Universitye of Biskra
Department of Mathematics

L3 Mathematics
Module: ODEs

Interrogation 02

Exercise:-01 ____________ **06 points**

1. Find the general solution of the given differential equation.:

$$y'' + y = 3\sin(2t) + t\cos(2t)$$

2. Use the method of variation of parameters to find the general solution of the differential equation:

$$ty'' - (1+t)y' + y = t^2e^{2t}; \qquad t > 0; \qquad y_1(t) = 1 + t$$

Exercise:- 02 ____________ **04 points**

Solve the given initial value problem. Sketch the graph of the solution and describe its behavior for increasing t.

$$y'' + 4y' + 4y = 0; \qquad y(-1) = 2; \quad y'(-1) = 1$$

Solution

Because this ODE is linear, the general solution can be expressed as a sum of the complementary solution $y_c(t)$ and the particular solution $y_p(t)$.

$$y(t) = y_c(t) + y_p(t)$$

The complementary solution satisfies the associated homogeneous equation.

$$y_c'' + y_c = 0 \tag{1}$$

This is a homogeneous ODE with constant coefficients, so the solution is of the form $y_c = e^{rt}$.

$$y_c = e^{rt} \quad \to \quad y_c' = re^{rt} \quad \to \quad y_c'' = r^2e^{rt}$$

Substitute these expressions into the ODE.

$$r^2e^{rt} + e^{rt} = 0$$

Divide both sides by e^{rt}.

$$r^2 + 1 = 0$$
$$r = \{-i, i\}$$

Two solutions to equation (1) are then $y_c = e^{-it}$ and $y_c = e^{it}$. By the principle of superposition, the general solution is a linear combination of these two.

$$\begin{aligned}
y_c(t) &= C_1e^{-it} + C_2e^{it} \\
&= C_1[\cos(-t) + i\sin(-t)] + C_2[\cos(t) + i\sin(t)] \\
&= C_1[\cos(t) - i\sin(t)] + C_2[\cos(t) + i\sin(t)] \\
&= C_1\cos t - iC_1\sin t + C_2\cos t + iC_2\sin t \\
&= (C_1 + C_2)\cos t + (-iC_1 + iC_2)\sin t \\
&= C_3\cos t + C_4\sin t
\end{aligned}$$

On the other hand, the particular solution satisfies

$$y_p'' + y_p = 3\sin 2t + t\cos 2t.$$

There are two terms on the right side. For the first one, since only even derivatives are present, we will include $A\sin 2t$ in the trial solution. For the second one, we will include $Bt\cos 2t + C\sin 2t$. The trial solution is thus $y_p(t) = A\sin 2t + Bt\cos 2t + C\sin 2t$. Substitute this into the ODE to determine A and B and C.

$$(A\sin 2t + Bt\cos 2t + C\sin 2t)'' + (A\sin 2t + Bt\cos 2t + C\sin 2t) = 3\sin 2t + t\cos 2t$$

$$(2A\cos 2t + B\cos 2t - 2Bt\sin 2t + 2C\cos 2t)' + (A\sin 2t + Bt\cos 2t + C\sin 2t) = 3\sin 2t + t\cos 2t$$

$$(-4A\sin 2t - 2B\sin 2t - 2B\sin 2t - 4Bt\cos 2t - 4C\sin 2t) + (A\sin 2t + Bt\cos 2t + C\sin 2t) = 3\sin 2t + t\cos 2t$$

$$(-4A - 2B - 2B - 4C + A + C)\sin 2t + (-4B + B)t\cos 2t = 3\sin 2t + t\cos 2t$$

For this equation to be true, A and B and C must satisfy the following system of equations.

$$-4A - 2B - 2B - 4C + A + C = 3$$
$$-4B + B = 1$$

Solving it yields $A + C = -5/9$ and $B = -1/3$, which means

$$\begin{aligned} y_p(t) &= A\sin 2t + Bt\cos 2t + C\sin 2t \\ &= (A + C)\sin 2t + Bt\cos 2t \\ &= -\frac{5}{9}\sin 2t - \frac{1}{3}t\cos 2t. \end{aligned}$$

Therefore,

$$y(t) = C_3\cos t + C_4\sin t - \frac{5}{9}\sin 2t - \frac{1}{3}t\cos 2t.$$

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Since one solution is known, the method of reduction of order can be applied to determine the general solution. Substitute $y(t) = c(t)(1+t)$ into the ODE and solve the resulting ODE for $c(t)$.

$$t[c(t)(1+t)]'' - (1+t)[c(t)(1+t)]' + [c(t)(1+t)] = t^2e^{2t}$$

Evaluate the derivatives.

$$\begin{aligned} t[c'(t)(1+t) + c(t)]' - (1+t)[c'(t)(1+t) + c(t)] + [c(t)(1+t)] &= t^2e^{2t} \\ t[c''(t)(1+t) + c'(t) + c'(t)] - (1+t)[c'(t)(1+t) + c(t)] + [c(t)(1+t)] &= t^2e^{2t} \\ t(1+t)c''(t) + 2tc'(t) - (1+t)^2c'(t) - \cancel{(1+t)c(t)} + \cancel{c(t)(1+t)} &= t^2e^{2t} \\ t(1+t)c''(t) - (1+t^2)c'(t) &= t^2e^{2t} \end{aligned}$$

Divide both sides by $t(1+t)$.

$$c''(t) - \frac{1+t^2}{t(1+t)}c'(t) = \frac{t}{1+t}e^{2t}$$

Use an integrating factor I to solve this ODE.

$$I = \exp\left[\int^t -\frac{1+s^2}{s(1+s)}\,ds\right] = \exp\left[\int^t \left(-1 - \frac{1}{s} + \frac{2}{1+s}\right)ds\right] = e^{-t-\ln t+2\ln(1+t)} = e^{-t}t^{-1}(1+t)^2$$

Multiply both sides of the previous equation by I.

$$\frac{(1+t)^2}{t}e^{-t}c''(t) - \frac{(1+t^2)(1+t)}{t^2}e^{-t}c'(t) = (1+t)e^t$$

The left side can be written as $d/dt[Ic'(t)]$ by the product rule.

$$\frac{d}{dt}\left[\frac{(1+t)^2}{t}e^{-t}c'(t)\right] = (1+t)e^t$$

Integrate both sides with respect to t.

$$\begin{aligned} \frac{(1+t)^2}{t}e^{-t}c'(t) &= \int^t (1+s)e^s\,ds + C_1 \\ &= \int^t e^s\,ds + \int^t se^s\,ds + C_1 \\ &= e^t + \int^t s\frac{d}{ds}(e^s)\,ds + C_1 \\ &= e^t + \left[s(e^s)\Big|^t - \int^t (1)e^s\,ds\right] + C_1 \\ &= e^t + (te^t - e^t) + C_1 \\ &= te^t + C_1 \end{aligned}$$

Divide both sides by I.

$$c'(t) = \frac{t^2}{(1+t)^2}e^{2t} + C_1\frac{te^t}{(1+t)^2}$$

Integrate both sides with respect to t once more.

$$c(t) = \frac{t-1}{2(1+t)}e^{2t} + C_1\frac{e^t}{1+t} + C_2$$

Therefore,

$$\begin{aligned} y(t) &= c(t)(1+t) \\ &= \left[\frac{t-1}{2(1+t)}e^{2t} + C_1\frac{e^t}{1+t} + C_2\right](1+t) \\ &= \frac{t-1}{2}e^{2t} + C_1e^t + C_2(1+t). \end{aligned}$$

The terms containing C_1 and C_2 are the second and first solutions, respectively, to the associated homogeneous equation, and the first term is the particular solution.

In each of Problems 11 through 14, solve the given initial value problem. Sketch the graph of the solution and describe its behavior for increasing t.

$$y'' + 4y' + 4y = 0, \qquad y(-1) = 2, \quad y'(-1) = 1$$

Solution

Since this is a linear homogeneous constant-coefficient ODE, the solution is of the form $y = e^{rt}$.

$$y = e^{rt} \quad \rightarrow \quad y' = re^{rt} \quad \rightarrow \quad y'' = r^2 e^{rt}$$

Substitute these expressions into the ODE.

$$r^2 e^{rt} + 4(re^{rt}) + 4(e^{rt}) = 0$$

Divide both sides by e^{rt}.

$$r^2 + 4r + 4 = 0$$
$$(r+2)^2 = 0$$
$$r = \{-2\}$$

One solution to the ODE is $y = e^{-2t}$. Because the ODE is homogeneous, any constant multiple of this is also a solution, that is, $y = ce^{-2t}$. According to the method of reduction of order, the general solution is found by allowing c to vary as a function of t.

$$y(t) = c(t)e^{-2t}$$

Substitute this expression for y into the original ODE to determine $c(t)$.

$$y'' + 4y' + 4y = 0 \quad \rightarrow \quad [c(t)e^{-2t}]'' + 4[c(t)e^{-2t}]' + 4[c(t)e^{-2t}] = 0$$

Evaluate the derivatives using the product rule.

$$[c'(t)e^{-2t} - 2c(t)e^{-2t}]' + 4[c'(t)e^{-2t} - 2c(t)e^{-2t}] + 4[c(t)e^{-2t}] = 0$$

$$[c''(t)e^{-2t} - 2c'(t)e^{-2t} - 2c'(t)e^{-2t} + 4c(t)e^{-2t}] + 4[c'(t)e^{-2t} - 2c(t)e^{-2t}] + 4[c(t)e^{-2t}] = 0$$

$$c''(t)e^{-2t} - \cancel{2c'(t)e^{-2t}} - \cancel{2c'(t)e^{-2t}} + \cancel{4c(t)e^{-2t}} + \cancel{4c'(t)e^{-2t}} - \cancel{8c(t)e^{-2t}} + \cancel{4c(t)e^{-2t}} = 0$$

$$c''(t)e^{-2t} = 0$$

Divide both sides by e^{-2t}.

$$c''(t) = 0$$

Integrate both sides with respect to t.

$$c'(t) = C_1$$

Integrate both sides with respect to t once more.

$$c(t) = C_1 t + C_2$$

The general solution is then

$$y(t) = C_1 te^{-2t} + C_2 e^{-2t}.$$

Differentiate it with respect to t.

$$y'(t) = C_1 e^{-2t} - 2C_1 te^{-2t} - 2C_2 e^{-2t}$$

Apply the initial conditions now to determine C_1 and C_2.

$$y(-1) = -C_1 e^2 + C_2 e^2 = 2$$
$$y'(-1) = C_1 e^2 + 2C_1 e^2 - 2C_2 e^2 = 1$$

Solving this system of equations yields $C_1 = 5/e^2$ and $C_2 = 7/e^2$. Therefore,

$$y(t) = \frac{5}{e^2} te^{-2t} + \frac{7}{e^2} e^{-2t}.$$

Take the limit of $y(t)$ as $t \to \infty$.

$$\begin{aligned}
\lim_{t\to\infty} y(t) &= \lim_{t\to\infty} \left(\frac{5}{e^2} te^{-2t} + \frac{7}{e^2} e^{-2t} \right) \\
&= \lim_{t\to\infty} \left(\frac{5}{e^2} t + \frac{7}{e^2} \right) e^{-2t} \\
&= \lim_{t\to\infty} \frac{\frac{5}{e^2} t + \frac{7}{e^2}}{e^{2t}} \\
&\overset{\frac{\infty}{\infty}}{\underset{\text{H}}{=}} \lim_{t\to\infty} \frac{\frac{5}{e^2}}{2e^{2t}} \\
&= 0
\end{aligned}$$