Exercise series N°1

Exercise 1 *:* Consider the following assertions:

A_1 : $\exists x \in \mathbb{R}$, $\forall y \in \mathbb{R}$: $x + y > 0$.

A_2 : $\forall x \in \mathbb{R}$, $\exists y \in \mathbb{R}$: $x + y > 0$.

A_3 : $\forall x \in \mathbb{R}$, $\forall y \in \mathbb{R}$: $x + y > 0$.

A_4 : $\exists x \in \mathbb{R}$, $\forall y \in \mathbb{R}$: $y^2 > x$.

1. Are assertions A_1, A_2, A_3 and A_4 true or false?
2. Give their negation.

Exercise 2 *:*

- If a and b are two positive or zero real numbers, show that:

$$\sqrt{a} + \sqrt{b} \leq 2\sqrt{a+b}.$$

- Prove by induction the following equalities:

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2} \quad \textit{and} \quad \sum_{k=0}^{n-1} 2^k = 2^n - 1, \quad \textit{with} \;\; n \in \mathbb{N}^*$$

- Show that $\sqrt{2}$ is not a rational number.

Exercise 3 Let x and $y \in \mathbb{R}$.

1. Show that the following relationships are always true:
   (a) If $|x| < y$ *then* $-y < x < y$
   (b) $|x + y| \leq |x| + |y|$.
   (c) $||x| - |y|| \leq |x - y|$.
2. Solve the following inequalities:
   (a) $|x - 2| > 5$.
   (b) $|x + 2| > |x|$.
   (c) $|2x - 1| < |x - 1|$.

Exercise 4 Determine (if they exist): the all upper and lower bounds, supremum, infimum, maximum, and minimum, of the following sets:

$$E_1 = \left\{1, \frac{1}{3}, \frac{1}{5}, ..., \frac{1}{2n+1}, ...; \;\; n \in \mathbb{N}\right\}, \qquad E_2 =]0, 5], \qquad E_3 = \left\{4 - \frac{1}{n}; n \in \mathbb{N}^*\right\},$$

$$E_4 = \left\{\frac{1}{2} + \frac{n}{2n+1}, \; \frac{1}{2} - \frac{n}{2n+1}; \;\; n \in \mathbb{N}^*\right\}$$

Exercise 5 Show that the following relationships are true.

- $x - 1 < E(x) \leq x$,
- $E(x) + E(y) \leq E(x + y)$,
- $E(x) - E(y) \geq E(x - y)$,
- $E\left(\frac{E(nx)}{n}\right) = E(x)$,

with $x,\ y \in \mathbb{R}$, $n \in \mathbb{N}^*$ and $E(.)$ is the integral part function.