Mechanics of the material point

Physics

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Chapter I: Dimensional analysis and Vector calculus

I

Physics describes material and space, their properties and their behaviour. Measurable properties are called **PHYSICAL DIMENSIONS.** When their measurement is expressed by a simple number, we speak of a **scalar quantity**. When a set of several numbers (vectors) is required to represent them, they are referred to as **vector quantities**.

A. Dimensional Analysis

1. Fundamental (primary/basic) physical quantities

The International System (IS) is made up of the units of the rationalised MKSA system (m: metre, kg: kilogram, s: second and a: ampere) and includes additional definitions for the unit of temperature and the unit of luminous intensity.

	Symbol	Dimensions
Length	m	L
Mass	Kg	M
Time	s	T
Electric curre	A	I
Temperature	K	H
Amount of su	mol	N
Luminous inte	cd	J

Table1

2. Derived (secondary) physical quantities

These are quantities whose definitions are based on other physical quantities (base quantities)

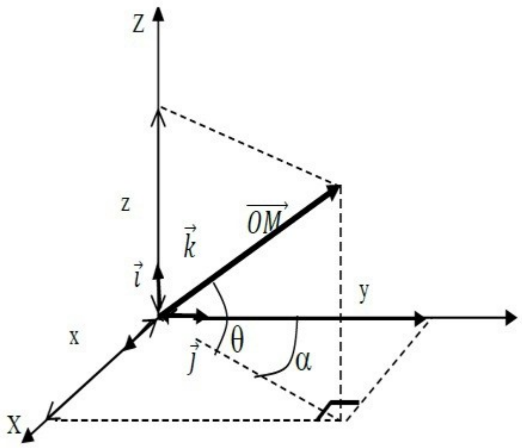
	Physical quantity	Dimenional formula
1	Area	$[M^0L^2T^0]$
2	volume	$[M^0L^3T^0]$
3	Density=mass/volume	$[M^1L^{-3}T^0]$
4	Speed or velocity=dx/dt	$[M^0LT^{-1}]$
5	Acceleration= dv/dt	$[M^0LT^{-2}]$
6	Force=mdv=ma	$[MLT^{-2}]$

Table2

B. Vector calculation

In physics, a number or numerical function uses vectors to model quantities that cannot be completely defined alone. For example, to specify a displacement, velocity, force or electric field, direction and sense are essential. Vectors are the opposite of scalar quantities described by a simple number, such as mass, temperature, etc.

$$\vec{OM} = x\,\vec{i} + y\,\vec{j} + z\,\vec{k}$$



1. The scalar product

Definition

The scalar product is an algebraic operation in addition to the laws that apply to vectors.

It associates two vectors with their product, which is a number (or scalar, hence the name). It allows you to exploit traditional Euclidean geometry concepts: lengths, angles, orthogonality.

Let $v=(v_{1,}v_{2,}v_3)$ and $w=(w_{1,}w_{2,}w_3)$ be vectors in R3. The dot product of v and w, denoted by v.w, is given by:

$$v.w=v_1w_1+v_2w_2+v_3w_3$$

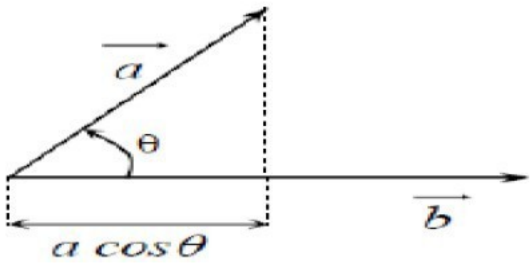
Similarly, for the vectors v=(v_1, v_2) and w=(w_1,w_2) in R^2, the dot product is

$$v.w=v_1w_1+v_2w_2$$

a) Theorem

Let v,w be nonzero vectors, and let θ be the angle between them, then:

$$\theta os \ =(v.w)/\|v\|\|w\|$$



b) Corollary

If θ is the angle between nonzero vectors v and w then:

$$\boldsymbol{v.w}\ is = \begin{cases} > 0\ for\ 0^{\circ} \leq \theta < 90^{\circ} \\ 0\ for\ \theta = 90^{\circ} \\ < 0\ for\ 90^{\circ} < \theta \leq 90^{\circ} \end{cases}$$

c) Theorem Basic Properties of the scalar product

For any vectors u, v, w, and scalar k, we have

- v.w=w.v **Commutative Law**
- (kv).w=v.(kw)=k(v.w) **Associative Law**
- v.0=0=0.v
- u.(v+w)=u.v+u.w **Distributive Law**
- (u+v).w=u.w+v.w **Distributive Law**
- $|v.w|=\|v\|\|w\|$ **Cauchy-Schwarz Inequality**

2. The vector product

Definition

The vector product is a vector operation performed in three-dimensional oriented Euclidean spaces.

Let $v=(v_{1,} v_{2,} v_3)$ and $w=(w_{1,} w_{2,} w_3)$ be vectors in R^3.The dot product of v and , denoted by v×w, is the vector in R^3 given by:

$$v \times w = \begin{pmatrix} v_3 w_3 - v_3 w_3 \\ v_2 w_1 - v_1 w_2 \\ v_1 w_2 - v_2 w_1 \end{pmatrix}$$

$$v \times w = \begin{vmatrix} i & j & k \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} = \begin{vmatrix} v_2 & v_3 \\ w_2 & w_3 \end{vmatrix} i - \begin{vmatrix} v_1 & v_3 \\ w_1 & w_3 \end{vmatrix} j + \begin{vmatrix} v_1 & v_2 \\ w_1 & w_2 \end{vmatrix} k$$

$$=(v_3 w_3 - v_3 w_3) i - (v_2 w_1 - v_1 w_2) j + (v_1 w_2 - v_2 w_1) k$$

a) Theorem

If the cross product v×w of two nonzero vectors **v** and **w** is also nonzero vector, then it is perpendicular to both **v** and **w**.

If **θ** is the angle between nonzero vectors **v** and w in **R³** then:

$$v \times w = \| v \| \| w \| \sin\theta$$

b) Derivatives of the products of vectors

$$\frac{\partial}{\partial x}(\mathbf{A} \bullet \mathbf{B}) = \mathbf{A} \bullet \frac{\partial \mathbf{B}}{\partial x} + \frac{\partial \mathbf{A}}{\partial x} \bullet \mathbf{B}$$

$$\frac{\partial}{\partial y}(\mathbf{A} \bullet \mathbf{B}) = \mathbf{A} \bullet \frac{\partial \mathbf{B}}{\partial y} + \frac{\partial \mathbf{A}}{\partial y} \bullet \mathbf{B}$$

$$\frac{\partial}{\partial z}(\mathbf{A} \bullet \mathbf{B}) = \mathbf{A} \bullet \frac{\partial \mathbf{B}}{\partial z} + \frac{\partial \mathbf{A}}{\partial z} \bullet \mathbf{B}$$

$$\frac{\partial}{\partial x}(\mathbf{A} x \mathbf{B}) = \mathbf{A} x \frac{\partial \mathbf{B}}{\partial x} + \frac{\partial \mathbf{A}}{\partial x} x \mathbf{B}$$

$$\frac{\partial}{\partial y}(\mathbf{A} x \mathbf{B}) = \mathbf{A} x \frac{\partial \mathbf{B}}{\partial y} + \frac{\partial \mathbf{A}}{\partial y} x \mathbf{B}$$

$$\frac{\partial}{\partial z}(\mathbf{A} x \mathbf{B}) = \mathbf{A} x \frac{\partial \mathbf{B}}{\partial z} + \frac{\partial \mathbf{A}}{\partial z} x \mathbf{B}$$

Example

Determine dA if vector function $A(x, y, z) = (x^2 siny) i + (z^2 cosy) j - (xy^2) k$

▪ *Solution*

$$
\begin{aligned}
d\mathbf{A} &= \frac{\partial \mathbf{A}}{\partial x}dx + \frac{\partial \mathbf{A}}{\partial y}dy + \frac{\partial \mathbf{A}}{\partial z}dz = \left[(\sin y)\mathbf{i}\frac{d}{dx}(x^2) - (y^2)\mathbf{k}\frac{dx}{dx}\right]dx + \left[x^2\mathbf{i}\frac{d}{dy}(\sin y) + z^2\mathbf{j}\frac{d}{dy}(\cos y)\right]dy \\
&+ \left[(\cos y)\mathbf{j}\frac{d}{dz}(z^2)\right]dz \\
&= \left[(2x\sin y)\mathbf{i} - y^2\mathbf{k}\right]dx + \left[(x^2\cos y)\mathbf{i} - (z^2\sin y)\mathbf{j} - 2xy\mathbf{k}\right]dy + \left[(2z\cos y)\mathbf{j}\right]dz \\
&= (2x\sin y\,dx + x^2\cos y\,dy)\mathbf{i} + (2z\cos dz - z^2\sin y\,dy)\mathbf{j} - (y^2\,dx + 2xy\,dy)\mathbf{k}
\end{aligned}
$$

3. Gradient, divergence and curl

Gradient, divergence and curl are frequently used when dealing with variations of vectors using a vector operator designated by ∇(Pronounced Del) defined as follows:

$$\nabla = \partial/\partial x\, i + \partial/\partial y\, j + \partial/\partial z\, k$$

in a rectangular coordinate system.

a) Gradient

$$grad\ \phi = \nabla\ \phi = (\partial/\partial x\, i + \partial/\partial y\, j + \partial/\partial z\, k)\phi = \partial\phi/\partial x\, i + \partial\phi/\partial y\, j + \partial\phi/\partial z\, k$$

b) Divergence

$$div\,\mathbf{A} = \nabla \bullet \mathbf{A} = \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right)\bullet(A_x\mathbf{i} + A_y\mathbf{j} + A_z\mathbf{k}) = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

c) Curl (rotation)

$$
\begin{aligned}
curl\,\mathbf{A} &= \nabla\times\mathbf{A} = \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right)\times(A_x\mathbf{i} + A_y\mathbf{j} + A_z\mathbf{k}) \\
&= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix} = \mathbf{i}\begin{vmatrix} \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_y & A_z \end{vmatrix} - \mathbf{j}\begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial z} \\ A_x & A_z \end{vmatrix} + \mathbf{k}\begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ A_x & A_y \end{vmatrix} \\
&= \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right)\mathbf{i} - \left(\frac{\partial A_z}{\partial x} - \frac{\partial A_x}{\partial z}\right)\mathbf{j} + \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)\mathbf{k}
\end{aligned}
$$

C. Solved exercises

Exercise 1

Quantity **A** has a dimensional formula of **ML** and quantity **B** has a dimensional formula of **MT^{-1}**. If **A = √BC**,

what is the dimensional formula of **C**?

A. MLT

B. $ML^2 T$

C. LT^{-1}

D. $M^2 L^2 T^{-1}$

Solution

if:

$$A=\sqrt{BC}$$

then:

$$C=A^2/B$$

Therefore, the dimensional formula of C is

$$[C]=M^2L^2/MT^{-1}=ML^2T\,.$$

Exercise 2

If **F** is the gravitational force between two bodies of masses **m_1** and **m_2** separated by distance **r** and it is given

by **F= Gmm2**, what is the unit of the gravitational constant, **G**?

A. $m^3/kg.s$

B. $N.m^2/kg$

C. $m^2/kg.s^2$

D. All of the above

Solution

Applying dimensional analysis to the formula, we have:

$$[\mathrm{F}] = [\frac{GMm}{r^2}]$$

$$[\mathrm{F}] = \frac{[\mathrm{G}][\mathrm{M}][\mathrm{m}]}{[\mathrm{r}]^2}$$

$$[\mathrm{G}] = \frac{[\mathrm{F}][\mathrm{r}]^2}{[\mathrm{M}][\mathrm{m}]}$$

Substituting in the dimensions for force, distance, and mass, we get

$$[G]=MLT^{-2}L^2/M^2$$

$$[G]=M^{-1}L^3T^{-2.}$$

Replacing these with the SI units for each dimension, we get the unit for G as $m^3/kg.s^2$.

Exercise 3

If the velocity **V** (**in cm/s**) of a particale is given in term of time (in sec) by the equation **V=atb/(t+c)**

Give the dimensions of **a**, **b** and **c**.

Solution

$$[V]=[at]=[b][t+c]-1/[t+c]=[t]=[C]$$

$$[v]=[a][t]=[a]=[v][t]^{-1}/[V]=LT^{-1}$$

$$[a]=LT^{-1}T^{-1}=LT^{-2}$$

$$[t]=[C]=[C]=T$$

$$[V]=[b][t]^{-1}=[b][c]^{-1}$$

$$[b]=[V][t]=LT^{-1}T=L$$

$$[a]=LT^{-2};[b]=L;[c]=T$$

Exercise 4

A force $\vec{F}=3\vec{i}+4\vec{j}+5\vec{k}$ (N) acting on a body produces a velocity $\vec{v}=2\vec{i}-\vec{j}+3\vec{k}$ (m.s-¹), calculate the power.

Solution

$$W = \vec{F}\vec{r} \quad ; \mathrm{P} = \frac{W}{t} = \vec{F}.\frac{\vec{r}}{t}$$

$$\frac{\vec{r}}{t} = \vec{v}$$

$$P = \vec{F}.\vec{V}$$

$$\mathrm{P}= \begin{pmatrix}3\\4\\5\end{pmatrix}\begin{pmatrix}2\\-1\\3\end{pmatrix} = 3(2) +4(-1) +5(3) = 6-4+15 = 17$$