . . P_n$ as its column vectors.

Step III: The matrix $P^{-1}AP$ will then be diagonal with $\lambda_1, \lambda_2 \dots \dots . . \lambda_n$ as its successive diagonal entries, where λ_i is the eigenvalue corresponding to P_i, $i = 1, 2, \dots \dots \dots . . n$.

Example 6.3: Find a matrix P that diagonalizes

$$A = \begin{pmatrix} 2 & 1 & 1 \\ 2 & 3 & 4 \\ -1 & -1 & -2 \end{pmatrix}$$

Tutorial (Matrices)

Q1. Show that the square matrix $A = \begin{pmatrix} 3 & 2 & 4 \\ 1 & 5 & 3 \\ -1 & 8 & 2 \end{pmatrix}$ is a singular matrix.

Q2. If $A = \begin{pmatrix} 1 & 4 & 3 \\ 6 & 2 & 5 \\ 1 & 7 & 0 \end{pmatrix}$ determine (i) $|A|$ (ii) *Adj A*

Q3. Find the inverse of the matrix $A = \begin{pmatrix} 2 & 1 & 4 \\ 3 & 5 & 1 \\ 2 & 0 & 6 \end{pmatrix}$

Q4. If $A = \begin{pmatrix} 1 & 0.5 \\ 0.5 & 0.1 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix}$ determine

(i) B^{-1} (ii) AB (iii) $B^{-1}A$

Q5. Consider the matrix $A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 1 & 5 & 7 \end{pmatrix}$

(i) Compute $|A|$ (ii) find adj A

(iii) Verify $A(adj\ A) = |A|$ I (iv) Find A^{-1}

Q6. Find the possible value of x can take, given that

$A = \begin{pmatrix} x^2 & 3 \\ 1 & 3x \end{pmatrix}$ $B = \begin{pmatrix} 3 & 6 \\ 2 & x \end{pmatrix}$ such that $AB = BA$.

Q7. If $A = \begin{pmatrix} 3 & 2 \\ -4 & 1 \end{pmatrix}$ find the values of m and n given that $A^2 = mA + nA$

Q8. Find the echelon form of matrix:

$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 2 & 3 & 4 & 5 \\ 4 & 9 & 16 & 25 \end{pmatrix}$ Hence discuss (1) unique solution (ii) many solutions and (iii) No solutions of the following system and solve completely.

$x + y + z = 1$

$2x + 3y + 4z = 5$

$4x + 9y + 16z = 25$

Q9. If matrix A is $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ p & q & r \end{pmatrix}$ and I, the unit matrix of order 3, show that $A^3 = pI + qA + rA^2$.

Q10. Let A be a square matrix

a. Show that
$$(I - A)^{-1} = 1 + A + A^2 + A^3 \quad if \quad A^4 = 0$$

b. Show that
$$(I - A)^{-1} = 1 + A + A^2 + A^3 + \cdots \ldots . A^n \quad if \quad A^{n+1} = 0$$

Q11. Find values of a,b and c so that the graph of the polynomial $p(x) = ax^2 + bx + c$ passes through the points (1,2), (-1,6) and (2,3).

Q12. Find values of a,b and c so that the graph of the polynomial $p(x) = ax^2 + bx + c$passes through the points (-1,0) and has a horizontal tangent at (2,-9).

Q13. Let $\begin{pmatrix} a & 0 & b & 2 \\ a & a & 4 & 4 \\ 0 & a & 2 & b \end{pmatrix}$ be the augmented matrix for a linear- system. For what value of a and b does the system have

a. a unique solution
b. a one- parameter solution
c. a two parameter- solution
d. no solution

Q14. Find a matrix K such that $AKB = C$ given that

$$A = \begin{pmatrix} 1 & 4 \\ -2 & 3 \\ 1 & -2 \end{pmatrix} \qquad B = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & -1 \end{pmatrix} \qquad C = \begin{pmatrix} 8 & 6 & -6 \\ 6 & -1 & 1 \\ -4 & 0 & 0 \end{pmatrix}$$

a. For the triangle below, use trigonometry to show

$$b \cos\gamma \ + \ c \cos\beta \ = a$$

$$c \cos\alpha \ + \ a \cos\gamma \ = b$$

$$a \cos\beta \ + \ b \cos\alpha \ = c$$

And then apply Crame's Rule to show

$$\cos\alpha = \frac{b^2 + c^2 - a^2}{2bc}$$

Use the Cramer's rule to obtain similar formulas for $\cos\beta$ and $\cos\gamma$.