Chapter 1

SINGLE, DOUBLE AND TRIPLE INTEGRALS

In this chapter I designates a non-trivial interval of $\mathbb{R}$ and $\mathbb{K}$ the set of real numbers or complexes.

1.1 Simple integrals

1.1.1 Reminders

Définition 1.1.1 (Primitive) *Let f and F be functions of I in K, F is a primitive of f on I when F is differentiable on I and $\forall x \in I, \acute{F}(x) = f(x)$.*

Proposition 1.1.1 *If f admits an antiderivative on I then it admits an infinity of them all equal to a constant.*

Proposition 1.1.2 *Let f and $g \in \mathcal{F}(I, \mathbb{K})$, F be a primitive of f on I and G be a primitive of g on I.*

1) $\forall \alpha, \beta \in \mathbb{K}, (\alpha F + \beta G)$ is an antiderivative of $(\alpha f + \beta g)$ on I.

2) $\mathcal{R}e(F)$ (resp. $\mathcal{I}m(F)$) is an antiderivative on I of $\mathcal{R}e(f)$ (resp. $\mathcal{I}m(f)$).

Exemple 1.1.1 (Search for a primitive of f by transforming expressions) *1) $f : t \longrightarrow \dfrac{1}{t^4 - 1}$ we decompose into simple elements*

2) $f : t \longrightarrow \tan^2 t$ we reveal a usual primitive

3) $f : t \longrightarrow \sin^4 t$ we linearize

4) $f : t \longrightarrow t\cos(\omega x)e^{\alpha x}$ use $f(x) = \mathcal{R}e(e^{\alpha+i\omega x})$

Exemple 1.1.2 *$f : t \longrightarrow \cos^2 t \sin^3 t$ we make $\acute{u}u^n$ appear. This method can replace linearization for products of the type $\cos^p x \sin^q x$ with p or q impairs.*

Définition 1.1.2 (Integral) *Let $a, b \in \mathbb{R}, a \leq b$ and $f : [a, b] \to \mathbb{R}$ be a continuous function. The integral from a to b of f is the real denoted $\int_a^b f(t)dt$ which is equal to the algebraic area of the domain delimited by the curve representative of f, the axis (Ox) and the lines $x = a$ and $x = b$, expressed in area unit*

- Extension to any two reals a and b: If $b < a$, we set $\int_a^b f(t)dt = -\int_b^a f(t)dt$.

- Extension to functions with complex values: Let $f \subset \mathcal{F}(I, \mathbb{C})$ is continuous, for all real numbers a and b of I, we set $\int_a^b f(t)dt = \int_a^b \mathcal{R}e(f)(t)dt + i\int_a^b \mathcal{I}m(f)(t)dt$.

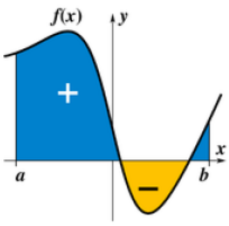

Figure 1.1: Integral Definition.

Remarque 1.1.1 *The integration variable is silent i.e. $\int_a^b f(t)dt = \int_a^b f(x)dx = \int_a^b f(u)du = ...$*

Proposition 1.1.3 (Properties of the integral) *Let f and g be continuous on I with values in $\mathbb{K}$ and a, b and c three real numbers of I.*

$\int_a^a f(t)dt = 0$ et $\int_a^b f(t)dt = -\int_b^a f(t)dt$.

Chasles relation: $\int_a^b f(t)dt = \int_a^c f(t)dt + \int_c^b f(t)dt$

Linearity: $\forall \alpha, \beta \in \mathbb{R}, \int_a^b [\alpha f(t) + \beta g(t)]\, dt = \alpha \int_a^b f(t)dt + \beta \int_a^b g(t)dt$

Positivity: Si $a \leq b$ et $f \geq 0$ sur $[a, b]$ alors $\int_a^b f(t)dt \geq 0$

Growth of the integral: if $a \leq b$ and $f \leq g$ on $[a, b]$ then $\int_a^b f(t)dt \leq \int_a^b g(t)dt$

Triangle inequality: If $a \leq b$ *then* $\left|\int\limits_a^b f(t)dt\right| \leq \int\limits_a^b |f(t)|\, dt$

1.1.2 Link between integrals and primitives of a function

Théoreme 1.1.1 (Fundamental theorem of analysis) *Let* f *be a continuous function on* I *with values in* $\mathbb{K}$, $F : x \longrightarrow \int\limits_a^x f(t)dt$ *is the unique primitive of* f *on* I *which cancels out at* a.

Consequences:

- Any continuous function on I admits an infinity of primitives on I. Let a be fixed in I, the set of primitives of f on I is $\{\ x \longrightarrow \int\limits_a^x f(t)dt + k\ ,\ k$ describes $\mathbb{K}\}$.
- Let G be any primitive of f on I and $a \in I$, we have: $\forall x \in I, G(x) = G(a) + \int\limits_a^x f(t)dt$.
- The notation: $\int\limits_I f = \int\limits_I f(t)dt$ denotes any primitive of f on I. For example: $\int_{\mathbb{R}} xdx = \frac{x^2}{2} + k$, where $k \in \mathbb{R}$. Attention, here the integration variable is no longer silent.

Corollaire 1.1.1 *Let* f *be a continuous function on* I,$\forall a, b \in I, \int\limits_a^b f(t)dt = [F(t)]_a^b = F(b) - F(a)$ *where* F *is any primitive of* f *on* I.

1.1.3 Integration by parts formula

Définition 1.1.3 *Let* $f : I \to \mathbb{K}$, *we say that* f *is of class* C^1 *on* I *when* f *is differentiable on* I *with* f *continues on* I. *The set of functions of class* C^1 *on* I *with values in* $\mathbb{K}$ *is denoted* $C^1(I, \mathbb{K})$.

Théoreme 1.1.2 *If* u *and* v *are two functions of class* C^1 *on* I *then* $\forall a, b \in I, \int\limits_a^b u(t)\acute{v}(t)dt = [u(t)v(t)]_a^b - \int\limits_a^b \acute{u}(t)v(t)dt$

Exemple 1.1.3 (Classic examples for the calculation of integral) $\int\limits_0^1 xe^x dx$ *and* $\int\limits_1^e t^2 \ln t dt$

Exemple 1.1.4 (Classic examples for calculating primitive) $\int\limits_1^x \ln t dt$ *et* $\int x \arctan x dx$. *We can directly use the IPP formula when* u *and* v *are of class* C^1 *on* I: I: $\forall x \in I, \int u(t)\acute{v}(t)dt = u(t)v(t) - \int \acute{u}(t)v(t)dt$ *(Be careful to validate the hypotheses of the theorem).*

1.1.4 Variable change formula

Théoreme 1.1.3 *Let f continue on I and $\varphi : [a,b] \to I$, of class C^1 on $[a,b]$. We have*

$$\int_a^b f(\varphi(x))\acute{\varphi}(x)dx \underset{\overleftarrow{(2)}}{\overset{\overrightarrow{(1)}}{=}} \int_{\varphi(a)}^{\varphi(b)} f(t)dt.$$

1.1.5 Applications

Application to the calculation of integrals

1^{st} case: We want to use the change of variable in the sense (1): We set $\varphi(x) = t$

Method: • We replace $\varphi(x)$ by t

- We replace $\acute{\varphi}(x)dx$ by dt.
- We modify the limits of the integral.
- Example: $\int_0^1 \frac{dx}{chx}$ by setting $e^x = t$

2^{nd} case: We want to use the change of variable in the direction (2): We set $t = \varphi(x)$

Method: • We determine a and b and we verify that φ is of class C^1 on $[a,b]$.

- We replace $\varphi(x)$ by t.
- We replace dt by $\acute{\varphi}(x)dx$.
- We modify the limits of the integral.
- Example: $\int_0^1 \sqrt{1-t^2}dt$ by setting $t = \sin x$. Be careful to validate the hypotheses of the theorem.

Application to the calculation of primitives

1^{st} case: We want to use the change of variable in the direction (1)

Méthode : • On pose le changement de variable choisi: avec de classe C^1 sur un intervalle de $\mathbb{R}$, à valeurs dans I

- We then have: $dt = \acute{\varphi}(x)dx$.
- We obtain: $\int f(\varphi(x))\acute{\varphi}(x)dx = \int f(t)dt = F(t) = F(\varphi(x))$ where F is an antiderivative of f on I.

Exemple 1.1.5 $\int \frac{dx}{1-\sin x}$ *on* $]0;\pi[$ *by setting* $\tan(x/2)=t$.

2^{nd} case: We want to use the change of variable in the sense (2): We must use a bijective change of variable in order to be able to return to the initial variable.

Method: • We set: $t=\varphi(x)$ with φ bijective of J on I, where J interval of and of class C^1 on J.

• On a alors: $dt=\acute{\varphi}(x)dx$

• We obtain: $\int f(t)dt = \int f(\varphi(x))\acute{\varphi}(x)dx = G(x) = G(\varphi^{-1}(t))$ where G is an antiderivative of $(fo\varphi)x\acute{\varphi}$ on J.

Exemple 1.1.6 $\int \sqrt{t^2-3}$ *on* $I=\left]-\sqrt{3},\sqrt{3}\right[$, *setting* $t-3\sin x=\varphi(x)$.

To know how to do without help: Primitive of $f:x\mapsto \frac{1}{ax^2+bx+c}$ on an interval I where $ax^2+bx+c\neq 0$

1^{sr} case: ax^2+bx+c has two real roots x_1 and x_2: We decompose f into simple elements: $\forall x\in I, \frac{A}{x-x_1}+\frac{B}{x-x_2}$ with real A and B. We obtain $\forall x\in I, \int f(x)dx = A\ln|x-x_1|+B\ln|x-x_1|$.

Exemple 1.1.7 $\int \frac{dx}{x^2-1}$ *on* $]-1,1[$.

2^{nd} case: ax^2+bx+c has a double real root x_0: $\forall x\in I, f(x)=\frac{A}{(x-x_0)^2}$ with real A. We obtain $\forall x\in I, \int f(x)dx=\frac{-A}{(x-x_0)}$

Exemple 1.1.8 $\int \frac{dx}{4x^2+4x+1}$ *on* $]0,+\infty[$

3^{rd} case: ax^2+bx+c has no real roots: $\Delta=b^2-4ac<0$. We write ax^2+bx+c in canonical form

$$ax^2+bx+c=a\left[\left(x+\frac{b}{2a}\right)^2-\frac{\Delta}{4a^2}\right]-a\left[\left(x+\frac{b}{2a}\right)^2-A^2\right]$$

with A real, $A>0$. We set $x+\frac{b}{2a}=t$, change of variable therefore affines C^1 and bijective of $\mathbb{R}$ on $\mathbb{R}$. We obtain $\forall x\in I$,

$$\int f(x)dx=\frac{1}{a}\int\frac{dt}{1+t^2}=\frac{1}{aA}\arctan\left(\frac{t}{A}\right)-\frac{1}{aA}\arctan\left(\frac{x+\frac{b}{2a}}{A}\right)$$

1.2 Double integrals

Multiple integrals constitute the generalization of so-called simple integrals: that is to say the integrals of a function of a single real variable. Here we focus on generalization to functions with a greater number of variables (two or three). Recall that a real function f, defined on an interval $[a, b]$, is said to be Riemann integrable if it can be framed between two staircase functions; hence any continuous function is integrable. The integral of f over $[a, b]$, denoted $\int_a^b f(t)dt$, is interpreted as the area between the graph of f, the axis (XoX) and the lines of equations $x = a, a = b$. By subdividing $[a, b]$ into n subintervals $[x_{i-1}, x_i]$ of the same length $\Delta x = \frac{b-a}{n}$, we define the integral of f over $[a, b]$ by:

$$\int_a^b f(x)dx = \lim_{n \longrightarrow +\infty} \sum_{i=1}^{n} f(a_i)(x_i - x_{i-1}), \qquad a_i \in [x_{i-1}, x_i]$$

where $f(a_i)(x_i - x_{i-1})$ represents area of the base rectangle $[x_{i-1}, x_i]$ and height $f(a_i)$:

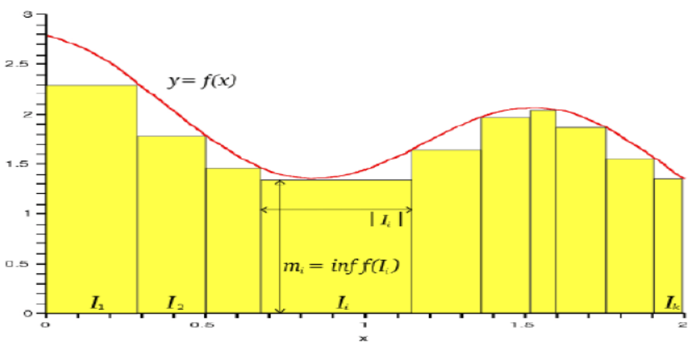

Figure 1.2: Principle of double integral.

1.2.1 Principle of the double integral on a rectangle

Let f be the real function of the two variables x and y, continuous on a rectangle $D = [a, b] \times [c, d]$ of $\mathbb{R}^2$. Its representation is a surface S in the space provided with the reference $\left(O, \overrightarrow{i}, \overrightarrow{j}, \overrightarrow{k}\right)$. We divide D into sub-rectangles, in each sub-rectangle $[x_{i-1}, x_i] \times [y_{i-1}, y_i]$ we choose a point $M(x, y)$ and we calculate the image of (x, y) for the function f. The sum of the volumes of the columns whose base is sub-rectangles and the height $f(x, y)$ is an approximation of the volume between the plane $Z = 0$ and the surface S. When the grid becomes sufficiently "fine" so that the diagonal of each sub-rectangle tends towards 0, this volume will be the limit of the Riemann sums and we note it:

$$\iint_D f(x,y)\,dxdy = \lim_{n\longrightarrow+\infty} \frac{(b-a)(d-c)}{n^2} \sum_{1\le i\le n;1\le j\le n} f\left(a+i\frac{(b-a)}{n}, c+j\frac{(d-c)}{n}\right).$$

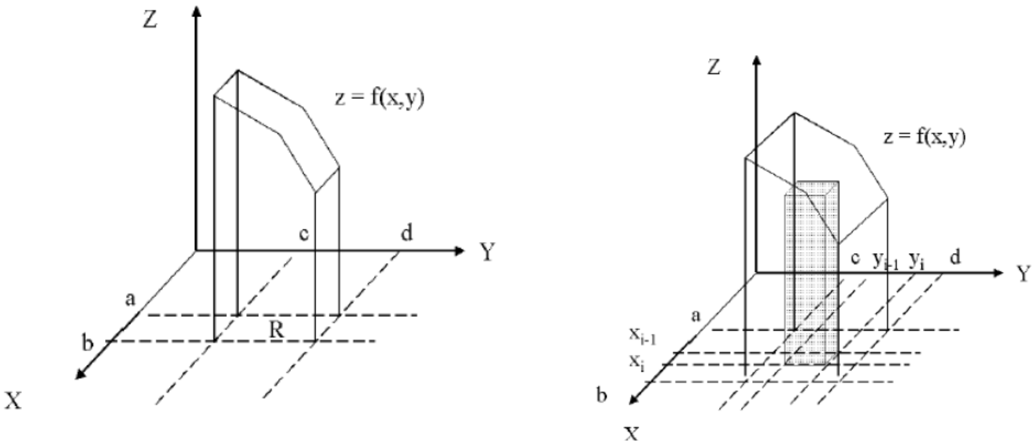

Figure 1.3: Double integral.

Exemple 1.2.1 *Using the definition, calculate* $\iint_{[0,1]\times[0,1]} (x+2y)\,dxdy$.

Remarque 1.2.1 *A priori, the double integral is made to calculate volumes, just as the simple integral was made to calculate an area.*

In a double integral, the terminals at x *and* y *must always be arranged in ascending order.*

Théoreme 1.2.1 *Let* D *be a bounded domain of* $\mathbb{R}^2$. *Then any continuous function* $f : D \longrightarrow \mathbb{R}$ *is integrable in the Riemann sense.*

1.2.2 Properties of double integrals

1. The double integral over a domain D is linear:

$$\iint_D (\alpha f+\mu g)(x,y)\,dxdy = \alpha \iint_D f(x,y)\,dxdy + \mu \iint_D g(x,y)\,dxdy.$$

2. If D and $\acute{D}$ are two domains such that $D \cap \acute{D} = \left\{ \begin{array}{ll} \emptyset, & \text{or} \\ \text{a curve,} & \text{or} \\ \text{isolated points,} & \text{or} \end{array} \right\}$, then:

$$\iint_{D \cup \acute{D}} f(x,y)\,dxdy = \iint_{D} f(x,y)\,dxdy + \iint_{\acute{D}} f(x,y)\,dxdy.$$

3. If $f(x,y) \geq 0$ at any point in D, with f not identically zero, then $\iint_D f(x,y)\,dxdy$ is strictly positive.

4. Si $\forall (x,y) \in D, f(x,y) \leq g(x,y)$, then $\iint_D f(x,y)\,dxdy \leq \iint_{\acute{D}} g(x,y)\,dxdy$.

5. $\left|\iint_D f(x,y)\,dxdy\right| \leq \iint_D |f(x,y)|\,dxdy$.

1.2.3 Fubini formulas

Théoreme 1.2.2 *Let f be a continuous function on a rectangle $D = [a,b] \times [c,d]$ of $\mathbb{R}$. We have*

Théoreme 1.2.3 *Let f be a continuous function on a rectangle $D = [a,b] \times [c,d]$ of $\mathbb{R}$. We have:*

$$\iint_D f(x,y)\,dxdy = \int_c^d \left[\int_a^b f(x,y)\,dx\right] dy.$$

So we calculate a double integral over a rectangle by calculating two single integrals:

- By first integrating with respect to x between a and b (leaving y constant). The result is a function of y.
- By integrating this expression of y between c and d. Alternatively, we can do the same by integrating first at y and then at x.

Exemple 1.2.2 *Calculation of $I = \iint_{[0,\frac{\pi}{2}]\times[0,\frac{\pi}{2}]} \sin(x+y)\,dxdy$. According to Fubini, we have:*

$$I = \int_0^{\frac{\pi}{2}} \left[\int_0^{\frac{\pi}{2}} \sin(x+y)\,dx\right] dy = \int_0^{\frac{\pi}{2}} \left[\int_0^{\frac{\pi}{2}} \sin(x+y)\,dy\right] dx = \int_0^{\frac{\pi}{2}} (\cos y + \sin y)\,dy = [\sin y - \cos y]_0^{\frac{\pi}{2}} = 2.$$

In this example x and y play the same role.

Exemple 1.2.3 *Calculation of* $I = \iint\limits_{[0,1]\times[2,5]} \frac{1}{(1+x+2y)^2} dxdy$. *Let's calculate*

$$I = \int_2^5 \left[\int_0^1 \frac{1}{(1+x+2y)^2} dx \right] dy = \int_2^5 \left[\frac{1}{(1+x+2y)} \right]_0^1 dy$$
$$= \frac{1}{2} \left[\ln(1+2y) - \ln(2+2y) \right]_2^5 = \frac{1}{2} \ln \frac{11}{10}.$$

Special case: If $g : [a,b] \longrightarrow \mathbb{R}$ and $h : [c,d] \longrightarrow \mathbb{R}$ are two continuous functions, then $\iint\limits_{[a,b]\times[c,d]} g(x)h(y)dxdy = \left(\int_a^b g(x)dx \right) \left(\int_c^d h(y)dy \right)$.

Exemple 1.2.4 *Calculate the integral* $I = \iint\limits_{[0,\frac{\pi}{2}]\times[0,\frac{\pi}{2}]} \sin(x)\cos(y)dxdy$.

Théoreme 1.2.4 *Let* f *be a continuous function on a bounded domain* D *of* $\mathbb{R}^2$. *The double integral* $I = \iint\limits_D f(x,y)\,dxdy$ *is calculated in one of the following ways:*

- If we can represent the domain D *in the form* $D = \left\{(x,y) \in \mathbb{R}^2 / f_1(x) \le y \le f_2(x), a \le x \le b\right\}$ *then*

$$\iint\limits_D f(x,y)\,dxdy = \int_a^b \left[\int_{f_1(x)}^{f_2(x)} f(x,y)\,dy \right] dx.$$

- If we can represent the domain D *in the form* $D = \left\{(x,y) \in \mathbb{R}^2 / g_1(x) \le x \le g_2(x), c \le y \le d\right\}$, *then:*

$$\iint\limits_D f(x,y)\,dxdy = \int_c^d \left[\int_{g_1(x)}^{g_2(x)} f(x,y)\,dx \right] dy.$$

- If both representations are possible, the two results are obviously equal.

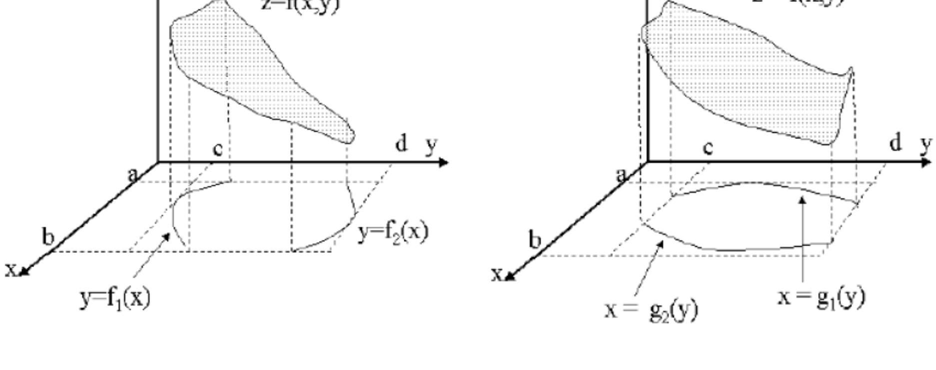

Figure 1.4: Theorem ilustration.

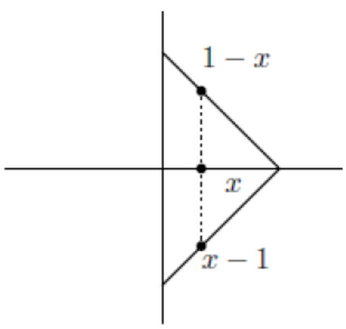

Figure 1.5: The domain D.

Exemple 1.2.5 *Calculate the integral* $\iint\limits_D (x^2+y^2)\,dxdy$ *with* D *is the triangle with vertices* $(0,1),(0,-1)$ *and* $(1,0)$. *For this we will define* D *analytically by the inequalities:*

$$D=\left\{(x,y)\in\mathbb{R}^2/x-1\le y\le 1-x, 0\le x\le 1\right\}$$

$$\iint\limits_D (x^2+y^2)\,dxdy=\int\limits_0^1\left[\int\limits_{x-1}^{1-x}(x^2+y^2)\,dy\right]dx=\int\limits_0^1\left[x^2y+\frac{y^3}{3}\right]_{x-1}^{1-x}dx=\frac{1}{3}.$$

Exemple 1.2.6 *Calculate* $I=\iint\limits_D (x+2y)\,dxdy$ *on the domain* D *formed by the union of the left part of the unit disk and the triangle of vertices* $(0,-1),(0,1)$ *and* $(2,1)$. *We have*

$$I=\int\limits_{-1}^{1}\left[\int\limits_{-\sqrt{1-y^2}}^{\sqrt{y+1}}(x+2y)\,dx\right]dy=\int\limits_{-1}^{1}\left(3y+3y^2+2y\sqrt{1-y^2}\right)dy=2.$$

Exemple 1.2.7 *Calculate the integral* $I=\iint\limits_D e^{x^2}dxdy$ *where* $D=\left\{(x,y)\in\mathbb{R}^2/0\le y\le x\le 1\right\}$. *The domain is the interior of the triangle limited by the* x *axis, the line* $x=1$ *and the line* $y=x$. *In this case we are obliged to integrate first with respect to* y *then with respect to* x, *because the primitive of the function is not expressed using the usual functions. Hence* $I=\int\limits_0^1\left[\int\limits_0^x e^{x^2}dy\right]dx=\int\limits_0^1 xe^{x^2}dx=\frac{e-1}{2}$.

Exemple 1.2.8 *Calculate* $I=\int\limits_0^4\left[\int\limits_{2x}^{8}\sin\left(y^2\right)dy\right]dx=\int\limits_0^8\left[\int\limits_0^{\frac{y}{2}}\sin\left(y^2\right)dx\right]dy=\frac{1}{4}\int\limits_0^8 2y\sin\left(y^2\right)dy=\frac{1-\cos 64}{4}$.

1.2.4 Change of variable

We will have a result similar to that of the simple integral, where the change of variable $x = \varphi(t)$ required us to replace the "dx" by $\acute{\varphi}(t)$. It is the Jacobian which will play the role of the derivative[1].

Théoreme 1.2.5 *Let $(u,v) \in \Delta \longrightarrow (x,y) = \varphi(u,v) \in D$ be a bijection of class C^1 from domain Δ to domain D. Let $|J_\varphi|$ the absolute value of the determinant of the Jacobian matrix of φ. So, we have:*

$$\iint_D f(x,y)\,dxdy = \iint_\Delta f\circ\varphi(u,v)\,|J_\varphi|\,dudv.$$

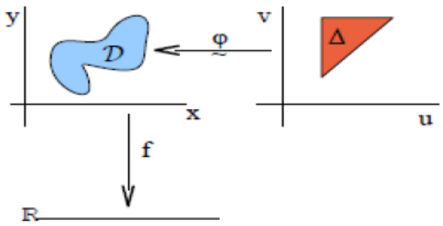

Figure 1.6: Changement de variable pour les intégrales doubles.

Exemple 1.2.9 *Calculate $\iint_D (x-1)^2\,dxdy$ on the domain with*

$$D = \left\{(x,y) \in \mathbb{R}^2 / -1 \leq x+y \leq 1, -2 \leq x-y \leq 2\right\}.$$

By changing the variable $u = x+y, v = x-y$. The domain D in (u,v) is therefore the rectangle $\{-1 \leq u \leq 1, -2 \leq v \leq 2\}$. We also have $x = \dfrac{u+v}{2}, y = \dfrac{u-v}{2}$. The Jacobian of this change of

[1] We call the Jacobian matrix $\varphi : \mathbb{R}^n \longrightarrow \mathbb{R}^p$ of the matrix with p rows and n columns:

$$J_\varphi = \begin{pmatrix} \frac{\partial\varphi_1}{\partial x_1} & \frac{\partial\varphi_1}{\partial x_2} & \cdots\cdots & \frac{\partial\varphi_1}{\partial x_n} \\ \frac{\partial\varphi_2}{\partial x_1} & \frac{\partial\varphi_2}{\partial x_1} & \cdots\cdots & \frac{\partial\varphi_2}{\partial x_n} \\ . & . & \cdots\cdots & . \\ . & . & \cdots\cdots & . \\ \frac{\partial\varphi_p}{\partial x_1} & \frac{\partial\varphi_p}{\partial x_2} & \cdots\cdots & \frac{\partial\varphi_p}{\partial x_n} \end{pmatrix}$$

The first column contains the partial derivatives of the coordinates of φ with respect to the first variable x_1, the second column contains the partial derivatives of the coordinates of φ with respect to the second variable x_2 and so on.

variables is $J = \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix} = \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{pmatrix}$ *whose determinant is* $\frac{-1}{2}$. *And so*

$$I = \frac{1}{8} \int_{-2}^{2} \left[\int_{-1}^{1} (u+v-2)^2 \, du \right] dv = \frac{136}{3}.$$

Remarque 1.2.2 *- If* $|\det(J_\varphi)| = 1$*, we obtain* $\iint\limits_D f(x,y)\,dxdy = \iint\limits_\Delta f[\varphi(u,v)]\,dudv$

- This allows us to use symmetries: if for example $\forall (x,y) \in D, (-x,y) \in D$ *et* $f(-x,y) = f(x,y)$ *then* $\iint\limits_D f(x,y)\,dxdy = 2\iint\limits_{\acute{D}} f(x,y)\,dxdy$*, where* $\acute{D} = D \cap (\mathbb{R}^+ \times \mathbb{R})$.

Changing variable to polar coordinates

Let $\varphi : \mathbb{R}^2 \longrightarrow \mathbb{R}$ be such that $(r,\theta) \longrightarrow (r\cos\theta, r\sin\theta)$. Then φ is of class C^1 on $\mathbb{R}^2$, and its Jacobian is $J_\varphi(r,\theta) = \begin{vmatrix} r\cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r$. Then

$$I = \iint\limits_D f(x,y)\,dxdy = \iint\limits_\Delta g(r,\theta)\,rdrd\theta.$$

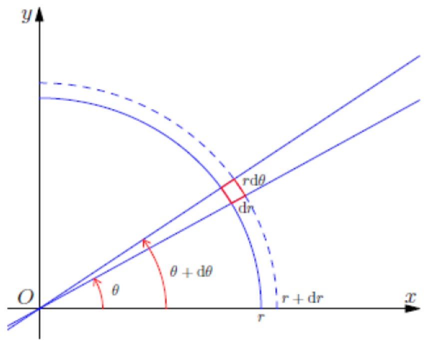

Figure 1.7: Changing variable to polar coordinates.

Exemple 1.2.10 *1) Calculate by passing in polar coordinates* $I = \iint\limits_D \frac{1}{x^2+y^2}dxdy$ *where* $D = \{(x,y) : 1 \le x^2+y^2 \le 4, x \ge 0, y \ge 0\}$ *which represents a quarter of the part between the two circles centered at the origin and with radii 1 and 2 (ring). From where*

$$I = \iint\limits_D \frac{1}{x^2+y^2}dxdy = \int_0^{\frac{\pi}{2}} \int_1^2 \frac{r}{r^2} drd\theta = \frac{\pi}{2}\ln 2.$$

2) Calculate the volume of a sphere $V = \iint\limits_{x^2+y^2<R^2} \sqrt{R^2 - x^2 - y^2}dxdy$ *and since the function is even with respect to the two variables,* $V = 8\int\limits_0^{\frac{\pi}{2}}\int\limits_0^R \sqrt{R^2 - r^2}rdrd\theta = \frac{4}{3}\pi R^2.$

1.2.5 Applications

1. **Calculation of area of a domain D:** We have seen that $\iint\limits_D f(x,y)\,dxdy$ measures the volume under the representation of f and above D. We also have the possibility of using the double integral to calculate the area itself of domain D. To do this, simply take $f(x,y) = 1$. Thus, the area A of the domain is $A = \iint\limits_D dxdy = \iint\limits_\Delta rdrd\theta$.

   Exemple 1.2.11 *Calculate the area delimited by the ellipse with equation* $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. *Let us note the area ofthis ellipse A, therefore* $A = \iint\limits_{\frac{x^2}{a^2}+\frac{y^2}{b^2}<1} dxdy$. *By symmetry and passing to generalized polar coordinates:* $x = ar\cos\theta, y = br\sin\theta$, *we obtain* $A = 4\int\limits_0^{\frac{\pi}{2}}\int\limits_0^1 abrdrd\theta = \pi ab$.

2. **Calculation of the area of a surface:** We call D the region of the XOY plane delimited by the projection onto the XOY plane of the surface representative of a function f, denoted $\sum$. The surface area of $\sum$ delimited by its projection D on the plane XOY is given by $A = \iint\limits_D \sqrt{\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 + 1dxdy}$

   Exemple 1.2.12 *Let's calculate the area of the paraboloid* $\sum = \left\{(x,y,z) : z = x^2 + y^2, 0 \leq z \leq h\right\}$. *Since the surface* $\sum$ *is equal to the graph of the function* $f(x,y) = x^2 + y^2$ *defined above the domain* $D = \left\{(x,y) : x^2 + y^2 \leq h\right\}$. *From where:*

$$Aire\left(\sum\right) = \iint\limits_D \sqrt{4x^2 + 4y^2 + 1}dxdy = 2\pi \int_0^{\sqrt{h}} \sqrt{4r^2 + 1}rdr = \frac{\pi}{6}(4h+1)^{3/2}.$$

3. **Mass and centers of inertia:** If we note $\rho(x,y)$ s the surface density of a plate Δ, its mass is given by the formula $M = \iint\limits_\Delta \rho(x,y)\,dxdy$. And its center of inertia $G = (x_G, y_G)$

is such that:

$$x_G = \frac{1}{M}\iint\limits_{\Delta} x\rho(x,y)\,dxdy$$
$$y_G = \frac{1}{M}\iint\limits_{\Delta} y\rho(x,y)\,dxdy$$

Exemple 1.2.13 *Determine the center of mass of a thin triangular metal plate whose vertices are at* $(0,0), (1,0)$ *et* $(0,2)$*, knowing that its density is* $\rho(x,y) = 1+3x+y$.

$$M = \iint\limits_{\Delta} \rho(x,y)\,dxdy = \int_0^1\int_0^{2-2x} (1+3x+y)\,dxdy = \frac{8}{3}$$
$$x_G = \frac{1}{M}\iint\limits_{\Delta} x\rho(x,y)\,dxdy = \int_0^1\int_0^{2-2x} x(1+3x+y)\,dxdy = \frac{3}{8}$$
$$y_G = \frac{1}{M}\iint\limits_{\Delta} y\rho(x,y)\,dxdy = \int_0^1\int_0^{2-2x} y(1+3x+y)\,dxdy = \frac{11}{16}$$

4. **<u>The moment of inertia:</u>** The moment of inertia of a point mass M with respect to an axis is defined by Mr^2, where r is the distance between the mass and the axis. We extend this notion to a metal plate which occupies a region D and whose density is given by $\rho(x,y)$, the moment of inertia of the plate with respect to the axis $(\acute{X}OX)$ is: $I_x = \iint\limits_D y^2\rho(x,y)\,dxdy$. Similarly, the moment of inertia of the plate with respect to the axis $(\acute{y}Oy)$ is: $I_y = \iint\limits_D x^2\rho(x,y)\,dxdy$. It is also interesting to consider the moment of inertia relative to the origin: $I_O = \iint\limits_D \left(x^2+y^2\right)\rho(x,y)\,dxdy$.

1.3 Triple integrals

The principle is the same as for double integrals, If $(x,y,z) \longrightarrow f(x,y,z) \in \mathbb{R}$ is a continuous function of three variables on a domain D of $\mathbb{R}^3$, we define $\iiint\limits_D f(x,y,z)\,dxdydz$ as sum limit of the form:

$$\sum_{i,j,k} f(u_i,v_j,w_k)(x_i - x_{i-1})(y_j - y_{j-1})(z_k - z_{k-1})$$

Remarque 1.3.1 *We have the same algebraic properties of double integrals: linearity, ...*

1.3.1 Formules de Fubini

1. **On a parallelepiped:** Fubini's theorem applies quite naturally when $D = [a,b] \times [c,d] \times [e,f]$, we come down to calculating three simple integrals:

$$\iiint_D f(x,y,z)\, dxdydz = \int_a^b \left[\int_c^d \left[\int_e^f f(x,y,z)\, dz\right] dy\right] dx = \int_e^f \left[\int_c^d \left[\int_a^b f(x,y,z)\, dx\right] dy\right] dz = \ldots$$

Exemple 1.3.1 *Calculate* $\iiint_{[0,1]\times[1,2]\times[1,3]} (x + 3yz)\, dxdydz.$

2. **On any bounded domain:** o establish the treatment of the search for the integration bounds. For a certain fixed x, varying between $x_{\min}$ and $x_{\max}$, we cut out a surface D_x in D. We can then represent in the YOZ plane, then the treatment on D_x is done as with double integrals: $I = \int_{x_{\min}}^{x_{\max}} \left[\int_{y_{\min}}^{y_{\max}} \left[\int_{z_{\min}}^{z_{\max}} f(x,y,z)\, dz\right] dy\right] dx$. Of course, we can swap the roles of x, y and z.

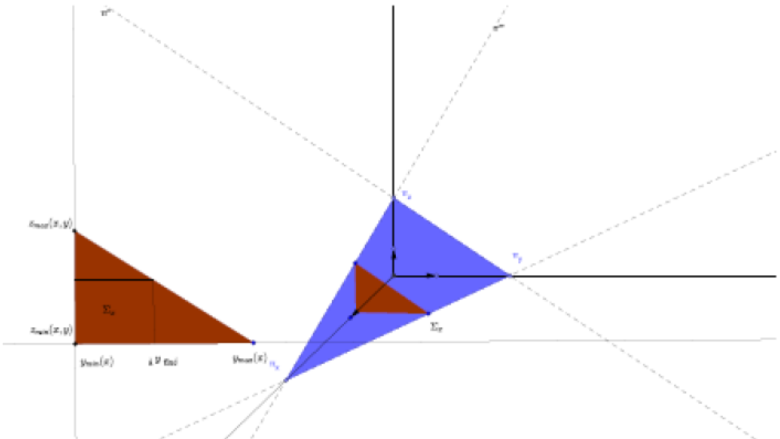

Figure 1.8: Triple integral.

Exemple 1.3.2 *Calculate* $I = \iiint_D (x^2 + yz)\, dxdydz$ *in the domain*

$$D = \{(x,y,z) : x \geq 0, y \geq 0, z \geq 0, x + y + 2h \leq 1\}$$

$$I = \iiint_D (x^2 + yz)\, dxdydz = \int_0^{1/2} \left[\int_0^{1-2z} \left[\int_0^{1-2z-x} (x^2 + yz)\, dy\right] dx\right] dz = \frac{1}{96}.$$

1.3.2 Changing variables

If we have a bijective map φ and class C^1 from domain Δ to domain D, defined by: $(u,v,w) \longrightarrow \varphi(u,v,w) = (x,y,z)$. The formula for changing variables is: $\iiint\limits_D f(x,y,z)\,dxdydz = \iiint\limits_\Delta f \circ \varphi(u,v,w)\,|J_\varphi(u,v,w)|\,dudvdw$. By noting $|J_\varphi|$ the absolute value of the determinant of the Jacobian.

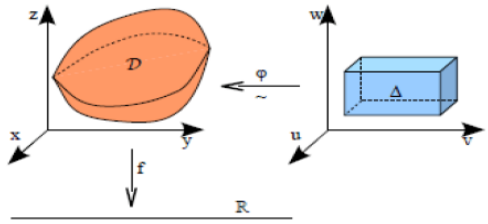

Figure 1.9: Changing variables for triple integrales.

1. **Calculation in cylindrical coordinates:** In dimension 3, the cylindrical coordinates are given by:

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{cases}$$

The determinant of the Jacobian matrix of $\varphi(r,\theta,z) \longrightarrow (x,y,z)$ is:

$$|J_\varphi| = \begin{vmatrix} r\cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = r dr d\theta dz$$

So we have

$$I = \iiint\limits_D f(x,y,z)\,dxdydz = \iiint\limits_\Delta g(r,\theta,z)\,r dr d\theta dz = \int_{\theta_{\min}}^{\theta_{\max}} \left[\int_{r_{\min}}^{r_{\max}} \left[\int_{z_{\min}}^{z_{\max}} g(r,\theta,z)\,r dz\right] dr\right] d\theta.$$

Exemple 1.3.3 *Calculate* $I = \iiint\limits_V \left(x^2+y^2+1\right) dxdydz$ *or*

$$D = \left\{(x,y,z) : x^2+y^2 \le 1,\ \text{and } 0 \le z \le 2\right\}$$

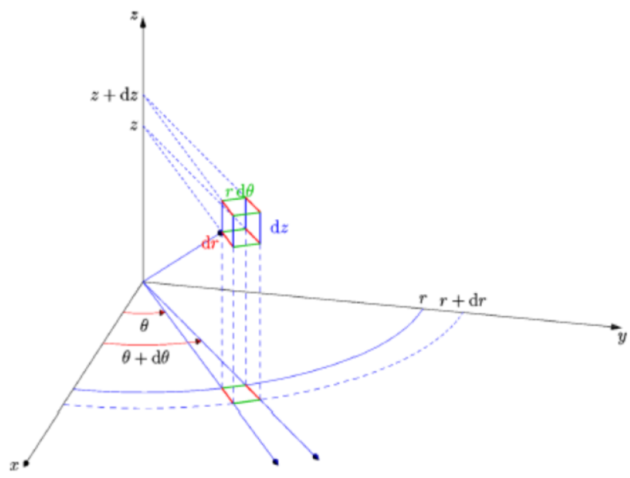

Figure 1.10: Principle of calculation of the Jacobian in cylindrical coordinates.

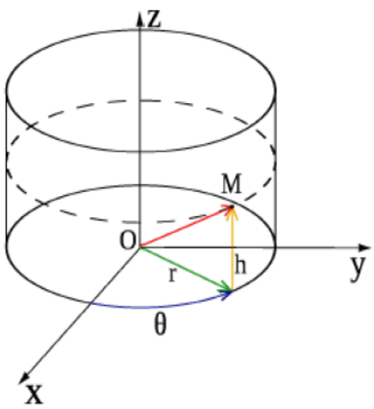

Figure 1.11: Cylindrical coordinates.

$$I=\int_0^2\int_0^{2\pi}\int_0^1 r\left(r^2+1\right)drd\theta dz=\int_0^{2\pi}d\theta\int_0^2 dz\left[\frac{1}{4}\left(r^2+1\right)^2\right]_0^1=4\pi.$$

2. **Calculation in spherical coordinates:** In dimension 3, the spherical coordinates are given by:

$$\begin{cases} x=r\sin\theta\cos\varphi \\ y=r\sin\theta\sin\varphi \\ z=r\cos\theta \end{cases}$$

The determinant of the Jacobian matrix of $\varphi(r,\theta,\varphi)\longrightarrow(x,y,z)$ is

$$|J_\varphi|=\begin{vmatrix} \sin\theta\cos\varphi & r\cos\theta\cos\varphi & -r\sin\theta\sin\varphi \\ \sin\theta\sin\varphi & r\cos\theta\sin\varphi & r\sin\theta\cos\varphi \\ \cos\theta & -r\sin\theta & 0 \end{vmatrix}=r^2\sin\theta drd\theta d\varphi.$$

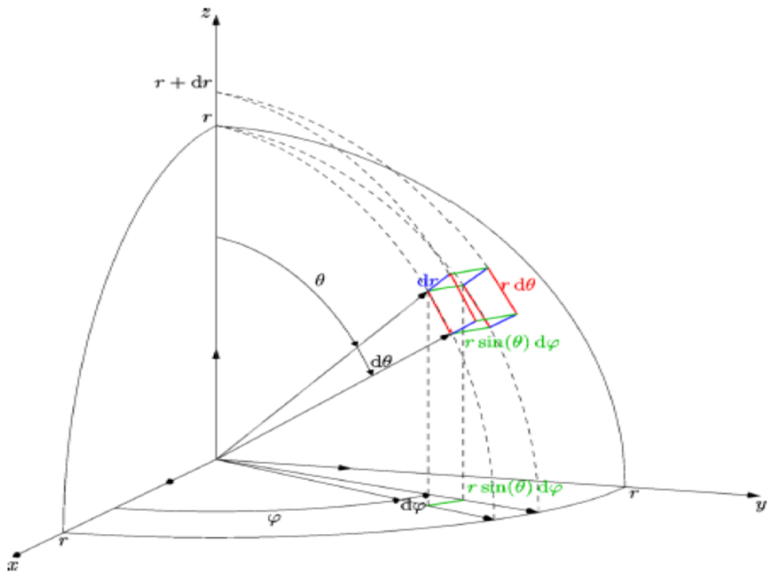

Figure 1.12: Principle of calculation of the Jacobian in spherical coordinates.

So we have

$$I = \iiint_D f(x, y, z)\, dxdydz = \iiint_\Delta g(r, \theta, z)\, r^2 \sin\theta dr d\theta d\varphi.$$

Exemple 1.3.4 *Calculate* $I = \iiint_D z dxdydz$, *or*

$$D = \left\{(x, y, z) : x^2 + y^2 + z^2 \leq R^2, \text{ and } z \geq 0\right\}.$$

The domain is the upper hemisphere (centered at the origin and of radius R), passing to spherical coordinates:

$$I = \int_0^{2\pi} d\varphi \int_0^{\frac{\pi}{2}} \cos\theta \sin\theta d\theta \int_0^R r^2 dr = \frac{\pi}{3} R^3$$

1.3.3 Applications

1. **Volume:** The volume of a body is given by $V = \iiint_D dxdydz$ such that D is the domain delimited by this body.

Exemple 1.3.5 *Calculate the volume of a sphere,* $V = \iiint_{x^2+y^2+z^2<R^2} dxdydz$, *according to*

the property of symmetry: $V = 8 \iiint\limits_{D} dxdydz$ *where*

$$D = \left\{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 \leq R^2, \ \textit{and } x \geq 0, y \geq 0, z \geq 0\right\}$$

from where $V = 8 \int_0^{\frac{\pi}{2}} d\varphi \int_0^{\frac{\pi}{2}} \sin\theta d\theta \int_0^R r^2 dr = \dfrac{4\pi}{3} R^3.$

2. **Mass, center and moments of inertia**: Let μ be the density of a solid which occupies region V, then its mass is given by

$$M = \iiint\limits_{V} \mu(x, y, z)\, dxdydz.$$

The center of mass $G = (x_G, y_G, z_G)$ has coordinates.

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$$x_G = \frac{1}{M} \iiint\limits_{V} x\mu(x, y, z)\, dxdydz.$$
$$y_G = \frac{1}{M} \iiint\limits_{V} y\mu(x, y, z)\, dxdydz.$$
$$z_G = \frac{1}{M} \iiint\limits_{V} z\mu(x, y, z)\, dxdydz.$$

The moments of inertia with respect to the three axes are:

$$I_x = \iiint\limits_{V} \left(y^2 + z^2\right) \mu(x, y, z)\, dxdydz.$$
$$I_y = \iiint\limits_{V} \left(x^2 + z^2\right) \mu(x, y, z)\, dxdydz.$$
$$I_z = \iiint\limits_{V} \left(y^2 + x^2\right) \mu(x, y, z)\, dxdydz.$$

Exemple 1.3.6 *Determine the center of mass of a solid of constant density, bounded by the parabolic cylinder* $x = y^2$ *and the planes* $x = z, z = 0$ *and* $x = 1$.
The mass is $= \int\limits_{-1}^{1} \left[\int\limits_{y^2}^{1} \left[\int\limits_{0}^{x} \mu dz\right] dx\right] dy = \dfrac{4\mu}{5}$, *due to symmetry of the domain and* μ *with*

Figure 1.13: A solid bounded by the parabolic cylinder $x = y^2$ and the planes $x = z, z = 0$ and $x = 1$.

respect to the OXZ plane, we has

$$y_G = \frac{1}{M}\iiint\limits_V y\mu dxdydz = \frac{\mu}{M}\int\limits_{-1}^{1}\left[\int\limits_{y^2}^{1}\left[\int\limits_{0}^{x} xdz\right]dx\right]dy = \frac{5}{7}.$$

$$z_G = \frac{1}{M}\iiint\limits_V z\mu dxdydz = \frac{\mu}{M}\int\limits_{-1}^{1}\left[\int\limits_{y^2}^{1}\left[\int\limits_{0}^{x} zdz\right]dx\right]dy = \frac{5}{14}.$$